Introduction to the Standard Model of Particle Physics

The first lecture

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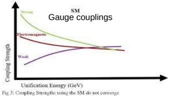
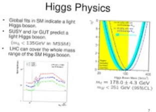
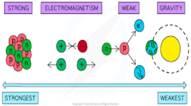
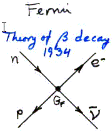
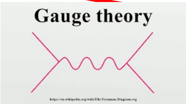
The Standard Model in three hours

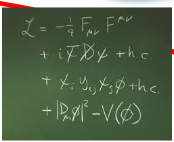


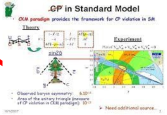






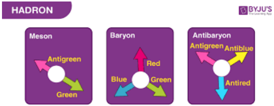


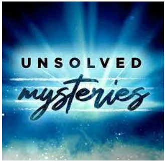


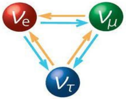


anomaly

$$\partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$







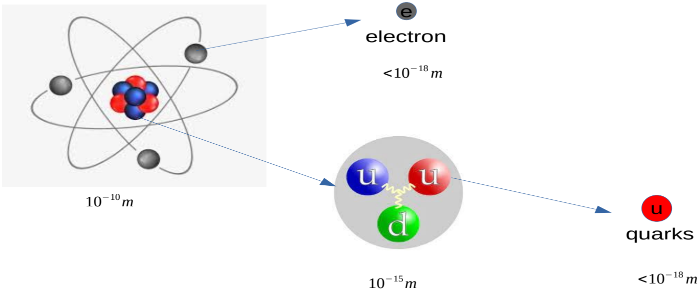
References

- Particle Data Group
  https://pdg.lbl.gov/
- Books:
  Mathew D. Schwartz "Quantum Field Theory and the Standard Model"
  C. P. Burgess and Guy D. Moore "The Standard Model: A Primer"
  Michael E. Peskin and Daniel V. Schroeder "An Introduction to Quantum Field Theory"
- Online Lectures:
  Prof. Leonard Susskind's lectures on the SM
  Prof. Michael E. Peskin's lectures on youtube
- Ask teachers, researchers and friends

Lecture 1

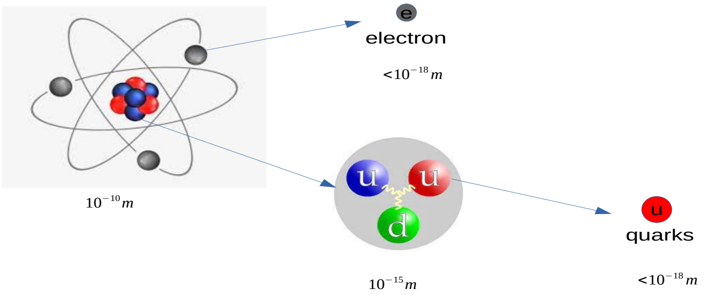
What are elementary particles ?

Atom and subatomic particles



What are elementary particles ?

Atom and subatomic particles



In natural unit: $\hbar = c = k_B = e_0 = 1$,

[E] = [p] = eV, [Length] = [Time] = eV^{-1}

	Atom	nucleon	quark (u,d)	electron
size	500 MeV^{-1}	5 GeV^{-1}	$<$ 5 TeV^{-1}	$<$ 5 TeV^{-1}
mass	n GeV	1 GeV	$u(d) \sim 2(5)$ MeV	0.5 MeV

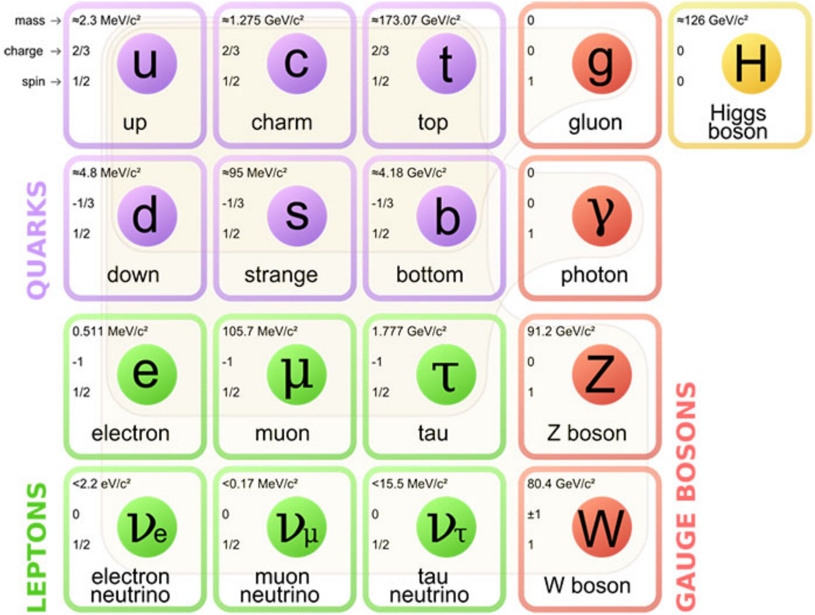
What are elementary particles ?

An elementary particle is a particle with no internal structure. It is considered as a point-like object.

Here are properties of elementary particles

1. Electric charge (quantized, $q = 0, \pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$)
2. Spin (scalar $s = 0$, Fermion $s = \frac{1}{2}$, vector boson $s = 1$)
3. Mass (large range: $0 \rightarrow$ eV $\rightarrow$ GeV)
4. life-time (τ, s): stable ($\tau_{\gamma,u,e} \rightarrow \infty$), unstable $\tau_{W,Z,t,H} \sim 10^{-25} s$
5. anti particle: carry opposite quantum number e^-, e^+
6. Fundamental Interactions (electromagnetic, weak,strong)
7. Some other quantum numbers (color charge, lepton, baryon, strange, CP)

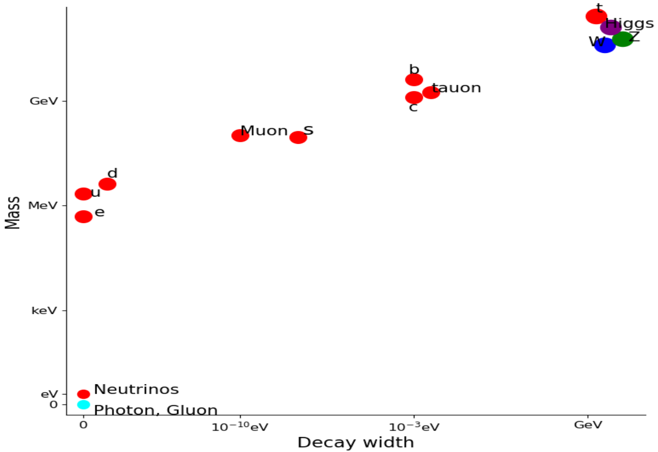
Particle content of the SM



Mass vs Decaywidth

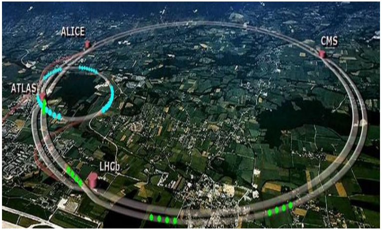
life-time and decay width

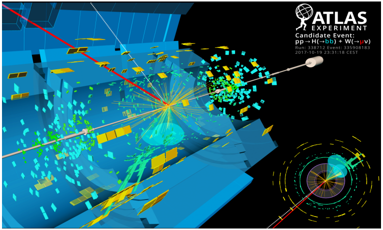
$$\tau = \frac{\hbar}{\Gamma}$$



Where do we observe elementary particles?

- Natural sources: Sun, explosion, cosmic rays $\rightarrow \nu, \gamma,\ e, \mu$
- Laboratory:
  - Reactor: e, ν, γ
  - Accelerator: all elementary particles can be produced as long as it has enough energy $\rightarrow$ High Energy Physics
    Tevatron ($E_{CM} = 2$ TeV), Large Electron-Positron Collider (LEP, $E_{CM} = 209$ GeV)





Large Hadron Collider (LHC) $\quad E_{CM} = 7, 8, 13, 13.6, 14$ TeV

Energy mass relation: $E^2 = m^2 + p^2, \qquad p = m\gamma v,$
$\gamma = \frac{1}{\sqrt{1-v^2}}$

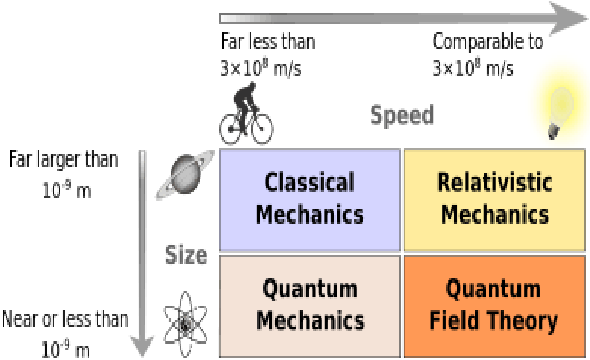
LEP: $E_e = 104.5$ GeV
$m_e = 0.5$ MeV
$v_e \sim 0.99999999999885$

LHC: $E_p = 7$ TeV
$m_p = 938$ MeV
$v_p \sim 0.99999999102$

High Energy Physics

Energy mass relation: $E^2 = m^2 + p^2, \qquad p = m\gamma v, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$

LEP: $E_e = 104.5$ GeV	LHC: $E_p = 7$ TeV
$m_e = 0.5$ MeV	$m_p = 938$ MeV
$v_e \sim 0.99999999999885$	$v_p \sim 0.99999999102$



HEP needs QFT

Uncertainty principle

$\Delta E\,.\,\Delta t \geq \hbar/2$

$\Delta p\,.\,\Delta x \geq \hbar/2$

consequence:

virtual particle

$E^2 \neq m^2 + p^2$

Quantum Mechanics vs Quantum Field Theory

QM

- Space and time are separated variables
- Hamiltonian: $H = \frac{p^2}{2m} + V(x)$
- Schoedinger equation: $i\hbar\frac{\partial\phi(x,t)}{\partial t} = H\phi(x,t)$
- Wave function: $\phi(x,t) = \sum_i c_i(E_i,x)e^{-iE_it/\hbar}$
- Conserved quantity: $-i\hbar\frac{\partial A}{\partial t} = [H,A] = 0$

===============

The Legende transformation
$H = \frac{\partial\mathcal{L}}{\partial(\dot{x}(t))}\dot{x}(t) - \mathcal{L}$
Lagrangian: $\mathcal{L}(x(t),\dot{x}(t))$

QFT

- Minkowski space and Lorentz invariance: $\mu,\nu = 0,1,2,3$
  $x^\mu x_\mu = x^\mu g_{\mu\nu}x^\nu = t^2 - \vec{x}^2$
  $x^\mu = (t,\vec{x}) \equiv (t,x,y,z)$,
  $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$
  $p^\mu = (E,p_x,p_y,p_z)$
  $\partial^\mu = (\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$
- Lagrangian: $\mathcal{L}(\phi(x),\partial^\mu\phi(x))$
- Euler-Lagrande equation: $\frac{\partial\mathcal{L}}{\partial\phi} - \partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} = 0$
- Conserved current:
  $\phi(x) \to \phi(x) + \Delta\phi(x)$
  $\partial^\mu j_\mu = 0,\ j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\Delta\phi(x)$

Lagrangian and Field

Lagrangian at zero temperature in Minkowski space (4 dimensional space):

- Classical Mechanics: $\mathcal{L} = T - V = \mathcal{L}(x(t), \dot{x}(t))$
- QFT: Lagrangian is a function of free fields and their derivatives $\mathcal{L}(\phi(x), \partial^\mu \phi(x))$, satisfing several requirements:
  - Lorentz invariant, $\partial^\mu \phi(x) \partial_\mu \phi(x), A^\mu(x) A_\mu(x), F^{\mu\nu} F_{\mu\nu}$
  - Mass dimension is 4 since $S = \int dx^4 \mathcal{L}$ is dimensionless
  - Your wished symmetries

Free field:

it is a function of space time and belongs to a representation of Lorentz group (scalar, vector, left (right) handed Weyl spinor, Dirac spinor)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E}} \left(a(p) e^{-ip^\mu x_\mu} + b^\dagger(p) e^{+ip^\mu x_\mu} \right)$$

$a(p), b(p)$: annihilation operators, $a^\dagger(p), b^\dagger(p)$ creation operators

Particle classification: Boson vs Fermion

Boson

- $H, \gamma, Z, W^{\pm}, g$
- Spin: integer (0,1)
- commutative:
  $\phi(x)\phi(y) = \phi(y)\phi(x)$
  $[\phi(x), \phi(y)] = 0$
- real field:
  particle $\equiv$ anti-particle
- complex field:
  particle $\neq$ anti-particle

Fermion

- Leptons, quarks
- Spin: half-integer ($\frac{1}{2}$)
- anti-commutative:
  $\psi(x)\psi(y) = -\psi(y)\psi(x)$
  $\{\psi(x), \psi(y)\} = 0$
- only complex field:
  particle $\neq$ anti-particle

Higher spin elementary particle: graviton (s=2), gravitino (s=3/2) they have not yet been observed.

Bosonic Field

Real scalar field: spin =0, H

- Free Lagrangian: $\phi(x)$ with mass dimension $[\phi(x)] = 1$
  $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2$
- Klein-Gordon equation $(\partial^\mu\partial_\mu - m^2)\phi(x) = 0$
- one component field = one degree of freedom (dof)
- $[a(p), a^\dagger(p')] = (2\pi)^3\delta^3(p - p')$, $[a(p), a(p')] = 0$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left(a(p)e^{-ip^\mu x_\mu} + a^\dagger(p)e^{+ip^\mu x_\mu}\right)$$

one particle state: $|p\rangle = \sqrt{2E}a^\dagger(p)|0\rangle \qquad a(p)|0\rangle = 0$

Bosonic Field

Vector field: spin=1, massless γ, g, massive $(Z, W^{\pm})$

- Free Lagrangian: $A^\mu(x)$ with mass dimension $[A^\mu] = 1$
  $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu$ where the field strength tensor
  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$
- Proca equation: $\partial_\nu F^{\mu\nu} + m^2 A^\mu = 0$
- Four component field $>$ 2 dof (massless), 3 dof (massive) $\rightarrow$ need gauge fixing condition $\partial^\mu A_\mu(x) = 0$
- $\epsilon^\mu(\lambda, p)$: polarization vector $\lambda = \pm 1$ for massless,
  $\lambda = -1, 0, 1$ for massive

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E}} \sum_\lambda \left(\epsilon^\mu(p,\lambda) a(p,\lambda) e^{-ip^\mu x_\mu} + \epsilon^{\mu *}(p,\lambda) b^\dagger(p,\lambda) e^{+ip^\mu x_\mu} \right)$$

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^\nu(p,\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}$$

Dirac field: spin=$\frac{1}{2}$, charged fermions $e, \mu, \tau, u, d, s, c, b, t$

- Free Lagrangian: $\psi(x)$ with mass dimension $[\psi(x)] = \frac{3}{2}$
  $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ where $\bar{\psi} = \psi^\dagger \gamma^0$
  Dirac matrix: γ_μ are 4×4 matrices, in chiral representation:
  $\gamma^\mu = \left(\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right) \equiv \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$
  $\sigma^{1,2,3}$ are three 2×2 Pauli matrices
- Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$
- Four complex component field $\psi(x) \equiv (\psi_1, \psi_2, \psi_3, \psi_4)^T$
- $$\psi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E}} \sum_{\lambda=\pm 1} \left(u(p,\lambda) a(p,\lambda) e^{-ip^\mu x_\mu} + v(p,\lambda) b^\dagger(p,\lambda) e^{+ip^\mu x_\mu} \right)$$

  $u(p,\lambda), v(p,\lambda)$ are Dirac spinors satisfying $(\gamma^\mu p_\mu - m)u = 0$ and $(\gamma^\mu p_\mu + m)v = 0$

$$\gamma_5 = i\gamma^0\gamma^1\gamma2\gamma3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [\gamma_5, \gamma_\mu] = 0$$

- Let's define left- and right-handed projectors

  $P_L = \frac{1-\gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \frac{1+\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$

  $P_L P_L = P_L, P_R P_R = P_R, P_L P_R = 0$

  One can construct:

  $$\psi_L = P_L\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_R = P_R\psi = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi = \psi_L + \psi_R$$

  ψ_L is left-handed Weyl spinor and ψ_R is right-handed Weyl spinor. (Weyl spinors are two component spinors.)

Left- and right-handed Weyl spinor

$$\gamma_5 = i\gamma^0\gamma^1\gamma 2\gamma 3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [\gamma_5, \gamma_\mu] = 0$$

- Let's define left- and right-handed projectors
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  $P_L P_L = P_L, P_R P_R = P_R, P_L P_R = 0$
  One can construct:
  $$\psi_L = P_L\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_R = P_R\psi = \begin{pmatrix} 0 \\ 0 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi = \psi_L + \psi_R$$
  ψ_L is left-handed Weyl spinor and ψ_R is right-handed Weyl spinor. (Weyl spinors are two component spinors.)
- We define chirality from $\gamma_5\psi_L = -\psi_L$, $\gamma_5\psi_R = \psi_R$
- Dirac field Lagrangian:
  $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = i\bar{\psi}_L\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R + m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$

mass term does not respect chirality

Massless Fermionic Field

Lagrangian

$\mathcal{L} = i\bar{\psi}_L \partial\!\!\!/ \psi_L + i\bar{\psi}_R \partial\!\!\!/ \psi_R$ Left- and right-handed Weyl spinor are independent. Chirality is Lorentz invariant and from Dirac equation one gets $(i, j, k = 1, 2, 3)$
$(\gamma^0|p| - \gamma^i p^i)u(p) = 0 \rightarrow (1 - \frac{\gamma^0\gamma^i p^i}{|p|})u(p) = 0$
Using $\gamma^0\gamma^i = \gamma^5\Sigma^i$ where $\Sigma^i = \frac{1}{2}\epsilon^{ijk}\sigma_{jk}$, $\sigma_{jk} = \frac{i}{2}[\gamma_i, \gamma_k]$ then
$\gamma_5 u(p) = \frac{\Sigma^i p^i}{|p|}u(p)$ where $\frac{\vec{\Sigma}\vec{p}}{|p|}$ is the helicity operator.
For massless particle helicity is identical to chirality.

Massless Fermionic Field

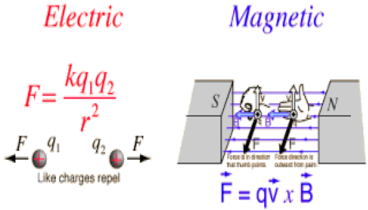
Lagrangian

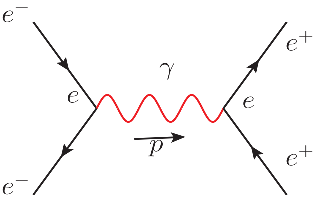
$\mathcal{L} = i\bar{\psi}_L \not{\partial} \psi_L + i\bar{\psi}_R \not{\partial} \psi_R$ Left- and right-handed Weyl spinor are independent. Chirality is Lorentz invariant and from Dirac equation one gets $(i, j, k = 1, 2, 3)$
$(\gamma^0 |p| - \gamma^i p^i) u(p) = 0 \rightarrow (1 - \frac{\gamma^0 \gamma^i p^i}{|p|}) u(p) = 0$
Using $\gamma^0 \gamma^i = \gamma^5 \Sigma^i$ where $\Sigma^i = \frac{1}{2} \epsilon^{ijk} \sigma_{jk}$, $\sigma_{jk} = \frac{i}{2}[\gamma_i, \gamma_k]$ then
$\gamma_5 u(p) = \frac{\Sigma^i p^i}{|p|} u(p)$ where $\frac{\vec{\Sigma}\vec{p}}{|p|}$ is the helicity operator.
For massless particle helicity is identical to chirality.

Neutrino in the SM

They are left-handed Weyl spinor and their anti-neutrinos are right-handed Weyl spinor. There is no right-handed neutrinos. As consequence neutrinos are massless in the SM.

- between particles carring electric charge
- Quantum view: charged particles interact by exchanging a virtual photon



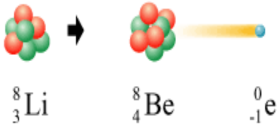


EM interaction can be described by a point-like interaction of charged particle with photon. Lagrangian can be written as

$$\mathcal{L}_{int} = e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x)$$

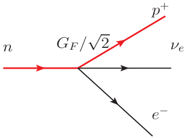
Weak interaction (1)

- Nuclear beta decay :
  $n \to p^+ + e^- + \nu_e$
  see Prof. Nakaya's lecture



- Enrico Fermi proposed (1934)
  $\frac{G_F}{\sqrt{2}}(\bar{p}(x)\gamma_\mu n(x))(\bar{e}(x)\gamma^\mu \nu(x))$
- Wu's experiment (1956) found parity violation in beta decay
- Sudarshan and Marshak (1957), Feynman and Gell-Mann (1958) suggested

$$\frac{G_F}{\sqrt{2}}(\bar{p}(x)\gamma_\mu(1-\gamma_5)n(x))(\bar{e}(x)\gamma^\mu(1-\gamma_5)\nu(x))$$



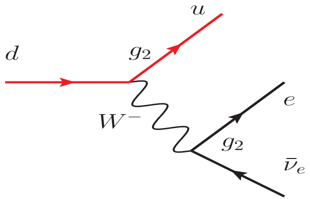
- Proton and neutron consist of quarks (1964, Gell-Mann and Zweig): $p = (uud)$ while $n = (ddu)$

$$\frac{G_F}{\sqrt{2}}(\bar{d}(x)\gamma_\mu(1-\gamma_5)u(x))(\bar{e}(x)\gamma^\mu(1-\gamma_5)\nu(x))$$

- The beta decay can be described exchanged by W. It can be written as

$$\mathcal{L}_{int} = \frac{g_2}{2\sqrt{2}}\bar{u}(x)\gamma^\mu(1-\gamma_5)d(x)W_\mu(x) + \frac{g_2}{2\sqrt{2}}\bar{e}(x)\gamma^\mu(1-\gamma_5)\nu(x)W_\mu(x)$$

$$\alpha_2 = \frac{g_2^2}{4\pi} \sim 3 \times 10^{-2}$$



Weak interaction (3)

Not only charged current interaction

$$\mathcal{L}_{int} = \frac{g_2}{2\sqrt{2}}\bar{u}(x)\gamma^\mu(1-\gamma_5)d(x)W_\mu(x) + \frac{g_2}{2\sqrt{2}}\bar{e}(x)\gamma^\mu(1-\gamma_5)\nu(x)W_\mu(x)$$

But also neutral current interaction

$$\mathcal{L}_{int} = \frac{g_2}{2\cos\theta_W}\bar{f}(x)\gamma^\mu(g_V^f - g_A^f\gamma_5)f(x)Z_\mu(x)$$

f: quark, charged leptons, neutrinos

g_V^f, g_A^f are coefficients depending on electric charge and isospin of fermion f

weak neutral current was confirmed in 1973, in a neutrino experiment in the Gargamelle buble chamber at CERN

W, Z were seen in 1983 at Super Proton Synchrotron at CERN

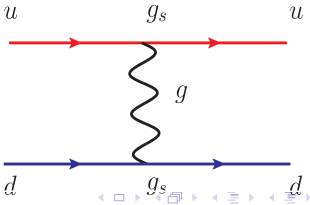
Strong interaction

- Old: Strong interaction between nucleons to form nuclei
- The quarks model of Gell-Mann and Zweig (1964) to explain the classification of hadrons
- New: all particles carrying color charge participate strong interaction (quarks, gluons)
- Lagrangian: simple guest based on Lorentz invariance

$$\mathcal{L}_{int} = g_s \bar{q} \gamma^\mu q G_\mu + g_s (\partial^\mu G^\nu) G_\mu G_\nu + g_s^2 G_\mu G_\nu G^\mu G^\nu$$

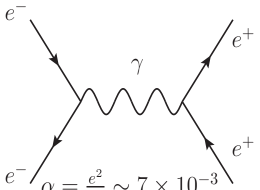
$G_\mu(x)$: gluon field $\qquad$ g_s: strong coupling

$$\alpha_s = \frac{g_s^2}{4\pi} \sim 1.2 \times 10^{-1}$$



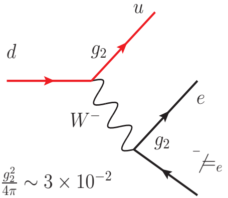
Fundamental interactions: recap

Electromagnetic



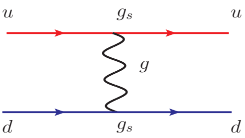
$\alpha = \frac{e^2}{4\pi} \sim 7 \times 10^{-3}$

Weak



$\alpha_2 = \frac{g_2^2}{4\pi} \sim 3 \times 10^{-2}$

Strong



$\alpha_s(M_Z) = \frac{g_s^2}{4\pi} \sim 1.2 \times 10^{-1}$

Gravitation

gravitational force: $F = G\frac{m_1 m_2}{r^2}$

$G \sim 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-1}$

Construct a dimensionless coupling

$\alpha_G = \frac{Gm_e^2}{\hbar c} \sim 1.7 \times 10^{-45}$

$\alpha_G \ll \alpha < \alpha_2 < \alpha_s$

Symmetry

- makes a theory more predictable
- leads to conservation laws
- makes computation simpler
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Familiar example:

- Space is isotropic $\to$ $SO(3)$ rotation group $\to$ conservation of angular momentum
- Space-time is homogeneous $\to$ translation group $\to$ conservation of energy and momentum
- $U(1)_Q$: $\psi_f \to e^{i\alpha q_f}\psi_f$, q_f is electric charge $\to$ conservation of electric charge

Question?

Can we use symmetry to build interactions between elementary particles?

Unitary transformations

Important in physics, geometry, informatics

- Preserve probability in quantum mechanics

$$|\langle\psi_2 U|U\psi_1\rangle|^2 = |\langle\psi_2|U^\dagger U|\psi_1\rangle|^2 = |\langle\psi_2|\psi_1\rangle|^2$$

  Unitary: $\langle\psi_2|U^\dagger U|\psi_1\rangle = \langle\psi_2|\psi_1\rangle$
  Anti-unitary: $\langle\psi_2|U^\dagger U|\psi_1\rangle^* = \langle\psi_2|\psi_1\rangle$

- Unitary

$$U^\dagger U = \mathbb{1}$$

- Special Unitary

$$\det U = 1$$

- Matrix representation $SU(N)$: $N \times N$ matrices with

$$U^\dagger U = \mathbb{1} \qquad \det U = 1$$

  $U = e^{i\alpha_a T_a}$, $\quad T_a + T_a^\dagger = 1$, $\mathrm{Tr}[T_a] = 0$,
  T_a: generators, Hermitian and traceless matrices.
  $a = 1, \dots, D$ where D is the number degree of freedom
  $D = N^2 - 1$

Abelian Gauge Symmetry (1)

- $U(1)_Q$ transformation: given a quantum field $\psi(x)$
$$\psi'(x) = e^{i\alpha g Q}\psi(x)$$
α is a continuous parameter $\in \mathbb{R}$, g is parameter characterizing the group, Q is the quantum number carried by the field $\psi(x)$. It acts on field, donot change space-time.
- Abelian/ commutative group:
$$e^{i\alpha g Q_1} e^{i\alpha g Q_2} = e^{i\alpha(Q_1+Q_2)g} = e^{i\alpha g Q_2} e^{i\alpha g Q_1}$$
- Global transformation: α is independent of space-time
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$
where $\bar{\psi} = \psi^\dagger \gamma^0$

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$\rightarrow$ invariant under $U(1)_Q$ $\rightarrow$ Dirac Lagrangian has $U(1)_Q$ symmetry
- Assume if there exists a term as $\bar{\psi}_1\phi\psi_2$ then we have $-Q_{\psi_1} + Q_\phi + Q_{\psi_2} = 0$. It means that Q quantum number is conserved.

Abelian Gauge Symmetry (2)

- If α is a function of space-time $\alpha(x) \rightarrow$ local transformation.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

is not invariant anymore.

What should we do to restore symmetry?

Abelian Gauge Symmetry (2)

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$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

is not invariant anymore.

What should we do to restore symmetry?

- Under $e^{i\alpha(x)gQ}$ transformation

$$\mathcal{L} \to \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - gQ\bar{\psi}\gamma^\mu\psi\partial_\mu\alpha(x)$$

- If we add one more term in the original Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + gQ\bar{\psi}\gamma^\mu A_\mu \psi \qquad (1)$$

and require that the new field A_μ transform as

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

(1) is invariant under local transformation.

- Rewrite (1) as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

with $D_\mu = \partial_\mu - igQA_\mu$: covariant derivative

Quantum Electrodynamics (QED)

- Lagrangian is invariant under a local transformation. There appears a new vector field A_μ and an interaction term.

$$\mathcal{L} = i\bar{\psi}\gamma^\mu(\partial_\mu - igQA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{i}{gQ}[D_\mu, D_\nu]$$

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Identifying g to elementary charge e, Q to electric charge quantum number, $Q = -1$ for electron, $Q = 2/3$ for up-type quark, $Q = -1/3$ for down-type quark, A_μ as photon field, we obtain a theory called Quantum Electrodynamics (QED)

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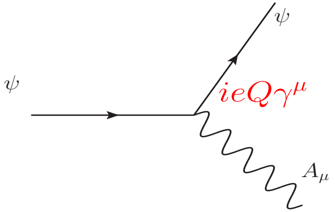
  Identifying g to elementary charge e, Q to electric charge quantum number, $Q = -1$ for electron, $Q = 2/3$ for up-type quark, $Q = -1/3$ for down-type quark, A_μ as photon field, we obtain a theory called Quantum Electrodynamics (QED)
- Photon is massless since the term $m_A A^\mu A_\mu$ is not gauge invariant.

- Photon is a real vector field.
- There is no self-coupling of photons



Non-Abelian gauge Symmetry (1): Yang-Mills Theory (1954)

It is not enough to describe all interactions by only Abelian gauge symmetry, we want to have multiple vector particles appears in the theory.

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It is not enough to describe all interactions by only Abelian gauge symmetry, we want to have multiple vector particles appears in the theory.

- $SU(N)$, $N \geq 2$: a set of fermionic fields belong to a fundamental representntation of $SU(N)$.

$$\Sigma = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} \rightarrow e^{ig_N \alpha_a(x) T^a} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}$$

$a = 1, \cdots, N^2 - 1$, T^a are $N \times N$ Hermitian matrices, generators of $SU(N)$ which obey the group algebra

$$[T^a, T^b] = i f^{abc} T^c \qquad f^{abc} \text{ group structure constant}$$

Non-Abelian gauge Symmetry (2)

- Lagrangian of the set of fermionic fields

$$\mathcal{L} = i\bar{\Sigma}\gamma^\mu D_\mu \Sigma - \bar{\Sigma} M \Sigma$$

M is $N \times N$ mass matrix.

- Under SU(N) local transformation

$$\mathcal{L} \to i\bar{\Sigma} U^\dagger \gamma^\mu D'_\mu U \Sigma - \bar{\Sigma} U^\dagger M U \Sigma$$

It is invariant if

$$\begin{aligned} D'_\mu &= U D_\mu U^\dagger \\ U^\dagger M U &= M \end{aligned}$$

$M = m\mathbb{1}$ means that all fields in the multiplet must have the same mass.

- Using the similarity with Abelian case, we set

$$D_\mu = \partial_\mu - ig_N T^a A^a_\mu, \quad a = 1, \cdots, N^2 - 1$$

The requirement $D'_\mu = U D_\mu U^\dagger$ means that

$$\begin{aligned} T.A'_\mu &= UT.A_\mu U^\dagger + \frac{i}{g_N} U \partial_\mu U^\dagger \\ &= T.A_\mu + T.\partial_\mu \phi + i[T.\partial_\mu \phi, T.A_\mu] + \cdots \end{aligned}$$

- How to construct field strength tensor?

$$[D_\mu, D_\nu] = -ig_N F^a_{\mu\nu} T^a$$

then $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_N f^{abc} A^b_\mu A^c_\nu$

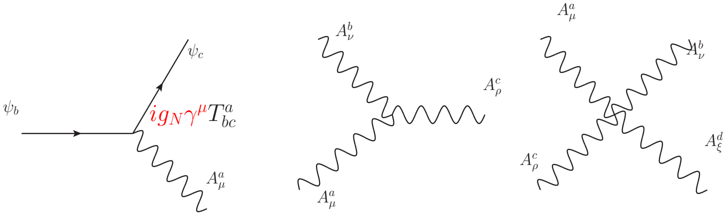
Non-Abelian gauge Symmetry (4)

- Lagrangian of the set of fermionic fields and new vector fields (gauge fields)

$$\mathcal{L} = i\bar{\Sigma}\gamma^{\mu} D_{\mu}\Sigma - m\bar{\Sigma}\Sigma - \frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a}$$

$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + g_N f^{abc} A^{b}_{\mu}A^{c}_{\nu}$

- All fields in the multiplet must have the same mass. Gauge fields are massless
- Number of new gauge fields $N^2 - 1$, they belong to the adjoin representation of the $SU(N)$ $\rightarrow$ real fields
- Trilinear and quartic gauge couplings exist



Discussions on Gauge symmetry

- Gauge fields are real vector fields and massless
- Interaction terms appear naturally with a common coupling for each group
- All fields in the multiplet must have the same mass

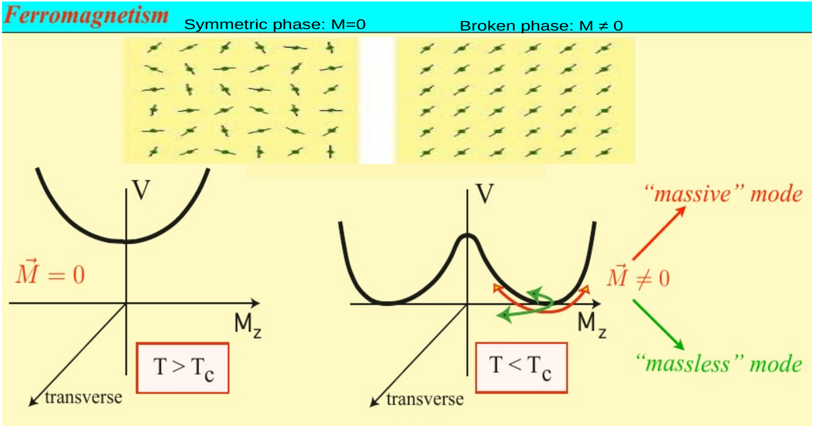
Is that enough to describe the electromagnetic, weak and strong interactions?

Symmetry breaking mechanism

Gauge symmetry must be broken for at least weak interaction.

- explicit breaking: add different mass terms for different fermions and for different gauge fields. This loose the predictive feature of the theory.
- spontaneous symmetry breaking?
  Lagrangian is invariant under the symmetry, but the ground state of the theory is not.
  $\rightarrow$ attractive idea

Spontaneous symmetry breaking in phase transition [L.D. Landau 1937]
Spontaneous symmetry breaking in field theory [Y. Nambu 1960]
Spontaneous symmetry breaking in the Standard Model [F. Englert, R. Brout 1964, P.W. Higgs 1964]



taken from the Nobel lecture by Francois Englert in 2013

Spontaneous breaking of discrete symmetries (1)

Consider a Lagrangian of a real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

where the potential is $V = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$

- proccesses a Z_2 symmetry $\phi \to -\phi$
- m, λ depend on temperature.

$$m^2(T) \sim c(T - T_C) + \cdots \to \begin{cases} \geq 0 & \text{if } T \geq T_C \\ < 0 & \text{if } T < T_C \end{cases}$$

- for $T < T_C$, $m^(T) = -\mu^2$, λ is positive. Find the extrema through

$$\frac{\partial V}{\partial \phi} = -\mu^2\phi + \frac{\lambda}{6}\phi^3 = 0 \to \begin{cases} \phi = 0 & \text{maximum} \\ \phi = \pm\sqrt{\frac{6\mu^2}{\lambda}} & \text{two possible minima} \end{cases}$$

the two minima correspond to two vacuum states $|0_-\rangle$, $|0_+\rangle$

Spontaneous breaking of discrete symmetries (2)

- At either minimum, the Z_2 symmetry is spontaneous broken.
- Take one minimum and expand ϕ around it $\phi = \sqrt{\frac{6\mu^2}{\lambda}} + \tilde{\phi}$

$$\mathcal{L} = \frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} + \frac{3m^4}{2\lambda} - \mu^2\tilde{\phi}^2 - \sqrt{\frac{\lambda}{6}}m\tilde{\phi}^3 - \frac{\lambda}{4!}\tilde{\phi}^4$$

- The $\tilde{\phi}$ field has a possitive mass squared μ^2
- The new Lagrangian is not invariant under symmetry $\tilde{\phi} \to -\tilde{\phi}$, so it seems Z_2 symmetry is broken. But it is in fact still invariant under the original $\phi \to -\phi$. Its manifestation presents in the relation between mass and couplings. Z_2 symmetry is not broken in Lagrangian, it is hidden.

If $\phi = v$ (v is a constant) and $\mathcal{L}(v) = 0$. One recall the vacuum expectation value $\langle 0|\phi|0\rangle = \int \mathcal{D}\phi e^{-i\int d^4x\mathcal{L}[\phi]}\phi = v$

At the two minima $\langle 0_+|\phi|0_+\rangle = \sqrt{\frac{6\mu^2}{\lambda}}$ and $\langle 0_-|\phi|0_-\rangle = -\sqrt{\frac{6\mu^2}{\lambda}}$

Spontaneous breaking of continuous global symmetries (1)

Consider a Lagrangian of a complex scalar field $\phi = \phi_1 + i\phi_2$

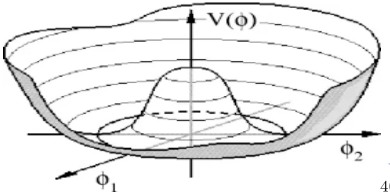
$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi - \frac{\lambda}{4} \phi^2 \phi^{*2}$$

where the potential is $V = m^2 \phi^* \phi + \frac{\lambda}{4} \phi^2 \phi^{*2}$

- proccesses a global $U(1)$ symmetry $\phi \to e^{i\alpha} \phi$
- $m^2 > 0, \lambda > 0$ the potential has stable minima at $\phi = 0$
- $m^2 < 0, \lambda > 0$ the potential has an unstable maximum at $\phi = 0$ and infinite number of equivalent minima with $|\phi| = \sqrt{\frac{2\mu^2}{\lambda}}$ where $\mu^2 = -m^2$.

infinite vacuum states (degenerate vacuum) $|0_\theta\rangle$

$\langle \phi \rangle = \langle 0_\theta | \phi | 0_\theta \rangle = \sqrt{\frac{2\mu^2}{\lambda}} e^{i\theta}$



Spontaneous breaking of continuous global symmetries (2)

- Choosing a particular vacuum means the spontaneous symmetry breaking.
- For convenience, take the vacuum $\langle 0_0|\phi|0_0\rangle = \sqrt{\frac{2\mu^2}{\lambda}} \equiv v/\sqrt{2}$ and expand around the VEV as

$$\phi = \frac{v+\sigma}{\sqrt{2}} e^{i\rho/(v)}$$

  We obtain

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\rho)^2 + \frac{\mu^4}{\lambda} - \mu^2\sigma^2 + \frac{1}{2}\sqrt{\mu^2\lambda}\sigma^3 + \frac{\lambda}{16}\sigma^4$$

- σ has mass $\sqrt{2}\mu$. ρ is massless and it is called Nambu-Goldstone boson.
- Remaining shift symmetry of the Lagrangian $\rho \to \rho + v\theta$ that forbids a mass term for ρ

Goldstone's Theorem

Spontaneous breaking of continuous global symmetries implies the existence of massless spin-0 Nambu-Goldstone bosons.
Proof given by Goldstone, Salam and Weinberg (1962)

- No observed massless scalars.
- What we observed are some pions with small masses.

Spontaneous breaking of gauge symmetries (1)

The Higgs mechanism is not fairly named since the idea was contributed from many people: Anderson, Brout, Englert, Ginzburg, Guralnik, Haga, Kibble, Landau and Higgs.

- Consider here only the Abelian gauge symmetry. For the non-Abelian we leave it to the Standard Model discussion.
- Lagriangian for complex scalar field with $U(1)$ gauge symmetry $\phi \to e^{ig\alpha(x)}\phi$

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + D_\mu\phi^* D^\mu\phi + \mu^2\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2$$

  where $D_\mu = \partial_\mu + igA_\mu$ and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.
- The potential has minimum at $\langle\phi\rangle = \frac{v}{\sqrt{2}}$ with $v = \sqrt{2\mu^2/\lambda}$.
- Expand around the VEV as

$$\phi = \frac{v+\sigma}{\sqrt{2}} e^{i\rho/(v)}$$

  Consider only the term

$$D_\mu\phi^* D^\mu\phi = \frac{(v+\sigma)^2}{2}\left[\frac{\partial_\mu\sigma}{v+\sigma} - i\frac{\partial_\mu\rho}{v} - igA_\mu\right]\left[\frac{\partial^\mu\sigma}{v+\sigma} + i\frac{\partial^\mu\rho}{v} + igA^\mu\right]$$

Spontaneous breaking of gauge symmetries (2)

Rewrite

$$D_\mu\phi^* D^\mu\phi = \frac{(v+\sigma)^2}{2}\left[\frac{\partial_\mu\sigma}{v+\sigma} - i\frac{\partial_\mu\rho}{v} - igA_\mu\right]\left[\frac{\partial^\mu\sigma}{v+\sigma} + i\frac{\partial^\mu\rho}{v} + igA^\mu\right]$$
$$= \frac{(v+\sigma)^2}{2}\left[\frac{(\partial^\mu\sigma)^2}{(v+\sigma)^2} + \left(\frac{(\partial^\mu\rho)}{v} + gA_\mu\right)^2\right]$$

- There is a mass term for gauge field:

$$\frac{v^2 g^2}{2} A^\mu A_\mu$$

$M_A = vg$

- There is a bilinear term mixing

$$M_A \partial_\mu \rho A^\mu$$

complicates the interpretation of physical spectrum.
$\rightarrow$ How to remove this term?

Spontaneous breaking of gauge symmetries (3)

- The remaining symmetry of the new Lagrangian is

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha(x) \text{ and } \rho \to \rho - v\alpha(x)$$

- Removing the bilinear term mixing by using a gauge fixing condition
  - Unitary gauge: $\rho = 0$, the Goldstone boson disappears from theory. Physical gauge. The Goldstone boson has been eaten by the gauge boson. Before massless gauge field has 2 dof, now it is massive and it has 3 dof.
  - t'Hooft Feymann gauge: adding a gauge fixing term to the Lagrangian

$$-\frac{1}{2}\left(\partial_\mu A^\mu - M_A \rho\right)^2$$

    In this gauge ρ has mass M_A.
  - This gauge fixing is not gauge invariant. To restore gauge invariance one has to add Faddeev-Popov ghost term

$$\bar{c}_A \frac{\delta(\partial_\mu A^\mu - M_A \rho)}{\delta\alpha} c_A$$

    $c_A, \bar{c}_A$ are ghost and anti-ghost field, scalars but have fermionic properties.

Summary of lecture 1