Università di Pisa

INFN Istituto Nazionale di Fisica Nucleare
SEZIONE DI PISA

# Neutrino Flavour Models

**Arsenii Titov**

*Dipartimento di Fisica “Enrico Fermi”, Università di Pisa, Italy*
*INFN, Sezione di Pisa, Italy*

*19th Recontres du Vietnam*

**Neutrino Workshop at IFIRSE**

Quy Nhon, Vietnam
18 July 2023

# Outline

- 3-neutrino mixing
- Flavour puzzle
- Non-Abelian discrete flavour symmetries
- Modular flavour symmetries
- Conclusions

# 3-neutrino mixing

Charged current weak interactions

$$-\mathscr{L}_{\rm CC} = \frac{g}{\sqrt{2}} \sum_{\ell=e,\mu,\tau} \overline{\ell_L}(x)\,\gamma_\alpha\,\nu_{\ell L}(x)\,W^{\alpha\dagger}(x) + \text{h.c.}$$

Mismatch between the interaction and mass eigenstates

$$\nu_{\ell L}(x) = \sum_{j=1}^{3} U_{\ell j}\,\nu_{jL}(x)\,, \qquad \nu_j \text{ has mass } m_j\,,\; j = 1,2,3$$

$U$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) neutrino mixing matrix

The standard parameterisation (adopted by the PDG)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\alpha_{21}}{2}} & 0 \\ 0 & 0 & e^{i\frac{\alpha_{31}}{2}} \end{pmatrix}$$

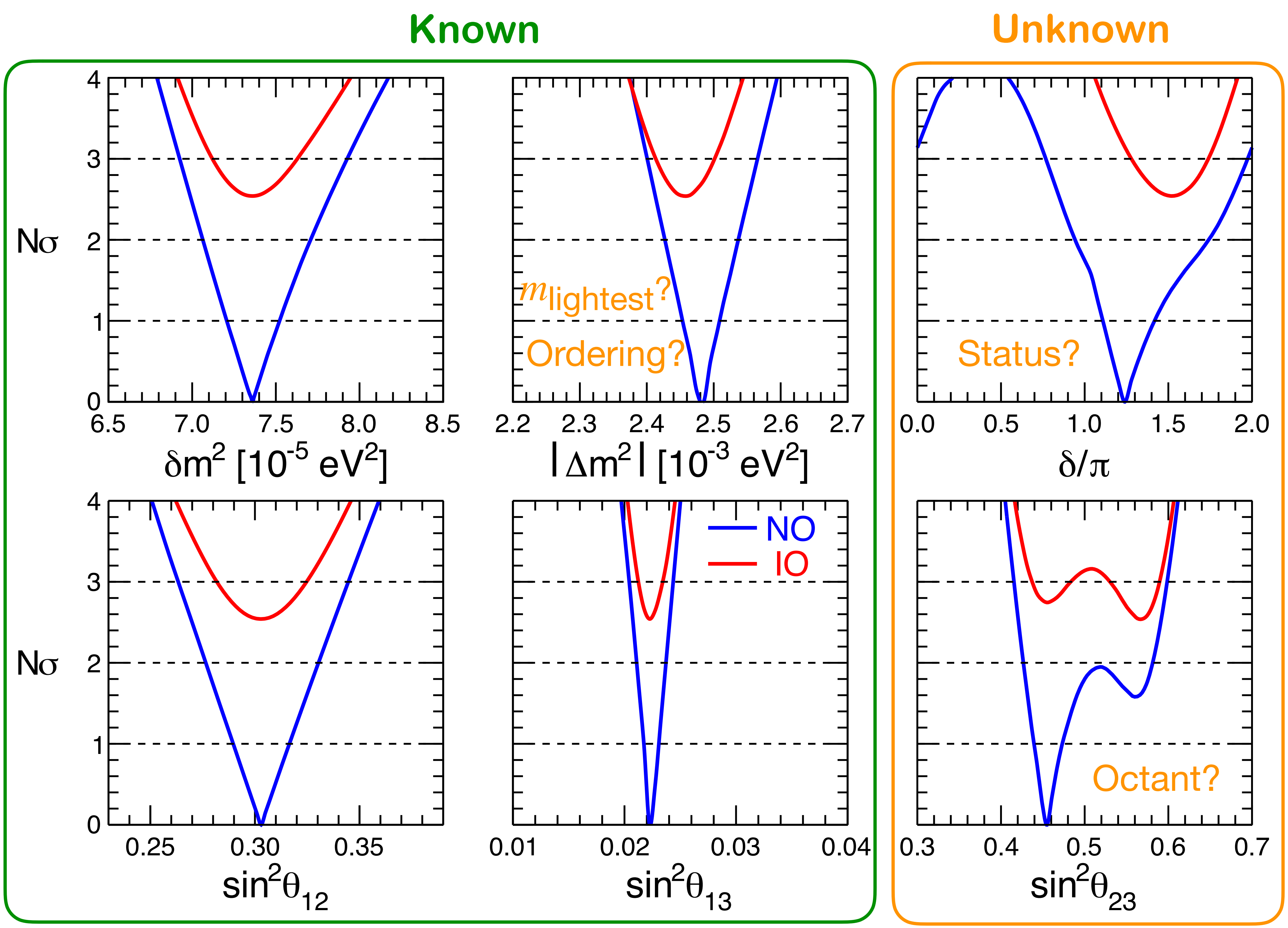
Atmospheric angle $\theta_{23}$

Reactor angle $\theta_{13}$
Dirac phase $\delta$

Solar angle $\theta_{12}$

Majorana phases
$\alpha_{21}$ and $\alpha_{31}$
(if $\nu$ are Majorana)

# Global fit to neutrino oscillation data




Capozzi et al., 2107.00532; see also Esteban et al., 2007.14792 and de Salas et al., 2006.11237

# Global fit to neutrino oscillation data

**Mass-squared differences**

$\delta m^2 \equiv m_2^2 - m_1^2 = 7.36 \times 10^{-5}$ eV$^2$ $(2.3\%)$ [b.f.v. (relative $1\sigma$ error)]

$\Delta m^2 \equiv m_3^2 - \left(m_1^2 + m_2^2\right)/2$

$= 2.485 \times 10^{-3}$ eV$^2$ $(1.1\%)$ for NO or $-2.455 \times 10^{-3}$ eV$^2$ $(1.1\%)$ for IO

**Mixing angles**

$\sin^2\theta_{12} = 0.303\ (4.5\%)$ -> Large

$\sin^2\theta_{23} = 0.455\ (6.7\%)$ for NO or $0.569\ (5.5\%)$ for IO -> Large (maximal?)

$\sin^2\theta_{13} = 0.0223\ (3\%)$ -> Small, but ≠ 0

**CP-violating (CPV) phases**

$\delta/\pi = 1.25\ (16\%)$ for NO or $1.52\ (9\%)$ for IO -> Maximal?

$\alpha_{21} = ?$

$\alpha_{31} = ?$

Capozzi et al., 2107.00532; see also Esteban et al., 2007.14792 and de Salas et al., 2006.11237

# Global fit to neutrino oscillation data

**Mass-squared differences**

$\delta m^2 \equiv m_2^2 - m_1^2 = 7.36 \times 10^{-5}$ eV$^2$ $(2.3\%)$ [b.f.v. (relative $1\sigma$ error)]

$\Delta m^2 \equiv m_3^2 - \left(m_1^2 + m_2^2\right)/2$

$= 2.485 \times 10^{-3}$ eV$^2$ $(1.1\%)$ for NO or $-2.455 \times 10^{-3}$ eV$^2$ $(1.1\%)$ for IO

**Mixing angles**

$\sin^2\theta_{12} = 0.303\ (4.5\%)$ -> Large

$\sin^2\theta_{23} = 0.455\ (6.7\%)$ for NO or $0.569\ (5.5\%)$ for IO -> Large (maximal?)

$\sin^2\theta_{13} = 0.0223\ (3\%)$ -> Small, but ≠ 0

**CP-violating (CPV) phases**

$\delta/\pi = 1.25\ (16\%)$ for NO or $1.52\ (9\%)$ for IO -> Maximal?

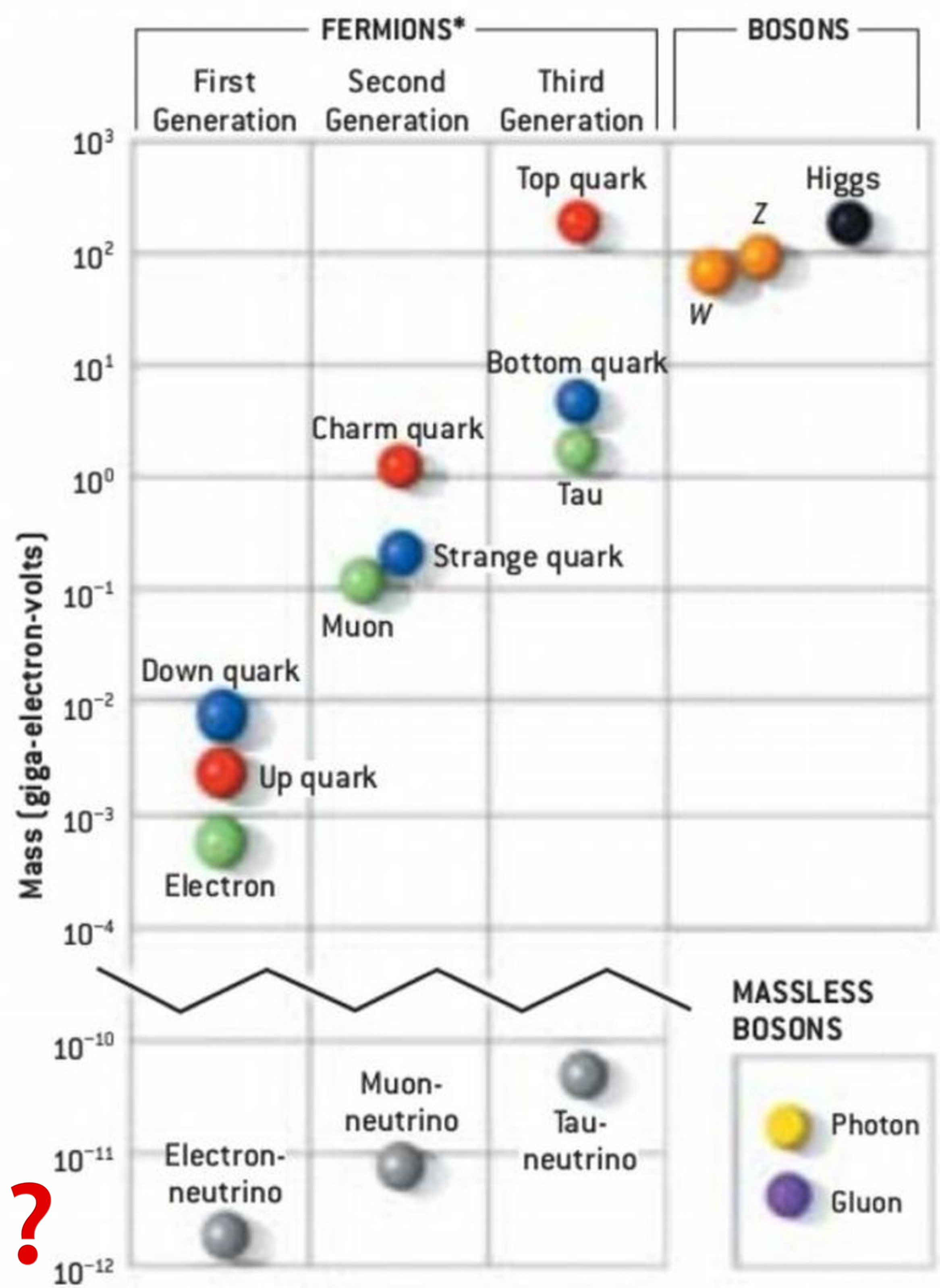
$\alpha_{21} = ?$

$\alpha_{31} = ?$

What are the data telling us?

Capozzi et al., 2107.00532; see also Esteban et al., 2007.14792 and de Salas et al., 2006.11237

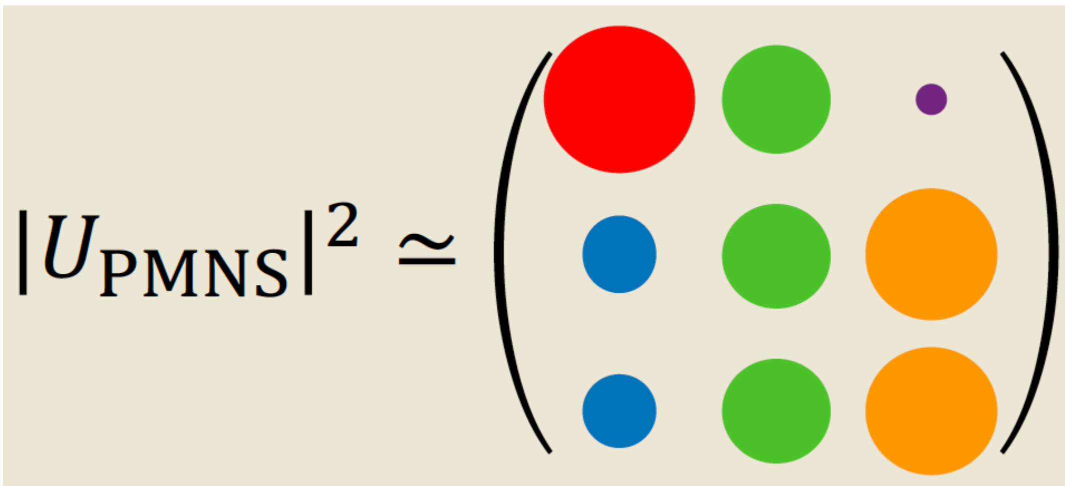
# Flavour puzzle: masses

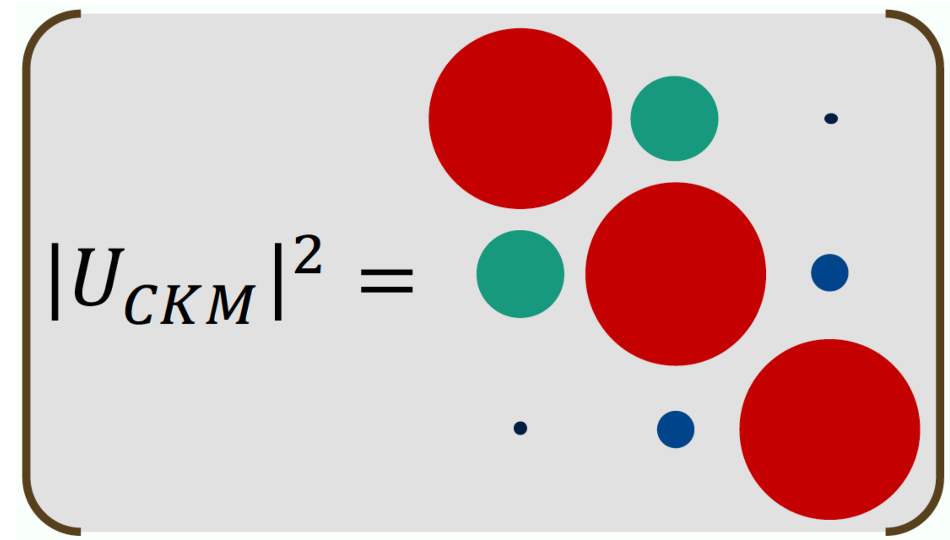



- What is the value of the lightest neutrino mass?
- Why is the mass of neutrino $\sim 10^7$ times smaller than that of electron?
- What is the mechanism of neutrino mass generation?
- Why are there 3 families?
- Is there any organising principle behind the values of fermion masses?

# Flavour puzzle: mixing and CP

2 large and 1 small (but non-zero) mixing angles -> very different from quarks



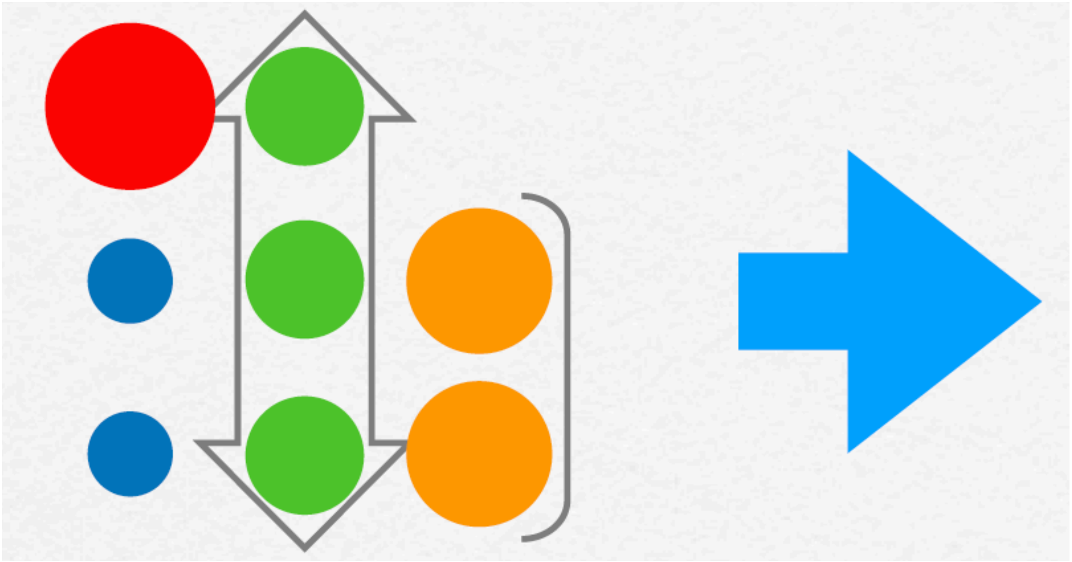

Images: Phill Litchfield

$$(s_{12}^2,\ s_{23}^2,\ s_{13}^2) \sim (0.3,\ 0.5,\ 0.022) \quad \text{vs} \quad (0.05,\ 1.8\times10^{-3},\ 1.4\times10^{-5})$$

- Why are these mixing patterns so different?
- Is there any organising principle behind the values of mixing parameters?
- What is the mechanism of CP violation? $(\delta_{\rm PMNS} = \mathcal{O}(1)? \quad \delta_{\rm CKM} \approx 1.2, \quad \bar{\theta} \approx 0)$

# Tri-bimaximal (TBM) mixing

Harrison, Perkins, Scott, hep-ph/0202074



$$U_{\rm TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix} \begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{pmatrix} \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\sin^2\theta_{23} = \frac{1}{2}$ — allowed at $2\sigma$

$\sin^2\theta_{13} = 0$ — excluded at many $\sigma$

$\sin^2\theta_{12} = \frac{1}{3}$ — allowed at $2\sigma$

# Flavour symmetry

At high energies, the theory is invariant under

$$\varphi(x) \to \rho(g)\,\varphi(x)\,, \quad g \in G_f \qquad \text{e.g.} \qquad \varphi = \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix}$$

representation of $G_f$

flavour symmetry group

At low energies, flavour symmetry has to be broken

Clever way of breaking:

$G_f$

$G_e \subset G_f$ residual symmetries $G_\nu \subset G_f$

$$\rho(g_e)^\dagger M_e M_e^\dagger \rho(g_e) = M_e M_e^\dagger \qquad \rho(g_\nu)^T M_\nu \rho(g_\nu) = M_\nu$$

$$U_e^\dagger M_e M_e^\dagger U_e = \text{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right) \qquad U_\nu^T M_\nu U_\nu = \text{diag}\left(m_1, m_2, m_3\right)$$

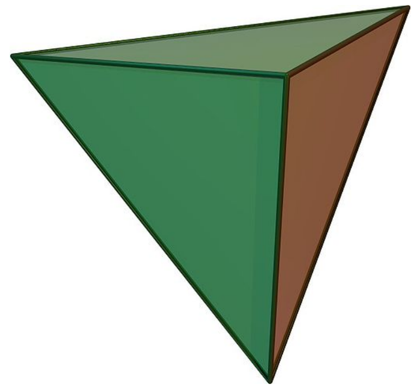
$$U_e^\dagger \rho(g_e) U_e = \rho(g_e)^{\text{diag}} \qquad U_\nu^\dagger \rho(g_\nu) U_\nu = \rho(g_\nu)^{\text{diag}}$$

$$U = U_e^\dagger U_\nu$$

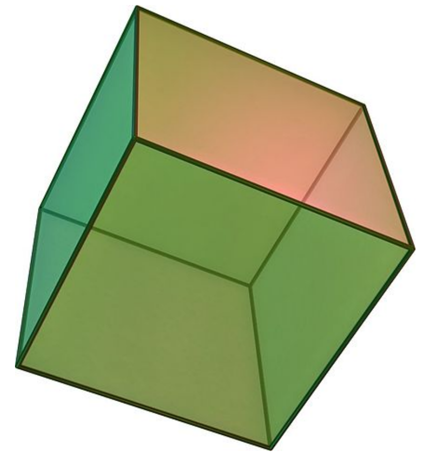
# Non-Abelian discrete symmetries

Images: WIKIPEDIA

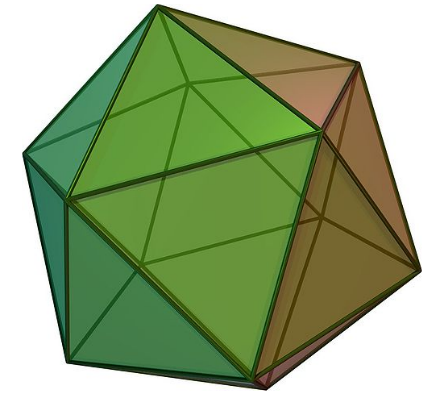
$S_3$ (6)



$A_4$ (12)



$S_4$ (24)



$A_5$ (60)

Generated by two elements $S$ and $T$

$$\langle S,\, T \mid S^2 = (ST)^3 = T^N = I \rangle \,, \quad N = 2\,,\, 3\,,\, 4\,,\, 5$$

Another convenient presentation for $S_4$

$$\langle S,\, T,\, U \mid S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = I \rangle$$

$A_4$ , $S_4$ , and $A_5$ admit a 3-dimensional irrep (unification of families)

Reviews: Altarelli, Feruglio, 1002.0211; Ishimori et al., 1003.3552; King, Luhn, 1301.1340; Petcov, 1711.10806; Feruglio, Romanino, 1912.06028

# Example: TBM mixing from S4

$$G_f = S_4$$

$$G_e = Z_3^T \qquad\qquad G_\nu = Z_2^S \times Z_2^U$$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad \rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \omega = e^{2\pi i/3}$$

$$U_e = \mathbb{1}$$

diagonalised by $U_\nu = U_{\mathrm{TBM}}$

$$U_{\mathrm{PMNS}} = U_e^\dagger U_\nu = U_{\mathrm{TBM}}$$

In concrete models, flavour symmetry breaking occurs spontaneously when flavons (scalar fields not charged under the SM) acquire VEVs

$$\langle \phi^e \rangle \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ preserves } T \quad \text{and} \quad \langle \phi^\nu \rangle \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ preserves } S \text{ and } U$$

# Reconciling TBM mixing with data

Break $T$: charged lepton corrections

$U_e$ free and $U_\nu = U_{\text{TBM}}$

Different ansatzes: $U_e^\dagger = U_{ij}\left(\theta_{ij}^e, \delta_{ij}^e\right)$, $U_e^\dagger = U_{ij}\left(\theta_{ij}^e, \delta_{ij}^e\right) U_{kl}\left(\theta_{kl}^e, \delta_{kl}^e\right)$, …

$$U_{13}\left(\theta^e, \delta^e\right) = \begin{pmatrix} \cos\theta^e & 0 & \sin\theta^e e^{-i\delta^e} \\ 0 & 1 & 0 \\ -\sin\theta^e e^{i\delta^e} & 0 & \cos\theta^e \end{pmatrix}$$

$\theta^e$ and $\delta^e$ are free parameters

Example: $U_e^\dagger = U_{12}\left(\theta^e, \delta^e\right)$

$$\sin\theta_{13} = \frac{1}{\sqrt{2}}\sin\theta^e$$

$$\sin^2\theta_{23} = \frac{1 - 2\sin^2\theta_{13}}{2(1-\sin^2\theta_{13})} \approx 0.489$$

$$\cos\delta = \frac{(1-2\sin^2\theta_{13})^{\frac{1}{2}}}{\sin 2\theta_{12}\sin\theta_{13}}\left[\frac{1}{3} + \left(\sin^2\theta_{12} - \frac{2}{3}\right)\frac{1-3\sin^2\theta_{13}}{1-2\sin^2\theta_{13}}\right] \approx -0.156 \Rightarrow \delta/\pi = 1.45$$

neutrino mixing sum rules

Marzocca, Petcov, Romanino, Sevilla, 1302.0423

Petcov, 1405.6006

Girardi, Petcov, **AT**, 1410.8056, 1504.00658

Girardi, Petcov, Stuart, **AT**, 1509.02502

# Reconciling TBM mixing with data

Break $U$: $G_\nu = Z_2^S$ (instead of $Z_2^S \times Z_2^U$)

$$U_{\text{PMNS}} = U_{\text{TBM}} U_{13}(\theta^\nu, \delta^\nu) = \begin{pmatrix} * & \sqrt{\frac{1}{3}} & * \\ * & \sqrt{\frac{1}{3}} & * \\ * & \sqrt{\frac{1}{3}} & * \end{pmatrix}$$

trimaximal mixing 2 (TM2)

Grimus, Lavoura, 0809.0226

$$\sin^2\theta_{12} = \frac{1}{3\left(1-\sin^2\theta_{13}\right)} > \frac{1}{3} \qquad \cos\delta = \frac{\left(1-2\sin^2\theta_{13}\right)\cot 2\theta_{23}}{\sin\theta_{13}\sqrt{2-3\sin^2\theta_{13}}}$$

Break $S$ and $U$, preserving $SU$: $G_\nu = Z_2^{SU}$

$$U_{\text{PMNS}} = U_{\text{TBM}} U_{23}(\theta^\nu, \delta^\nu) = \begin{pmatrix} \sqrt{\frac{2}{3}} & * & * \\ -\sqrt{\frac{1}{6}} & * & * \\ -\sqrt{\frac{1}{6}} & * & * \end{pmatrix}$$

trimaximal mixing 1 (TM1)

Albright, Rodejohann, 0812.0436

$$\sin^2\theta_{12} = \frac{1-3\sin^2\theta_{13}}{3\left(1-\sin^2\theta_{13}\right)} < \frac{1}{3} \qquad \cos\delta = -\frac{\left(1-5\sin^2\theta_{13}\right)\cot 2\theta_{23}}{2\sqrt{2}\sin\theta_{13}\sqrt{1-3\sin^2\theta_{13}}}$$

# Flavour models with CP

$S_4 \rtimes CP$

$G_e = Z_3$ and $G_\nu = Z_2 \times CP$

| Case | I | II | IV | V |
|---|---|---|---|---|
| $\sin^2\theta_{13}$ | $\frac{2}{3}\sin^2\theta$ | $\frac{2}{3}\sin^2\theta$ | $\frac{1}{3}\sin^2\theta$ | $\frac{1}{3}\sin^2\theta$ |
| $\sin^2\theta_{12}$ | $\frac{1}{2+\cos 2\theta}$ | $\frac{1}{2+\cos 2\theta}$ | $\frac{\cos^2\theta}{2+\cos^2\theta}$ | $\frac{\cos^2\theta}{2+\cos^2\theta}$ |
| $\sin^2\theta_{23}$ | $\frac{1}{2}$ | $\frac{1}{2}\left(1-\frac{\sqrt{3}\sin 2\theta}{2+\cos 2\theta}\right)$ | $\frac{1}{2}$ | $\frac{1}{2}\left(1-\frac{2\sqrt{6}\sin 2\theta}{5+\cos 2\theta}\right)$ |
| $\lvert\sin\delta_{CP}\rvert$ | 1 | 0 | 1 | 0 |

TM2 $(\sin^2\theta_{12} > 1/3)$ TM1 $(\sin^2\theta_{12} < 1/3)$

Feruglio, Hagedorn, Ziegler, 1211.5560

$A_5 \rtimes CP$

$G_e = Z_5$ $G_e = Z_3$ $G_e = Z_2 \times Z_2$ $G_\nu = Z_2 \times CP$

| Case | II | III | IV | V | VII-a | VII-b |
|---|---|---|---|---|---|---|
| $\sin^2\theta_{13}$ | $\frac{3-\varphi}{5}\sin^2\theta$ | $\frac{\varphi}{\sqrt{5}}\sin^2\theta$ | $\frac{\varphi}{\sqrt{5}}\sin^2\theta$ | $\frac{1-\sin 2\theta}{3}$ | $\frac{(\cos\theta-\varphi\sin\theta)^2}{4\varphi^2}$ | |
| $\sin^2\theta_{12}$ | $\frac{2\cos^2\theta}{3+2\varphi+\cos 2\theta}$ | $\frac{4-2\varphi}{5-2\varphi+\cos 2\theta}$ | $\frac{4-2\varphi}{5-2\varphi+\cos 2\theta}$ | $\frac{1}{2+\sin 2\theta}$ | $\frac{(\varphi\cos\theta+\sin\theta)^2}{4\varphi^2-(\cos\theta-\varphi\sin\theta)^2}$ | |
| $\sin^2\theta_{23}$ | $\frac{1}{2}$ | $\frac{1}{2}-\frac{\sqrt{3-\varphi}\sin 2\theta}{3\varphi-2+\varphi\cos 2\theta}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{(\varphi^2\cos\theta-\sin\theta)^2}{4\varphi^2-(\cos\theta-\varphi\sin\theta)^2}$ | $\frac{\varphi^2(\cos\theta+\varphi\sin\theta)^2}{4\varphi^2-(\cos\theta-\varphi\sin\theta)^2}$ |
| $\lvert\sin\delta_{CP}\rvert$ | 1 | 0 | 1 | 1 | 0 | |

$\varphi = \left(1+\sqrt{5}\right)/2$ is the golden ratio

Li, Ding, 1503.03711

# Compatibility with global data

$$\chi^2(\theta) = \left[\frac{\sin^2\theta_{12}(\theta) - \sin^2\theta_{12}}{\sigma(\sin^2\theta_{12})}\right]^2 + \left[\frac{\sin^2\theta_{13}(\theta) - \sin^2\theta_{13}}{\sigma(\sin^2\theta_{13})}\right]^2 + \left[\frac{\sin^2\theta_{23}(\theta) - \sin^2\theta_{23}}{\sigma(\sin^2\theta_{23})}\right]^2$$

1-parameter models compatible with global data at $3\sigma$

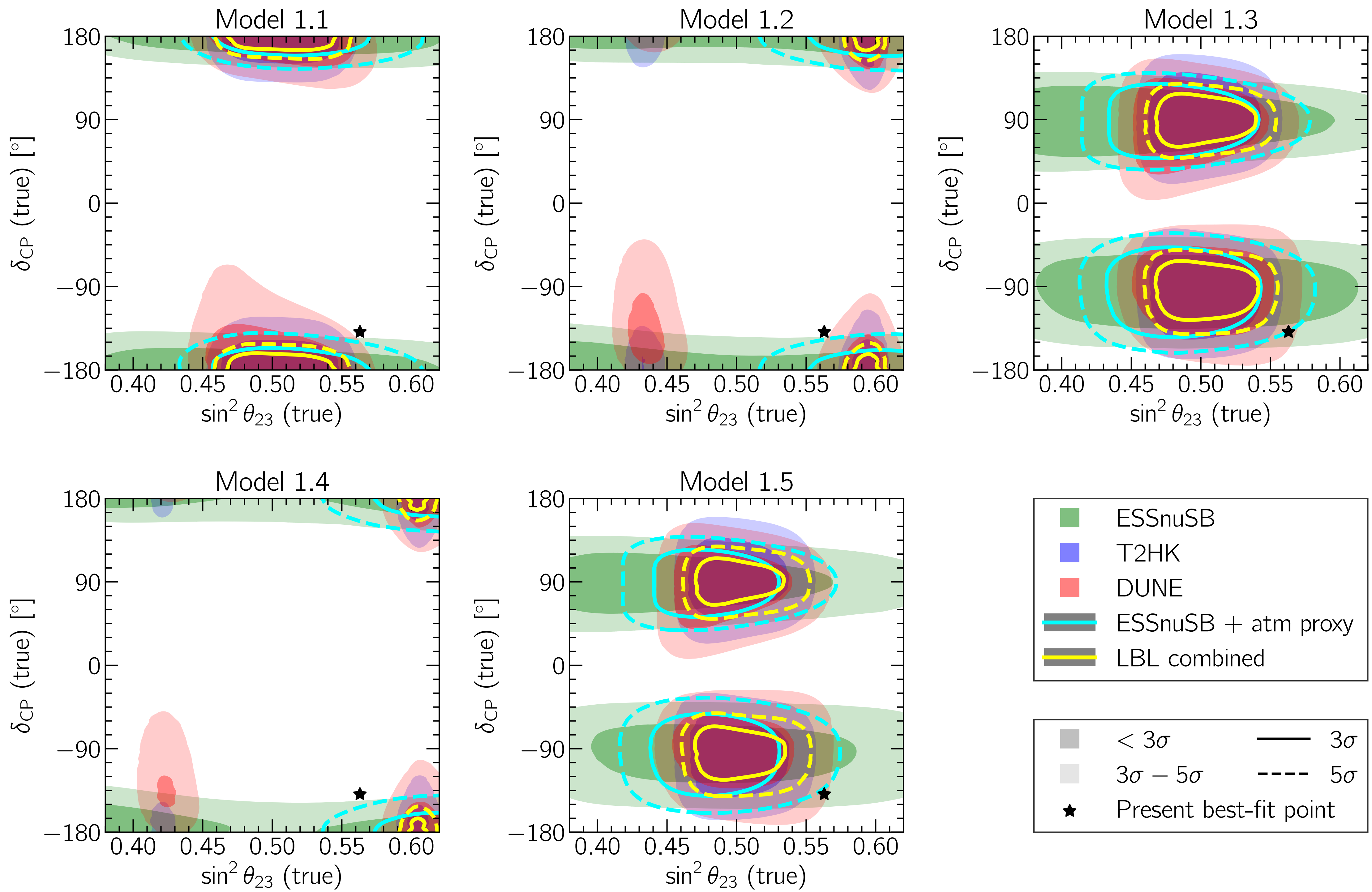
| Model | $\theta_{bf}$ | $\theta_{3\sigma}$ |
|---|---|---|
| 1.1 | 17.0° | (16.3°, 17.7°) |
| 1.2 | 169.9° | (169.4°, 170.4°) |
| 1.3 | 15.0° | (14.3°, 15.7°) |
| | 165.0° | (164.3°, 165.7°) |
| 1.4 | 169.5° | (169.0°, 170.0°) |
| 1.5 | 10.1° | (9.6°, 10.6°) |
| | 169.9° | (169.4°, 170.4°) |

| Model | Case [Ref.] | Group | $\sin^2\theta_{12}$ | $\sin^2\theta_{23}$ | $\delta_{CP}$ | $\chi^2_{min}$ |
|---|---|---|---|---|---|---|
| 1.1 | VII-b [25] | $A_5 \rtimes CP$ | 0.331 | 0.523 | 180° | 5.37 |
| 1.2 | III [25] | $A_5 \rtimes CP$ | 0.283 | 0.593 | 180° | 5.97 |
| 1.3 | IV [24] | $S_4 \rtimes CP$ | 0.318 | 1/2 | ±90° | 7.28 |
| 1.4 | II [24] | $S_4 \rtimes CP$ | 0.341 | 0.606 | 180° | 8.91 |
| 1.5 | IV [25] | $A_5 \rtimes CP$ | 0.283 | 1/2 | ±90° | 11.3 |

Blennow, Ghosh, Ohlsson, **AT**, 2004.00017

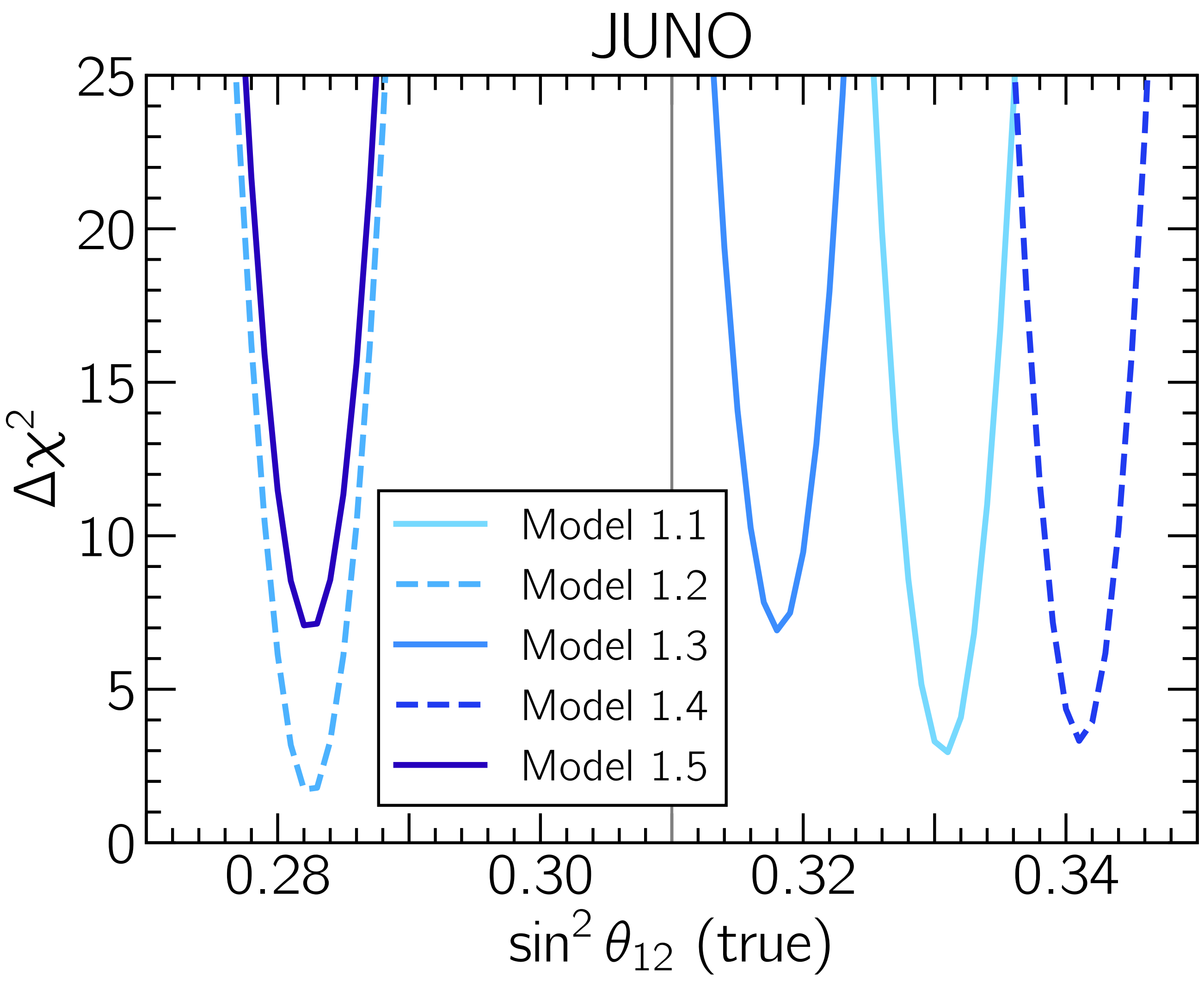
[24] Feruglio, Hagedorn, Ziegler, 1211.5560
[25] Li, Ding, 1503.03711

# Potential of future LBL experiments




Blennow, Ghosh, Ohlsson, **AT**, 2005.12277

# JUNO potential



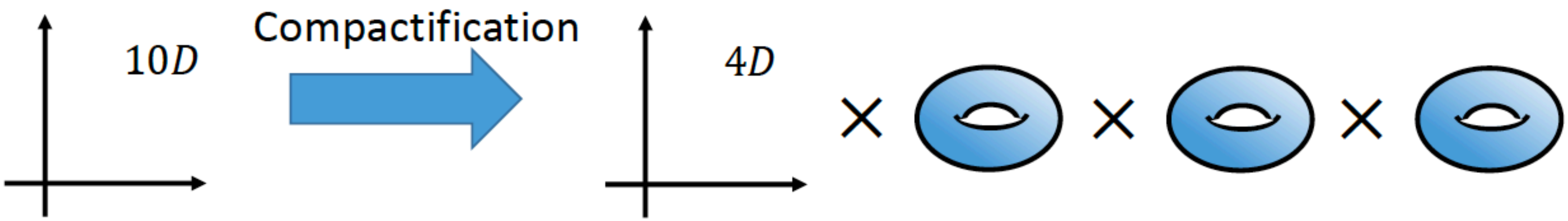

Blennow, Ghosh, Ohlsson, **AT,** 2005.12277

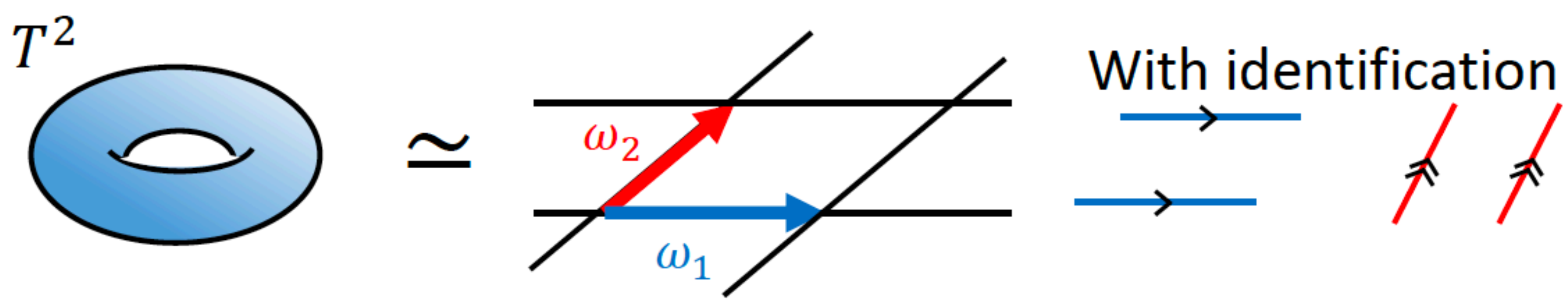
# Challenges

- Many symmetry groups, many models, which one is correct (if any)?
- Symmetry breaking typically relies on numerous flavons
- Elaborated potentials to get desirable vacuum alignment
- Higher-dimensional operators can spoil leading-order predictions
- Mainly mixing, and not masses
- What is the origin of discrete flavour symmetries?

# Modular invariance

Images: Takuya H. Tatsuishi




$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}(\tau) > 0$$

modulus

Lattice left invariant by modular transformations

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

Proposal to apply modular invariance to flavour physics: Feruglio, 1706.08749

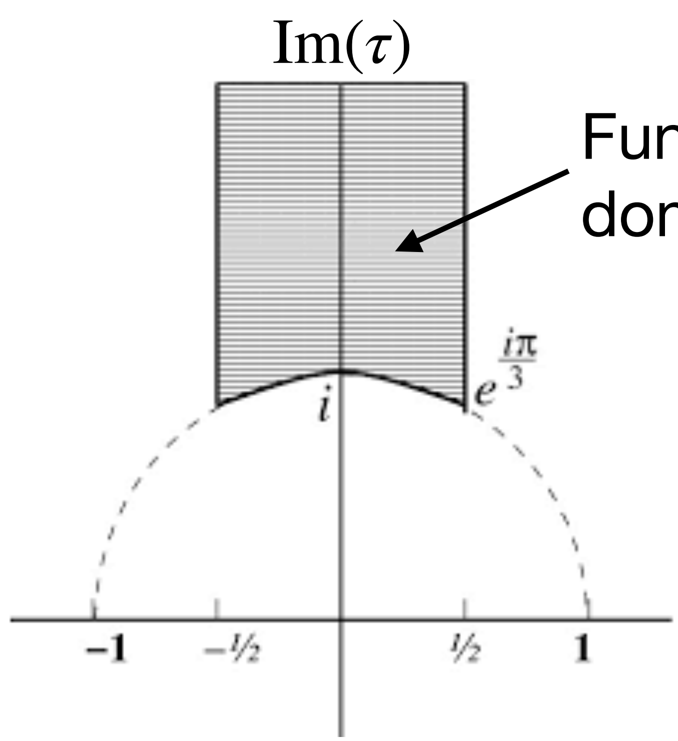
# Modular group

$$\overline{\Gamma} = \langle S, T \mid S^2 = (ST)^3 = I \rangle \cong PSL(2,\mathbb{Z})$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad\qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$




Infinite normal subgroups of $SL(2,\mathbb{Z})$, $N = 2, 3, 4, \ldots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Principal congruence subgroups of the modular group

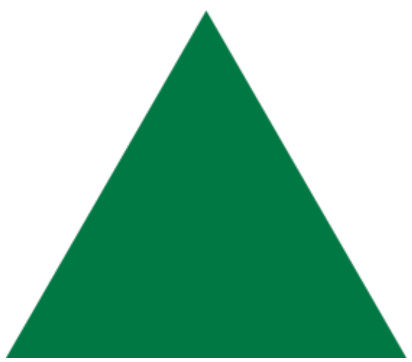
$$\overline{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \qquad \overline{\Gamma}(N) \equiv \Gamma(N), \quad N > 2$$

Finite modular groups $\quad \Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$

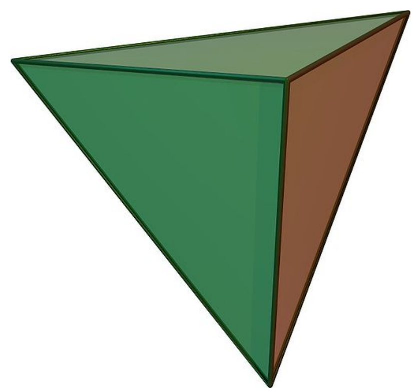
# Finite modular groups

$$\Gamma_N = \langle S,\, T \mid S^2 = (ST)^3 = T^N = I \rangle\,, \quad N = 2\,, 3\,, 4\,, 5$$

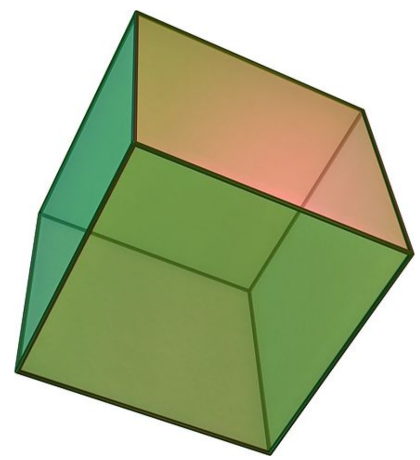
$\Gamma_2 \cong S_3$



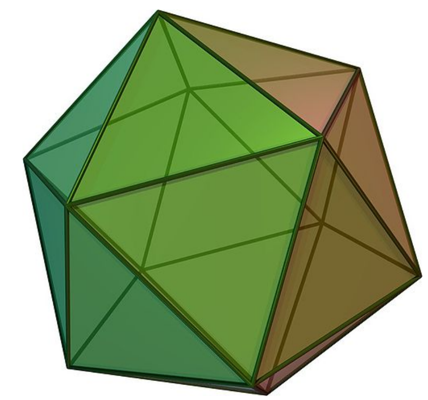
$\Gamma_3 \cong A_4$



$\Gamma_4 \cong S_4$



$\Gamma_5 \cong A_5$



Images: WIKIPEDIA

For $N > 5$ additional relations $f(S, T) = I$ needed to render $\Gamma_N$ finite

de Adelhart Toorop, Feruglio, Hagedorn, 1112.1340

# Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under $\overline{\Gamma}(N)$ as follows

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \quad \gamma \in \overline{\Gamma}(N)$$

$k$ is weight
non-negative even integer

$N$ is level
natural number

Modular forms of weight $k$ and level $N$ form a linear space $\mathscr{M}_k\left(\overline{\Gamma}(N)\right)$ of finite dimension. We can choose a basis in this space s.t. $F(\tau) \equiv \left(f_1(\tau)\,, f_2(\tau)\,, \ldots\right)^T$ transforms as

$$F\left(\gamma\tau\right) = \left(c\tau + d\right)^k \rho(\tilde{\gamma})\, F(\tau)\,, \quad \gamma \in \overline{\Gamma}$$

$\rho$ is a unitary representation of $\Gamma_N$

$\tilde{\gamma}$ represents the equivalence class of $\gamma$ in $\Gamma_N$

Feruglio, 1706.08749

# Modular-invariant SUSY theories

$\mathcal{N} = 1$ rigid SUSY matter action

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\; K(\tau, \overline{\tau}, \chi, \overline{\chi}) + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\; W(\tau, \chi) + \int \mathrm{d}^4x\, \mathrm{d}^2\overline{\theta}\; \overline{W}(\overline{\tau}, \overline{\chi})$$

Kähler potential — Superpotential — Chiral superfields

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \chi_I \to \left(c\tau + d\right)^{-k_I} \rho_I(\tilde{\gamma})\, \chi_I \end{cases}$$

Modular symmetry acts non-linearly

Invariance of the action under these transformations requires

$$\begin{cases} W(\tau, \chi) \to W(\tau, \chi) \\ K(\tau, \overline{\tau}, \chi, \overline{\chi}) \to K(\tau, \overline{\tau}, \chi, \overline{\chi}) + f_K(\tau, \chi) + \overline{f_K}(\overline{\tau}, \overline{\chi}) \end{cases}$$

Ferrara, Lust, Shapere, Theisen, PLB **225** (1989) 363

Ferrara, Lust, Theisen, PLB **233** (1989) 147

Feruglio, 1706.08749

# Modular-invariant SUSY theories

$$K(\tau, \bar{\tau}, \chi, \bar{\chi}) = -\Lambda_0^2 \log(-i\tau + i\bar{\tau}) + \sum_I \frac{|\chi_I|^2}{(-i\tau + i\bar{\tau})^{k_I}}$$

Minimal example

$$W(\tau, \chi) = \sum_n \sum_{\{I_1, \ldots I_n\}} g_{I_1 \ldots I_n} \left( Y_{I_1 \ldots I_n}(\tau)\, \chi_{I_1} \cdots \chi_{I_n} \right)_{\mathbf{1}}$$

Modulus-dependent (Yukawa) couplings

$$Y_{I_1 \ldots I_n}(\tau) \to (c\tau + d)^{k_Y} \rho_Y(\tilde{\gamma})\, Y_{I_1 \ldots I_n}(\tau)$$

$$k_Y = k_{I_1} + \ldots + k_{I_n} \geq 0$$

$$\rho_Y \otimes \rho_{I_1} \otimes \ldots \otimes \rho_{I_n} \supset \mathbf{1}$$

Yukawa couplings $Y_{I_1 \ldots I_n}(\tau)$ are modular forms!

Feruglio, 1706.08749

# Modular A4 symmetry

$\Gamma_3 = \langle S, T \mid S^2 = (ST)^3 = T^3 = I \rangle$

- 12 elements
- 4 irreps: $\mathbf{1}\,,\ \mathbf{1}'\,,\ \mathbf{1}''\,,\ \mathbf{3}$
- Space of the lowest non-trivial weight 2 modular forms has dimension 3
- 3 weight 2 modular forms arrange themselves in a triplet:

$$Y_{\mathbf{3}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

- $Y_i(\tau)$ are given in terms of the Dedekind eta function

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right) \ , \quad q = e^{2\pi i \tau}$$

- Products of $Y_i(\tau)$ generate modular forms of higher weights: 4, 6, 8, …

Feruglio, 1706.08749

# Modular forms of level 3 and weight 2

$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - 27\frac{\eta'(3\tau)}{\eta(3\tau)}\right] = 1 + 12q + 36q^2 + 12q^3 + \dots$$

$$Y_2(\tau) = -\frac{i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -6q^{1/3}\left(1 + 7q + 8q^2 + \dots\right)$$

$$Y_3(\tau) = -\frac{i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -18q^{2/3}\left(1 + 2q + 5q^2 + \dots\right)$$

Here $\omega = e^{\frac{2\pi i}{3}}$ and $q = e^{2\pi i\tau}$ ($|q| = e^{-2\pi\,\mathrm{Im}\tau} < 1$ since $\mathrm{Im}\,\tau > 0$)

Since modular forms are periodic

$$f(T^N\tau) = f(\tau + N) = (c\tau + d)^k f(\tau) = f(\tau)\,, \qquad T^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \overline{\Gamma}(N)\,,$$

they admit $q$-expansions: $f(\tau) = \sum_{n=0}^{\infty} a_n q_N^n\,,\quad q_N = e^{\frac{2\pi i\tau}{N}}$ ($N = 3$ in this example)

Feruglio, 1706.08749

# Feruglio's modular A4 model

$\Gamma_3 \cong A_4$ (level $N = 3$)

3 independent modular forms $Y_i(\tau)$ of weight $k = 2$ form a triplet of $A_4$

$$Y(\tau) = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \to (c\tau + d)^2 \rho(\gamma) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \qquad L = \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix} \to (c\tau + d)^{-1} \rho(\gamma) \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix}$$

$$W_\nu \supset \frac{1}{\Lambda} \left( Y(\tau)\, LL \right)_{\mathbf{1}} H_u H_u \qquad \Rightarrow \qquad M_\nu(\tau) = \frac{\mathrm{v}_u^2}{\Lambda} \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix}$$

$M_\nu$ depends on 3 real parameters: $\mathrm{Re}(\tau)$ , $\mathrm{Im}(\tau)$ and the overall scale $\mathrm{v}_u^2/\Lambda$

For $\langle \tau \rangle = 0.0111 + 0.9946\, i$

$\sin^2\theta_{12} = 0.295$ $\quad \sin^2\theta_{13} = 0.0447$ $\quad \sin^2\theta_{23} = 0.651$

$\delta/\pi = 1.55$ $\quad \alpha_{21}/\pi = 0.22$ $\quad \alpha_{31}/\pi = 1.80$

$m_1 = 0.0500$ eV $\quad m_2 = 0.0507$ eV $\quad m_3 = 0.0007$ eV (IO)

Feruglio, 1706.08749

# Modular S4 symmetry

$\Gamma_4 = \langle S, T \mid S^2 = (ST)^3 = T^4 = I \rangle$

- 24 elements
- 5 irreps: $\mathbf{1}$ , $\mathbf{1}'$, $\mathbf{2}$ , $\mathbf{3}$ , $\mathbf{3}'$
- Space of the lowest non-trivial weight 2 modular forms has dimension 5
- 5 weight 2 modular forms arrange themselves in a doublet and a triplet:

$$Y_{\mathbf{2}}(\tau) \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix} \qquad Y_{\mathbf{3}'}(\tau) \equiv \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$

- $Y_i(\tau)$ are given in terms of the Dedekind eta function

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right) \ , \quad q = e^{2\pi i \tau}$$

- Products of $Y_i(\tau)$ generate modular forms of higher weights: 4, 6, 8, …

Penedo, Petcov, 1806.11040

# Minimal modular S4 seesaw models

Seesaw type I models with no flavons

| | $E_1^c$ | $E_2^c$ | $E_3^c$ | $N^c$ | $L$ | $H_d$ | $H_u$ |
|---|---|---|---|---|---|---|---|
| $SU(2)_L \times U(1)_Y$ | $(\mathbf{1},+1)$ | $(\mathbf{1},+1)$ | $(\mathbf{1},+1)$ | $(\mathbf{1},0)$ | $(\mathbf{2},-1/2)$ | $(\mathbf{2},-1/2)$ | $(\mathbf{2},+1/2)$ |
| $\Gamma_4 \cong S_4$ | $\mathbf{1}$ or $\mathbf{1'}$ | $\mathbf{1}$ or $\mathbf{1'}$ | $\mathbf{1}$ or $\mathbf{1'}$ | $\mathbf{3}$ or $\mathbf{3'}$ | $\mathbf{3}$ or $\mathbf{3'}$ | $\mathbf{1}$ | $\mathbf{1}$ |
| $k_I$ | $k_1$ | $k_2$ | $k_3$ | $k_N$ | $k_L$ | 0 | 0 |

$$W = \sum_{i=1}^{3} \alpha_i \left(E_i^c\, L\, F_{E_i}(\tau)\right)_{\mathbf{1}} H_d + g\left(N^c\, L\, F_N(\tau)\right)_{\mathbf{1}} H_u + \Lambda\left(N^c\, N^c\, F_M(\tau)\right)_{\mathbf{1}}$$

Modular invariance imposes the following constraints on the weights:

$$\begin{cases} k_{\alpha_i} = k_i + k_L \\ k_g = k_N + k_L \\ k_\Lambda = 2\,k_N \end{cases} \quad \Leftrightarrow \quad \begin{cases} k_i = k_{\alpha_i} - k_g + k_\Lambda/2 \\ k_L = k_g - k_\Lambda/2 \\ k_N = k_\Lambda/2 \end{cases}$$

$$W = \lambda_{ij}(\tau)\, E_i^c\, L_j\, H_d + \mathcal{Y}_{ij}(\tau)\, N_i^c\, L_j\, H_u + \frac{1}{2} M_{ij}(\tau)\, N_i^c\, N_j^c$$

After integrating out heavy neutrinos and after EWSB

$$M_e = \mathrm{v}_d\, \lambda^\dagger \qquad M_\nu = -\,\mathrm{v}_u^2\, \mathcal{Y}^T M^{-1} \mathcal{Y}$$

Novichkov, Penedo, Petcov, **AT**, 1811.04933

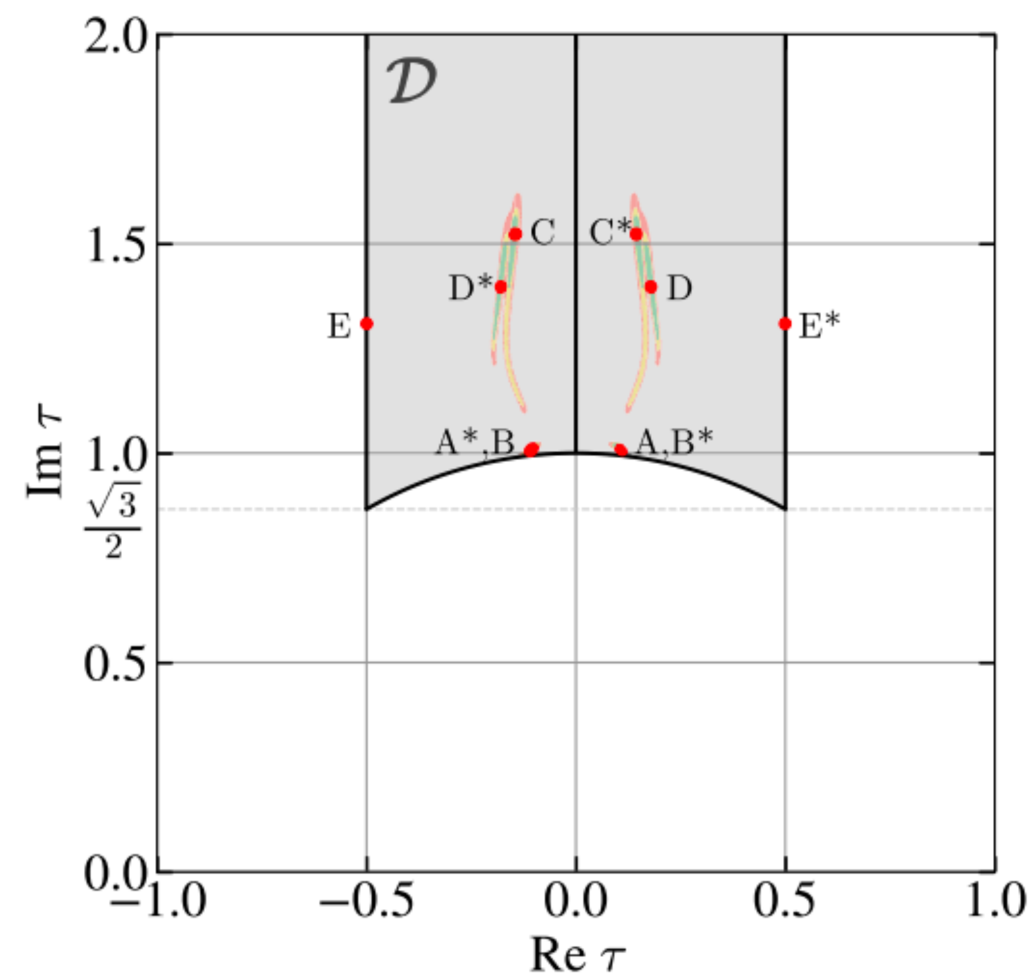
# Minimal modular S4 seesaw models

Charged leptons: $(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3}) = (2, 4, 4)$ Neutrinos: $(k_\Lambda, k_g) = (0, 2)$

$$W = \alpha \left(E_1^c L\, Y^{(2)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y^{(4)}_{\mathbf{3}}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y^{(4)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d$$
$$+ g \left(N^c L\, Y^{(2)}_{\mathbf{2}}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y^{(2)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}$$

## Solutions A and A*

| Input parameters | | Observables | | Predictions | |
|---|---|---|---|---|---|
| $\mathrm{Re}\,\tau$ | $\pm 0.1045$ | $m_e/m_\mu$ | 0.0048 | $m_1$ [eV] | 0.017 |
| $\mathrm{Im}\,\tau$ | 1.0100 | $m_\mu/m_\tau$ | 0.0565 | $m_2$ [eV] | 0.019 |
| $\beta/\alpha$ | 9.465 | $r$ | 0.0299 | $m_3$ [eV] | 0.053 |
| $\gamma/\alpha$ | 0.0022 | $\sin^2\theta_{12}$ | 0.305 | $\delta/\pi$ | $\pm 1.31$ |
| $\mathrm{Re}\,(g'/g)$ | 0.2330 | $\sin^2\theta_{13}$ | 0.0213 | $\alpha_{21}/\pi$ | $\pm 0.30$ |
| $\mathrm{Im}\,(g'/g)$ | $\pm 0.4924$ | $\sin^2\theta_{23}$ | 0.551 | $\alpha_{31}/\pi$ | $\pm 0.87$ |
| $v_d\alpha$ [MeV] | 53.19 | $\delta m^2$ [$10^{-5}$ eV$^2$] | 7.34 | $\lvert m_{ee}\rvert$ [eV] | 0.017 |
| $v_u^2 g_1^2/\Lambda$ [eV] | 0.0093 | $\lvert\Delta m^2\rvert$ [$10^{-3}$ eV$^2$] | 2.455 | $\sum_i m_i$ [eV] | 0.090 |
| | | $N\sigma$ | 0.02 | Ordering | NO |




8 (5) parameters vs 12 (9) observables

Novichkov, Penedo, Petcov, **AT**, 1811.04933

# Modular invariance and CP

- Modulus

$$\tau \xrightarrow{CP} -\tau^*$$

- Matter supermultiplets

$$\chi(x) \xrightarrow{CP} X\,\overline{\chi}(x_P)\,, \quad x_P = (t, -\mathbf{x})$$

In the symmetric basis where $\rho(S)^T = \rho(S)$ and $\rho(T)^T = \rho(T)$,
$X = \mathbb{1}$ (canonical CP basis)

- Modular form multiplets

$$Y(\tau) \xrightarrow{CP} Y(-\tau^*) = X\,Y^*(\tau) = Y^*(\tau) \text{ (in the symmetric basis)}$$

- Lagrangian couplings

$$g_i^* = g_i$$

Novichkov, Penedo, Petcov, **AT**, 1905.11970

# CP-conserving values of the modulus

$\tau$ and $\gamma\tau$ are physically equivalent, hence CP is preserved for

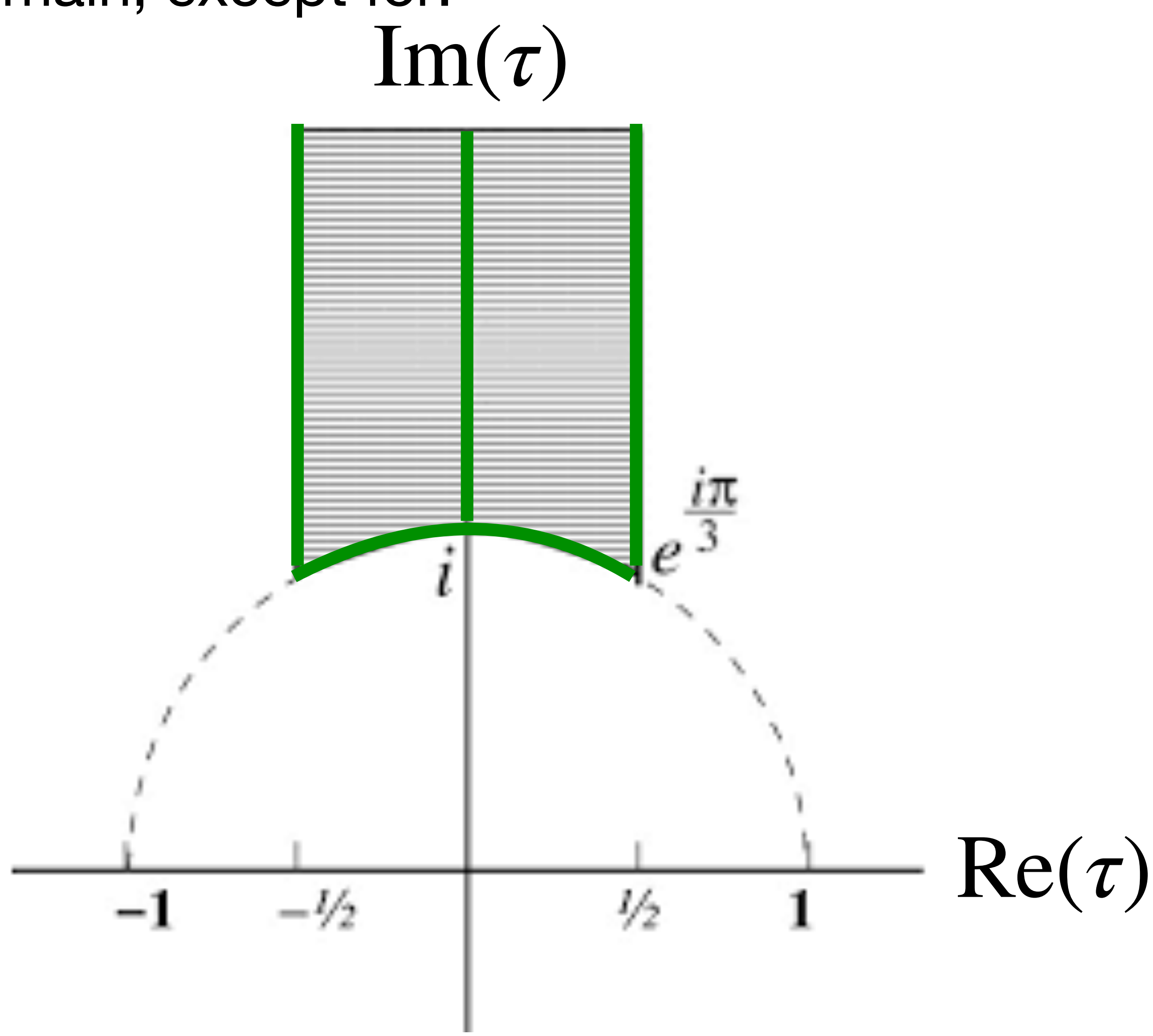
$$\tau \xrightarrow{CP} -\tau^* = \gamma\tau$$

CP is violated in the fundamental domain, except for:

1) $\tau = iy \xrightarrow{CP} iy$

2) $\tau = -\frac{1}{2} + i\,y \xrightarrow{CP} \frac{1}{2} + i\,y = T\tau$

3) $\tau = e^{i\varphi} \xrightarrow{CP} -e^{-i\varphi} = S\tau$




Novichkov, Penedo, Petcov, **AT**, 1905.11970

# Minimal modular S4 seesaw models

Charged leptons: $(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3}) = (2, 4, 4)$ Neutrinos: $(k_\Lambda, k_g) = (0, 2)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3'}}^{(4)}\right)_{\mathbf{1}} H_d$$

No CP

$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}$$

complex

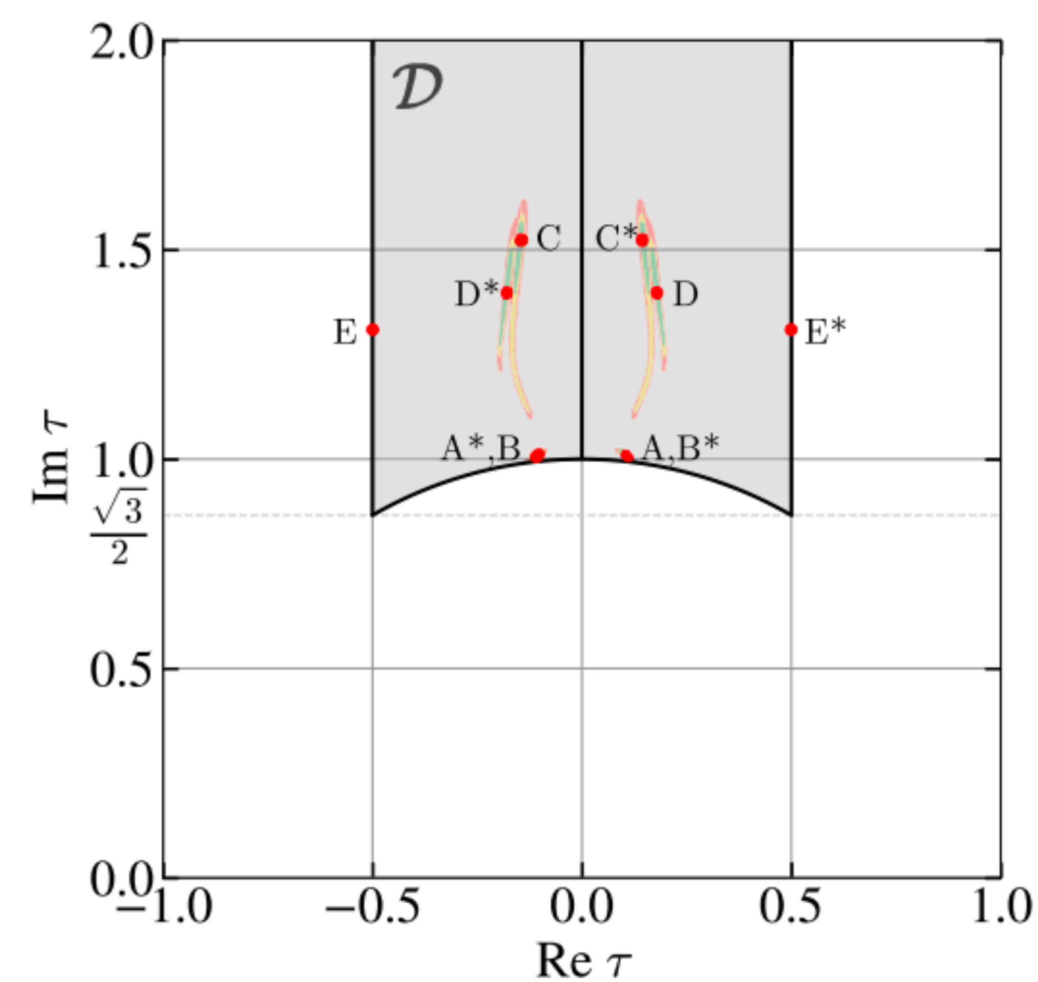
Solutions A and A*

| Input parameters | |
|---|---|
| $\mathrm{Re}\,\tau$ | $\pm 0.1045$ |
| $\mathrm{Im}\,\tau$ | $1.0100$ |
| $\beta/\alpha$ | $9.465$ |
| $\gamma/\alpha$ | $0.0022$ |
| $\mathrm{Re}\,(g'/g)$ | $0.2330$ |
| $\mathrm{Im}\,(g'/g)$ | $\pm 0.4924$ |
| $v_d \alpha$ [MeV] | $53.19$ |
| $v_u^2 g_1^2/\Lambda$ [eV] | $0.0093$ |

| Observables | |
|---|---|
| $m_e/m_\mu$ | $0.0048$ |
| $m_\mu/m_\tau$ | $0.0565$ |
| $r$ | $0.0299$ |
| $\sin^2\theta_{12}$ | $0.305$ |
| $\sin^2\theta_{13}$ | $0.0213$ |
| $\sin^2\theta_{23}$ | $0.551$ |
| $\delta m^2$ [$10^{-5}$ eV$^2$] | $7.34$ |
| $\lvert\Delta m^2\rvert$ [$10^{-3}$ eV$^2$] | $2.455$ |
| $N\sigma$ | $0.02$ |

| Predictions | |
|---|---|
| $m_1$ [eV] | $0.017$ |
| $m_2$ [eV] | $0.019$ |
| $m_3$ [eV] | $0.053$ |
| $\delta/\pi$ | $\pm 1.31$ |
| $\alpha_{21}/\pi$ | $\pm 0.30$ |
| $\alpha_{31}/\pi$ | $\pm 0.87$ |
| $\lvert m_{ee}\rvert$ [eV] | $0.017$ |
| $\sum_i m_i$ [eV] | $0.090$ |
| Ordering | NO |

8 (5) parameters vs 12 (9) observables




Novichkov, Penedo, Petcov, **AT**, 1811.04933

# Minimal modular S4 seesaw models

Charged leptons: $(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3}) = (2, 4, 4)$ Neutrinos: $(k_\Lambda, k_g) = (0, 2)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3'}}^{(4)}\right)_{\mathbf{1}} H_d$$

With CP

$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}$$

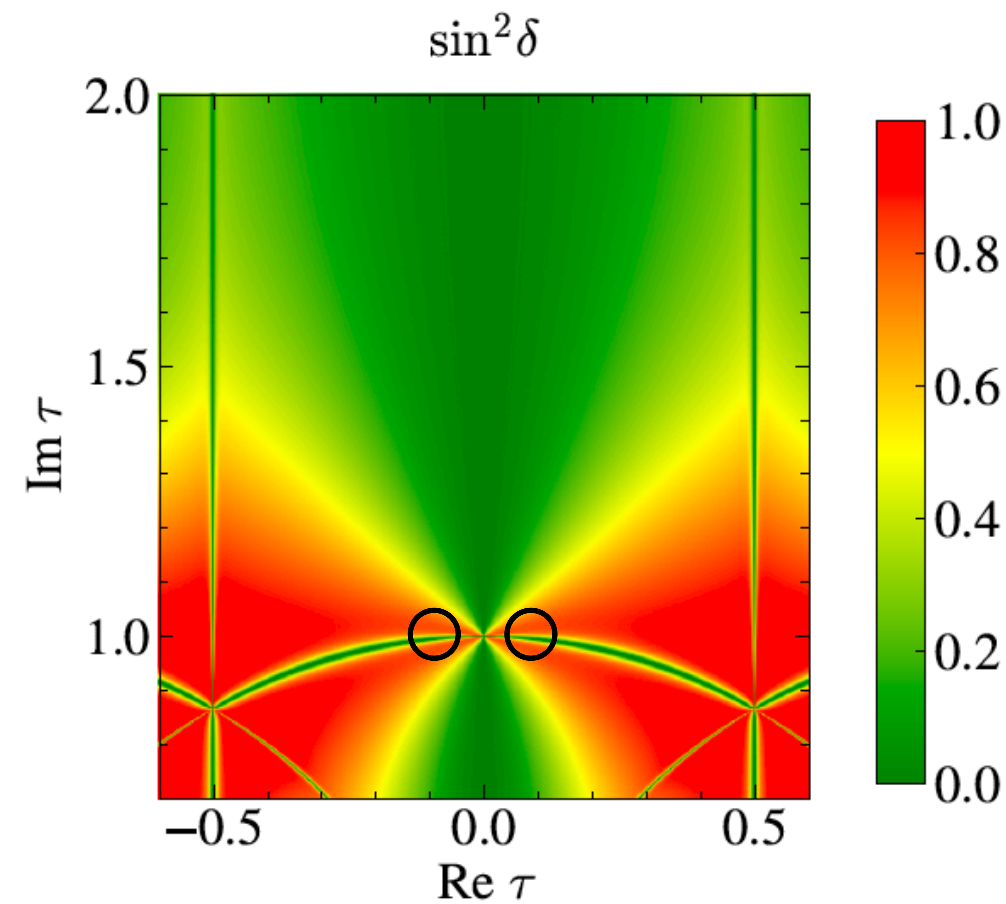
real

Solutions A and A*

| Input parameters | |
|---|---|
| $\mathrm{Re}\,\tau$ | $\pm 0.0992$ |
| $\mathrm{Im}\,\tau$ | 1.0160 |
| $\beta/\alpha$ | 9.348 |
| $\gamma/\alpha$ | 0.0022 |
| $g'/g$ | $-0.0209$ |
| $v_d\alpha$ [MeV] | 53.61 |
| $v_u^2 g_1^2/\Lambda$ [eV] | 0.0135 |

| Observables | |
|---|---|
| $m_e/m_\mu$ | 0.0048 |
| $m_\mu/m_\tau$ | 0.0576 |
| $r$ | 0.0298 |
| $\sin^2\theta_{12}$ | 0.305 |
| $\sin^2\theta_{13}$ | 0.0214 |
| $\sin^2\theta_{23}$ | 0.486 |
| $\delta m^2$ [$10^{-5}$ eV$^2$] | 7.33 |
| $\lvert\Delta m^2\rvert$ [$10^{-3}$ eV$^2$] | 2.457 |
| $N\sigma$ | 1.01 |

| Predictions | |
|---|---|
| $m_1$ [eV] | 0.012 |
| $m_2$ [eV] | 0.015 |
| $m_3$ [eV] | 0.051 |
| $\delta/\pi$ | $\pm 1.64$ |
| $\alpha_{21}/\pi$ | $\pm 0.35$ |
| $\alpha_{31}/\pi$ | $\pm 1.25$ |
| $\lvert m_{ee}\rvert$ [eV] | 0.012 |
| $\sum_i m_i$ [eV] | 0.078 |
| Ordering | NO |




7 (4) parameters vs 12 (9) observables

Novichkov, Penedo, Petcov, **AT**, 1905.11970

# Vacuum selection

In the considered bottom-up approach the VEV of $\tau$ is a free parameter

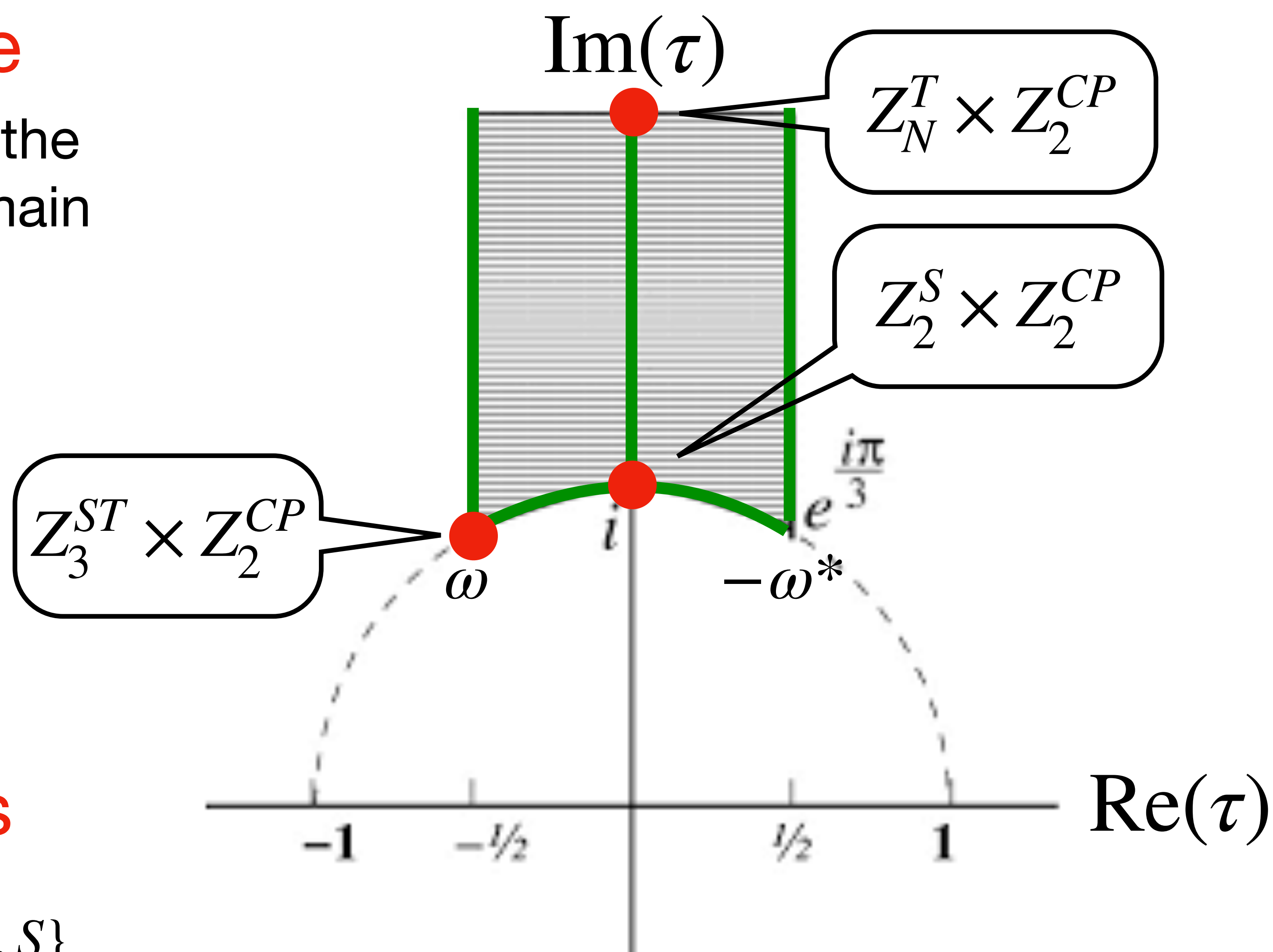
## Top-down conjecture

All extrema of the potential lie on the boundary of the fundamental domain and on the imaginary axis

M. Cvetic et al., NPB **361** (1991) 194

Recent studies find new, CP-violating minima

Novichkov, Penedo, Petcov, 2201.02020
Leedom, Righi, Westphal, 2212.03876



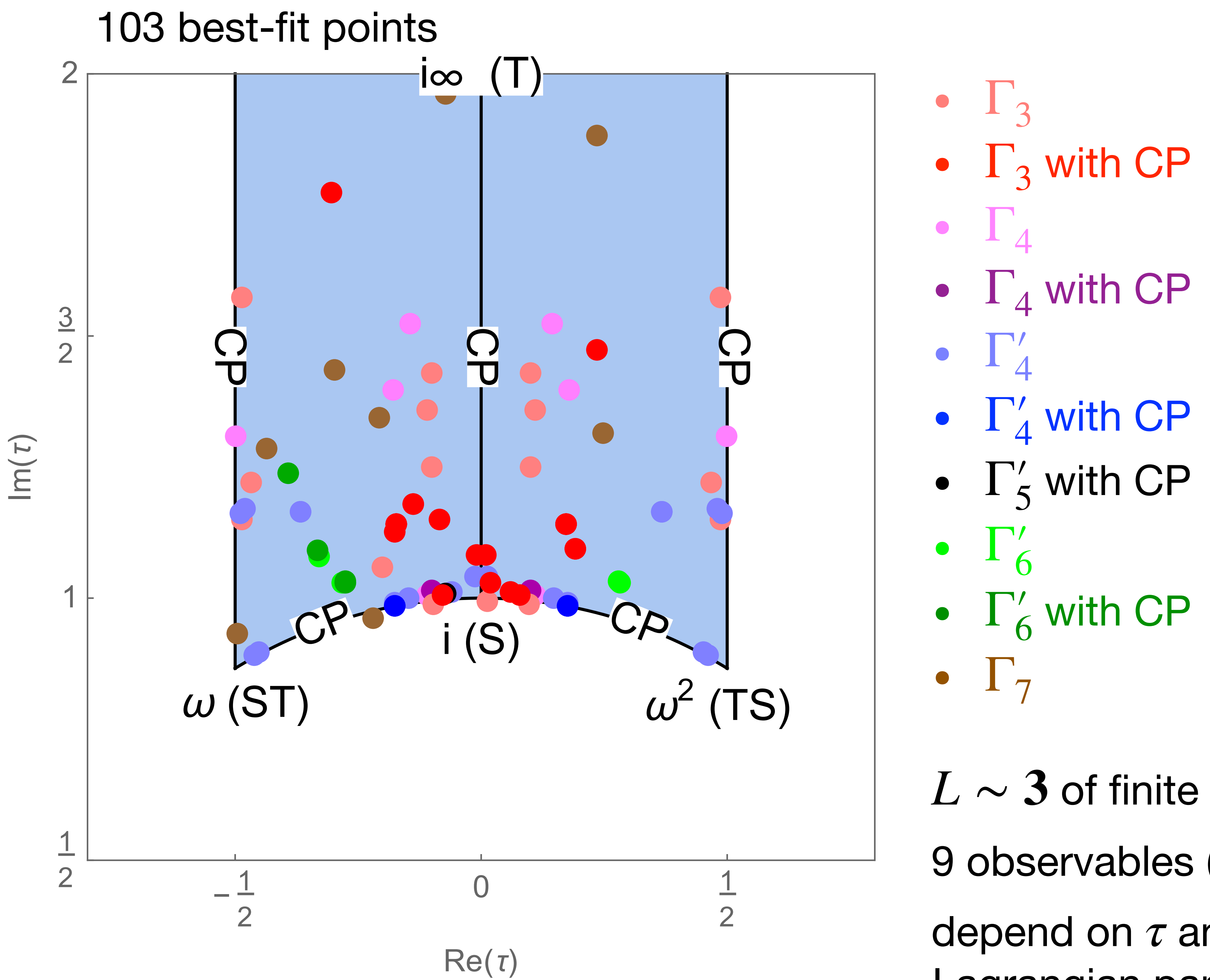

## Residual symmetries

$$\tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad Z_2^S = \{I, S\}$$

$$\tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega + 1} = \omega \quad \Rightarrow \quad Z_3^{ST} = \{I, ST, (ST)^2\}$$

$$\tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad Z_N^T = \{I, T, T^2, \ldots, T^N\}$$

# Selection of models




$L \sim \mathbf{3}$ of finite modular group $\Gamma_N^{(\prime)}$

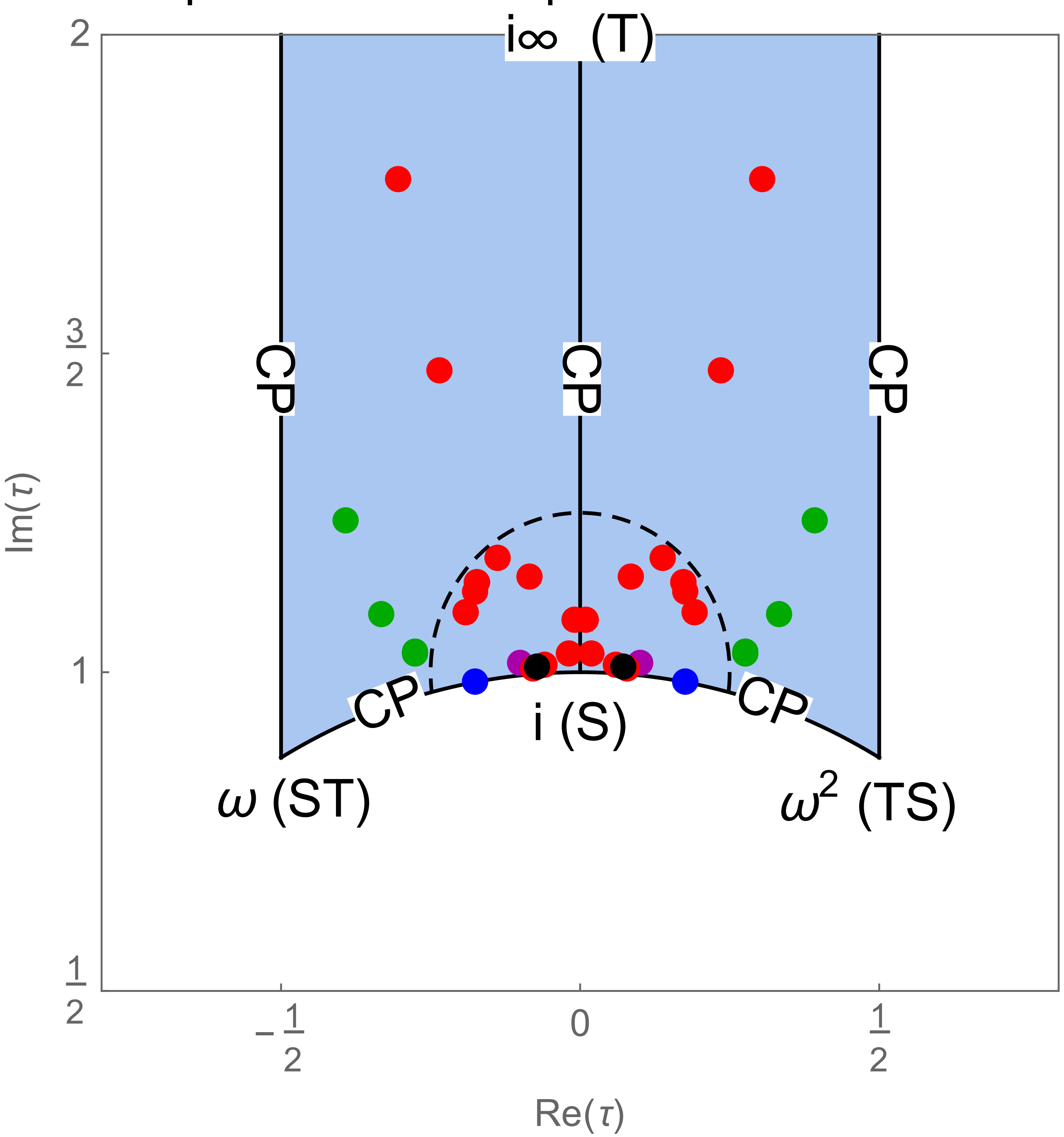
9 observables $(m_i\,,\ \theta_{ij}\,,\ \delta\,,\ \alpha_{ij})$

depend on $\tau$ and 2 or 3 additional Lagrangian parameters

Feruglio, 2211.00659

# Selection of models with CP

27 pairs of best-fit points




- $\Gamma_3$ with CP
- $\Gamma_4$ with CP
- $\Gamma_4'$ with CP
- $\Gamma_5'$ with CP
- $\Gamma_6'$ with CP

- - - $|\tau - i| = 0.25$

For 2/3 of points $|\tau - i| < 0.25$

$L \sim \mathbf{3}$ of finite modular group $\Gamma_N^{(\prime)}$

9 observables $(m_i\,,\ \theta_{ij}\,,\ \delta\,,\ \alpha_{ij})$ depend on $\tau$ and 2 or 3 additional Lagrangian parameters

Feruglio, 2211.00659

# Modular vs conventional discrete

Advantages

✓ Numerous scalar fields (flavons) → (single) modulus

✓ Complicated scalar potential → moduli space

✓ Yukawa couplings → modular forms (known functions of $\tau$)

✓ $A_4$, $S_4$, $A_5$ arise as quotient groups of the modular group

✓ Both mixing parameters and masses are predicted/constrained

Challenges

- What determines the level, weights and representations, $(N, k_I, \rho_I)$ tuple?
- Kinetic terms are not constrained in the bottom-up approach
- Dynamical selection of the vacuum $\langle \tau \rangle$
- Extension to the quark sector
- Is SUSY necessary?

# Aside remark: the strong CP problem

Published for SISSA by Springer

Received: *June 13, 2023*
Accepted: *June 16, 2023*
Published: *July 4, 2023*

## Modular invariance and the QCD angle

**Ferruccio Feruglio,[a] Alessandro Strumia[b] and Arsenii Titov[b]**

[a] *INFN, Sezione di Padova, Via Francesco Marzolo 8, Padova, Italy*
[b] *Dipartimento di Fisica, Università di Pisa, Largo Bruno Pontecorvo 3, Pisa, Italy*
*E-mail:* ferruccio.feruglio@pd.infn.it, alessandro.strumia@unipi.it, arsenii.titov@df.unipi.it

Abstract: String compactifications on an orbi-folded torus with complex structure give rise to chiral fermions, spontaneously broken CP, modular invariance. We show that this allows simple effective theories of flavour and CP where: i) the QCD angle vanishes; ii) the CKM phase is large; iii) quark and lepton masses and mixings can be reproduced up to order one coefficients. We implement such general paradigm in supersymmetry or supergravity, with modular forms or functions, with or without heavy colored states.

Keywords: CP Violation, Discrete Symmetries, Supersymmetry, Theories of Flavour

ArXiv ePrint: 2305.08908

# Conclusions

- We are still far from the Theory of Flavour
- Symmetries remain the best tool to approach the flavour puzzle
- Many viable models based on non-Abelian discrete symmetries broken to residual symmetries of the charged lepton and neutrino mass matrices
- The number of viable models will be reduced by future, more precise measurements of the neutrino mixing parameters, including $\delta_{\mathrm{CP}}$ (DUNE, T2HK, ESSnuSB, JUNO are crucial in this respect)
- Modular invariance is elegant, and it has a number of advantages over conventional discrete flavour symmetries
- More effort is needed towards deciphering the nature of flavour

# Backup slides

# Discrete symmetry and CP

Generalised CP (GCP) transformation

$$\varphi(x) \xrightarrow{CP} X\varphi^*(x_P) \qquad x = (t, \mathbf{x}) \qquad x_P = (t, -\mathbf{x})$$

unitary matrix

Consistency condition ($X$ is constrained by $G_f$)

$$X\rho^*(g)X^{-1} = \rho(g') \qquad g, g' \in G_f$$

Feruglio, Hagedorn, Ziegler, 1211.5560
Holthausen, Lindner, Schmidt, 1211.6953

If $G_e > Z_2$ and $G_\nu = Z_2 \times CP$, the mixing matrix is defined up to a real rotation

$$U_{\text{PMNS}} = U_{\text{fixed}} R_{ij}(\theta) \qquad R_{ij}(\theta) = U_{ij}(\theta, 0) \qquad \theta \text{ is a free real angle}$$

- 1 free parameter => higher predictive power
- Predictions for the Majorana phases

# Future neutrino oscillation experiments

| Experiment | Mass | Baseline | Power<br>POT | Running time |
|---|---|---|---|---|
| JUNO | 20 kt | 53 km | 36 $GW_{th}$ | 6 years |
| DUNE | 40 kt | 1300 km | 1.2 MW<br>$1.1\times10^{21}$ POT/y | 7 years |
| T2HK | 187 kt (×2) | 295 km | 1.3 MW<br>$2.7\times10^{21}$ POT/y | 10 years |
| ESSnuSB | 1 Mt | 540 km | 5 MW<br>$2.7\times10^{23}$ POT/y | 10 years |

Designs used in Blennow, Ghosh, Ohlsson, **AT**, 2005.12277

# Minimal modular S4 seesaw models

Systematic exploration of low weights $k_{\alpha_i}$, $k_g$, $k_N$

Higher weights => more free parameters in the superpotential

Majorana mass term for $N^c$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}' \otimes \mathbf{3}' = \mathbf{1} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{3}'$$

$$k_\Lambda = 0 \;\Rightarrow\; F_M = \text{const}: \quad \left(N^c N^c\right)_{\mathbf{1}} = N_1^c N_1^c + N_2^c N_3^c + N_3^c N_2^c \qquad M = 2\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$k_\Lambda = 2 \;\Rightarrow\; F_M = Y_{\mathbf{2}},\, Y_{\mathbf{3}'}: \quad \Lambda\left(N^c N^c Y_{\mathbf{2}}\right)_{\mathbf{1}} + \Lambda'\left(N^c N^c Y_{\mathbf{3}'}\right)_{\mathbf{1}} \qquad M = 2\Lambda \begin{pmatrix} 0 & Y_1 & Y_2 \\ Y_1 & Y_2 & 0 \\ Y_2 & 0 & Y_1 \end{pmatrix}$$

$$k_\Lambda = 4 \;\Rightarrow\; F_M = Y_{\mathbf{1}}^{(4)},\, Y_{\mathbf{2}}^{(4)},\, Y_{\mathbf{3}}^{(4)},\, Y_{\mathbf{3}'}^{(4)}:$$

$$\Lambda\left(N^c N^c Y_{\mathbf{1}}^{(4)}\right)_{\mathbf{1}} + \Lambda'\left(N^c N^c Y_{\mathbf{2}}^{(4)}\right)_{\mathbf{1}} + \Lambda''\left(N^c N^c Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} + \Lambda'''\left(N^c N^c Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}}$$

# Minimal modular S4 seesaw models

Charged-lepton Yukawa matrix

$$\left(k_{\alpha_1}, k_{\alpha_2}, k_{\alpha_3}\right) = (2, 4, 4) \;\Rightarrow\; \left(F_{E_1}, F_{E_2}, F_{E_3}\right) = \left(Y_{\mathbf{3'}}, Y^{(4)}_{\mathbf{3}}, Y^{(4)}_{\mathbf{3'}}\right):$$

$$\lambda = \begin{pmatrix} \alpha Y_3 & \alpha Y_5 & \alpha Y_4 \\ \beta\left(Y_1Y_4 - Y_2Y_5\right) & \beta\left(Y_1Y_3 - Y_2Y_4\right) & \beta\left(Y_1Y_5 - Y_2Y_3\right) \\ \gamma\left(Y_1Y_4 + Y_2Y_5\right) & \gamma\left(Y_1Y_3 + Y_2Y_4\right) & \gamma\left(Y_1Y_5 + Y_2Y_3\right) \end{pmatrix}$$

Number of free real parameters in $M_e$ and $M_\nu$

($\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$ + coupling constants in the superpotential)

| $k_g$ \ $k_\Lambda$ | 0 | 2 | 4 |
|---|---|---|---|
| 0 | 6 | 6 | 10 |
| 2 | 8 | 8 | 12 |
| 4, $\rho_N \neq \rho_L$ | 10 | 10 | 14 |
| 4, $\rho_N = \rho_L$ | 12 | 12 | 16 |

We aim to describe/predict 12 observables:

$$m_e, m_\mu, m_\tau$$

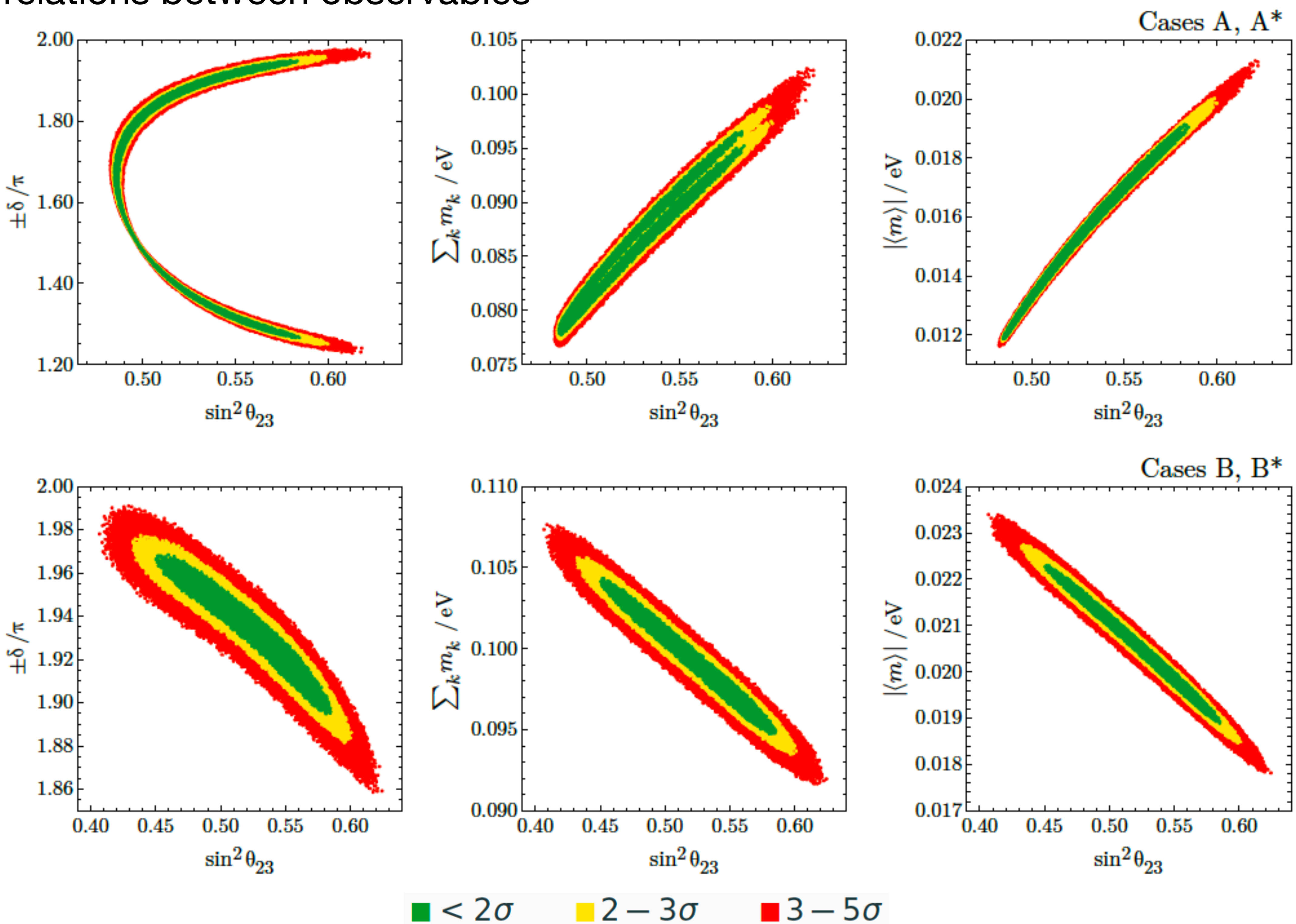
$$m_1, m_2, m_3 \quad \text{or} \quad m_3, m_1, m_2$$

$$\sin^2\theta_{12}, \;\sin^2\theta_{13}, \;\sin^2\theta_{23}$$

$$\delta, \alpha_{21}, \alpha_{31}$$

# Minimal modular S4 seesaw models

Correlations between observables



# Extended modular group

$CP \to \gamma \to CP^{-1}$ on the modulus

$$\tau \xrightarrow{CP} -\tau^* \xrightarrow{\gamma} -\frac{a\tau^*+b}{c\tau^*+d} \xrightarrow{CP^{-1}} \frac{a\tau-b}{-c\tau+d}$$

Outer automorphism of $\overline{\Gamma}$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to u(\gamma) \equiv CP\,\gamma\,CP^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$u(S) = S \qquad u(T) = T^{-1}$$

Extended modular group

$$\overline{\Gamma}^* = \left\langle \tau \xrightarrow{S} -1/\tau\,,\ \ \tau \xrightarrow{T} \tau+1\,,\ \ \tau \xrightarrow{CP} -\tau^* \right\rangle \simeq \overline{\Gamma} \rtimes Z_2^{CP}$$

$$\overline{\Gamma}^* \simeq PGL(2,\mathbb{Z}) \text{ with } CP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tau \to \frac{a\tau+b}{c\tau+d} \quad \text{if} \quad ad-bc=1 \qquad \text{and} \qquad \tau \to \frac{a\tau^*+b}{c\tau^*+d} \quad \text{if} \quad ad-bc=-1$$

# Implication of CP for couplings

$$W \supset \sum_s g_s \left(Y_s(\tau)\,\chi_1 \ldots \chi_n\right)_{\mathbf{1},s} \qquad \overline{W} \supset \sum_s g_s^* \overline{\left(Y_s(\tau)\,\chi_1 \ldots \chi_n\right)_{\mathbf{1},s}}$$

In a symmetric basis ($X = \mathbb{1}$)

$$g_s \left(Y_s(\tau)\chi_1 \ldots \chi_n\right)_{\mathbf{1},s} \xrightarrow{CP} g_s \left(Y_s^*(\tau)\overline{\chi}_1 \ldots \overline{\chi}_n\right)_{\mathbf{1},s} = g_s \overline{\left(Y_s(\tau)\chi_1 \ldots \chi_n\right)_{\mathbf{1},s}}$$

reality of Clebsch-Gordan coefficients (holds for $N \leq 5$)

$$g_s = g_s^*$$

Couplings must be real

# Sources of corrections

## SUSY breaking

Can be made negligible via separation of
SUSY-breaking scale and messenger scale

## Renormalisation group running

Small for $\tan\beta \lesssim 10\ (25)$ dependent on the model

Criado, Feruglio, 1807.01125

## Kähler potential

This is a problem in the bottom-up approach, since many terms allowed by modular invariance can be present in $K$

Feruglio, 1706.08749
Chen, Ramos-Sánchez, Ratz 1909.06910, + et al. 2108.02240
Feruglio, Gherardi, Romanino, **AT**, 2101.08718
Eclectic flavour symmetries: Nilles, Ramos-Sánchez, Vaudrevange, 2001.01736, 2004.05200

# Dimension of linear space of modular forms

| $N$ | $g$ | $d_{2k}(\Gamma(N))$ | $\mu_N$ | $\Gamma_N$ |
|---|---|---|---|---|
| 2 | 0 | $k+1$ | 6 | $S_3$ |
| 3 | 0 | $2k+1$ | 12 | $A_4$ |
| 4 | 0 | $4k+1$ | 24 | $S_4$ |
| 5 | 0 | $10k+1$ | 60 | $A_5$ |
| 6 | 1 | $12k$ | 72 | |
| 7 | 3 | $28k-2$ | 168 | |

$k(\text{this presentation}) \equiv 2\,k(\text{this table})$

Feruglio, 1706.08749

# Modular forms of level 2 and weight 2

**Level *N* = 2** $\left(\Gamma_2 \simeq S_3 : \quad S^2 = (ST)^3 = T^2 = I\right)$

| N\k | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 2 | 1 | **2** | 3 | 4 |

$S$ (up to multiplicative factors)

Seed functions: $\eta\left(\frac{\tau}{2}\right), \quad \eta\left(\frac{\tau+1}{2}\right), \quad \eta\left(2\tau\right)$

$T$

$$\eta(\tau) \equiv q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right) \,, \quad q = e^{2\pi i \tau},$$ is the Dedekind eta function

$$\eta(\tau+1) = e^{i\pi/12}\,\eta(\tau) \qquad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

Kobayashi, Tanaka, Tatsuishi, 1803.10391

# Modular forms of level 2 and weight 2

**Level *N* = 2** $\left(\Gamma_2 \simeq S_3 : \quad S^2 = (ST)^3 = T^2 = I\right)$

| N\k | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 2 | 1 | **2** | 3 | 4 |

$$Y(a_1, a_2, a_3 \,|\, \tau) \equiv \sum_{i=1}^{3} a_i \frac{\mathrm{d}}{\mathrm{d}\tau} \log \eta_i(\tau) \,, \qquad \sum_{i=1}^{3} a_i = 0$$

$$Y_{\mathbf{2}}(-1/\tau) = \tau^2 \rho(S)\; Y_{\mathbf{2}}(\tau) \qquad Y_{\mathbf{2}}(\tau+1) = \rho(T)\; Y_{\mathbf{2}}(\tau)$$

$$\rho(S) = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y_{\mathbf{2}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix} = c \begin{pmatrix} Y(1,1,-2\,|\,\tau) \\ Y(\sqrt{3}, -\sqrt{3}, 0\,|\,\tau) \end{pmatrix}$$

$S_3$ doublet of weight 2 modular forms

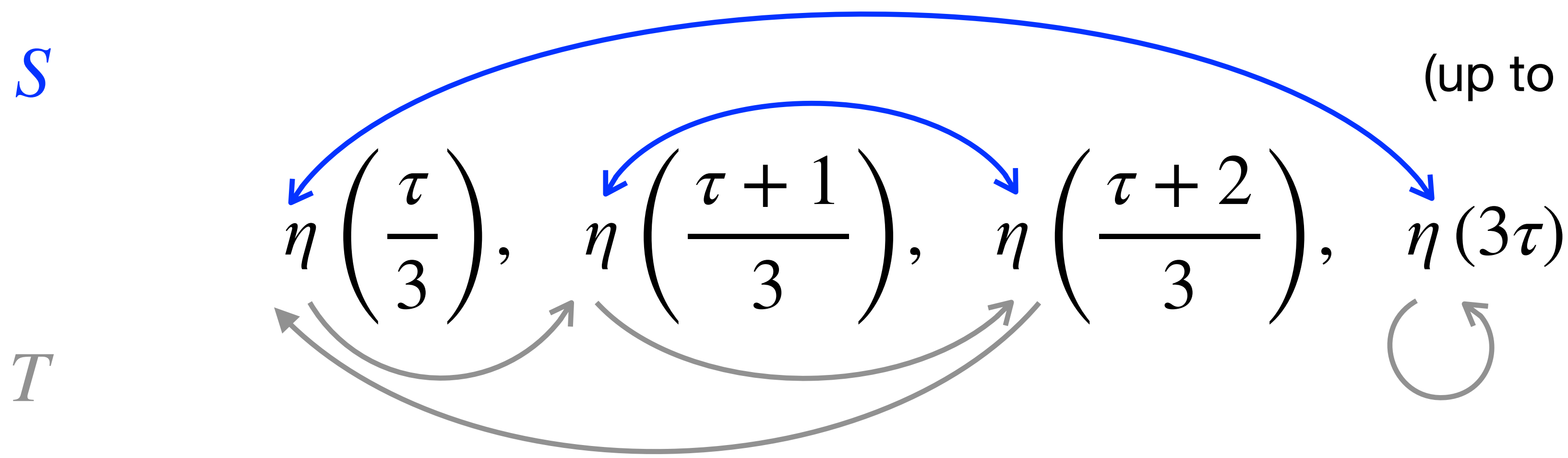
Kobayashi, Tanaka, Tatsuishi, 1803.10391

# Modular forms of level 3 and weight 2

**Level *N* = 3** $\left(\Gamma_3 \simeq A_4 : \quad S^2 = (ST)^3 = T^3 = I\right)$

| N\k | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 3 | 1 | **3** | 5 | 7 |

(up to multiplicative factors)




$$Y_{\mathbf{3}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = c \begin{pmatrix} Y(1,1,1,-3\,|\,\tau) \\ -2\,Y(1,\omega^2,\omega,0\,|\,\tau) \\ -2\,Y(1,\omega,\omega^2,0\,|\,\tau) \end{pmatrix} \qquad \omega = e^{2\pi i/3}$$

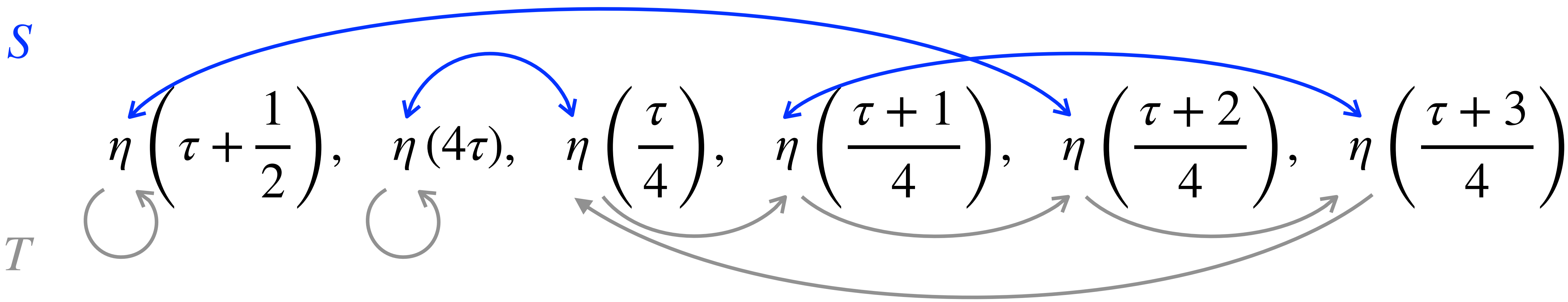
$A_4$ triplet of weight 2 modular forms

Feruglio, 1706.08749

# Modular forms of level 4 and weight 2

**Level $N$ = 4** $\left(\Gamma_4 \simeq S_4: \quad S^2 = (ST)^3 = T^4 = I\right)$

| N\k | 0 | 2 | 4 | 6 |
|---|---|---|---|---|
| 4 | 1 | **5** | 9 | 13 |




$$Y_{\mathbf{2}}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix} = c \begin{pmatrix} Y(1,1,\omega,\omega^2,\omega,\omega^2\,|\,\tau) \\ Y(1,1,\omega^2,\omega,\omega^2,\omega\,|\,\tau) \end{pmatrix}$$

$$Y_{\mathbf{3}'}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix} = c \begin{pmatrix} Y(1,-1,-1,-1,1,1\,|\,\tau) \\ Y(1,-1,-\omega^2,-\omega,\omega^2,\omega\,|\,\tau) \\ Y(1,-1,-\omega,-\omega^2,\omega,\omega^2\,|\,\tau) \end{pmatrix}$$

$S_4$ doublet and triplet $(\mathbf{3}')$ of weight 2 modular forms

Penedo, Petcov, 1806.11040