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Nguyen Thi Kim Ha

Contents

List of Abbreviations

SM	Standard Model
CC	Charged current
NC	Neutral current
PMNS matrix	Pontecorvo-Maki-Nakagawa-Sakata matrix
NO	Normal Ordering
IO	Inverted Ordering

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Introduction

The last decade has seen a revolution in neutrino physics. The establishment of a non-vanishing neutrino mass in neutrino oscillation experiments is one of the major achievements. Neutrinos are neutral leptons which are difficult to be detected, due to the fact that they interact very weakly with other matter. These tiny elementary particles travel through space at a speed close to the speed of light. There exist three types of neutrinos: electron neutrinos ν_e, muon neutrinos ν_μ and tau neutrinos ν_τ. These classifications are referred to as neutrinos's "flavors", and they may oscillate from one flavor to another. This phenomenon was first observed in 1998 by the Japanese Super-Kamiokande experiment and some others like Sudbury Neutrino Observatory (SNO), in which muon neutrinos generated in the atmosphere were found to "disappear", presumably turning into tau neutrinos.

Neutrino oscillation occurs when neutrinos have mass and non-zero mixing. Neutrino mixing is governed by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix which relates the mass eigenstates to the flavor eigenstates. The PMNS mixing matrix is constructed as the product of three independent rotations (a unitary matrix with three mixing angles and one phase). From there, the numbers are often shown in a graphical form called the unitary triangle, giving rise to CP violation (which corresponds to the vertex of the triangle), thus explaining the subtle differences between matter and anti-matter. The area of the triangle is a measurement of the amount of CP violation caused by the weak force. This CP violation partly explains why we live in a matter-dominated universe rather than one full of antimatter or radiation.

The thesis is organized as follows:

- Chapter 1: The Standard Model is introduced as the best description of the elementary particle at present. In this Model, neutrino mass is zero. However, the neutrino oscillations are detected by experiments, indicating that neutrino has mass and this is the only experimental evidence of physics beyond the description of the Standard Model so far.

- Chapter 2: The neutrino oscillation theory is introduced and the unitary mixing matrix is studied in detail. The unitary triangle representation is presented. Furthermore, the possibility of a non-unitary neutrino mixing matrix is introduced and the derived oscillation probability is shown.

- Chapter 3: The numerical calculation for presenting the unitary triangle with the current neutrino landscape is shown. A simple model for estimation of the uncertainty for the triangle vertex is introduced. Lastly, based on the predicted precision of oscillation parameters with the future neutrino experiments such as Hyper-Kamiokande and DUNE, we show the expectations for the unitary triangle representation in the future.

Chapter 1

Neutrino in Standard Model

1.1 Introduction to Standard Model

In particle physics, the elementary particles or fundamental particles can be divided into two groups: fermions (quarks, leptons, anti-quarks and anti-leptons) which generate matter and anti-matter in the Universe and bosons, including gauge bosons (which are force carriers that mediate interactions among fermions) and scalar bosons. Each fermion has its anti-fermion.

The Standard Model (SM) of particle physics explains how the basic building blocks of matter interact, governed by three of the four known fundamental forces (the electromagnetic, weak, and strong interactions, and not including the gravitational forces) in the Universe, as well as classifying all known elementary particles.

In the SM, electroweak interactions (including electromagnetism and the weak interactions) are combined with quantum chromodynamics. This theory is a gauge theory, which means that the fermions interact with each others by exchanging vector bosons. The electroweak $SU(2) \times U(1)$ section is called the Glashow - Weinberg - Salam model. The local symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$ with the subscripts C, L and Y denote color, left-handed charity and weak hypercharge, respectively. The Lagrangian of these bosons does not change under the $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge transformations, so such bosons are called gauge bosons. The bosons in the Standard Model are
- Photon, intermediate particles in electromagnetic interactions
- W and Z boson, intermediate particles in weak interactions
- 8 gluons, intermediate particles in strong interactions
- Higgs bosons, which create the masses of quarks and leptons (electron, muon and tau)
- Graviton is the hypothetical elementary particle that mediates the force of gravity,

but it is not included in the SM.

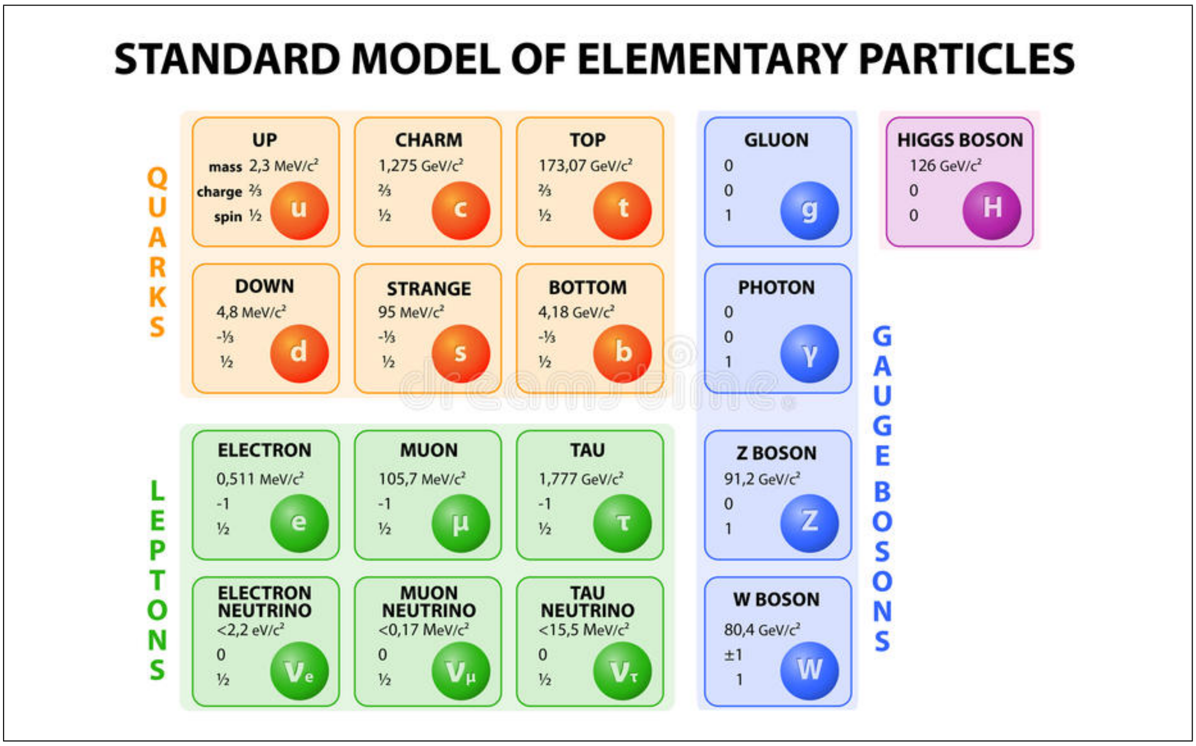

Figure 1.1: Elementary particles included in the Standard Model.

The standard model group corresponds to product of three groups, $SU(3) \times SU(2) \times U(1)$ where $SU(3)$ is the gauge group for strong interaction, $SU(2)$ is the gauge group describing the weak isospin and $U(1)$ describes the hypercharge.

The elementary particles are arranged as doublets for chiral left-handed fields and singlets for right-handed fields in the form [9]

$$\begin{pmatrix} u \\ d' \end{pmatrix}_L \quad \begin{pmatrix} c \\ s' \end{pmatrix}_L \quad \begin{pmatrix} t \\ b' \end{pmatrix}_L \quad \begin{pmatrix} e \\ \nu_e \end{pmatrix}_L \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}_L \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}_L$$
$$u_R \quad d_R \quad s_R \quad c_R \quad b_R \quad t_R \quad e_R \quad \mu_R \quad \tau_R.$$

According to the weak interaction, Lagrangian is represented by columns are $SU(2)$ doublets and the right-handed components of the fermion fields $(\nu'_{eR}, e'_{eR}, u'_{eR}, \ldots)$ are $SU(2)$ singlets.

$$\begin{array}{cccccc} \psi_{eL} = \begin{pmatrix} \nu'_{eL} \\ e'_L \end{pmatrix} & , & \psi_{\mu L} = \begin{pmatrix} \nu'_{\mu L} \\ \mu'_L \end{pmatrix} & , & \psi_{\tau L} = \begin{pmatrix} \nu'_{\tau L} \\ \tau'_L \end{pmatrix} \\ \psi_{1L} = \begin{pmatrix} u'_L \\ d'_L \end{pmatrix} & , & \psi_{2L} = \begin{pmatrix} c'_L \\ s'_L \end{pmatrix} & , & \psi_{3L} = \begin{pmatrix} t'_L \\ b'_L \end{pmatrix}. \end{array}$$

The Lagrangian in terms of the fermion fields for three generations, the boson fields and the Higgs doublet is shown below [17]

$$
\begin{aligned}
\mathcal{L} = {} & i \sum_{\alpha=e,\mu,\tau} \bar{l}_{\alpha L} \not{D} l_{\alpha L} + i \sum_{\alpha=1,2,3} \bar{q}_{\alpha L} \not{D} q_{\alpha L} \\
& + i \sum_{\alpha=e,\mu,\tau} \bar{l}_{\alpha R} \not{D} l_{\alpha R} + i \sum_{\alpha=d,s,b} \bar{d}_{\alpha R} \not{D} d_{\alpha R} + i \sum_{\alpha=u,c,t} \bar{u}_{\alpha R} \not{D} u_{\alpha R} \\
& - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
& + (D_\mu \phi)^\dagger (D_\mu \phi) - V(\phi) \\
& - \sum_{\alpha=e,\mu,\tau} (Y'^{l}_{\alpha\beta}\, \bar{l}_{\alpha L}\, \phi\, l_{\beta R} + Y'^{l}_{\alpha\beta}\, \bar{l}_{\beta R}\, \phi^\dagger\, l_{\alpha L}) \\
& - \sum_{\alpha=1,2,3} \sum_{\beta=d,s,b} (Y'^{d}_{\alpha\beta}\, \bar{q}_{\alpha L}\, \phi\, d_{\beta R} + Y'^{d}_{\alpha\beta}\, \bar{d}_{\beta R}\, \phi^\dagger\, q_{\alpha L}) \\
& - \sum_{\alpha=1,2,3} \sum_{\beta=u,c,t} (Y'^{u}_{\alpha\beta}\, \bar{q}_{\alpha L}\, \tilde{\phi}\, u_{\beta R} + Y'^{u}_{\alpha\beta}\, \bar{u}_{\beta R}\, \tilde{\phi}^\dagger\, q_{\alpha L}),
\end{aligned}
\tag{1.1}
$$

where
- The first two lines in Eq.(1.1) are the kinetic terms and the gauge couplings for SM fermions
- The third line is the kinetic and self-coupling terms for the gauge bosons
- The fourth line is the kinetic term and the potential for the SM Higgs, ϕ is the Higgs doublet and $\tilde{\phi} = i\tau_2\phi^*$
- The final three lines are the Higgs-fermion Yukawa couplings which generate the fermions masses and quark mixing. $Y'^{l}_{\alpha\beta}, Y'^{d}_{\alpha\beta}, Y'^{u}_{\alpha\beta}$ are the Yukawa couplings of leptons, d-like quarks and u-like quarks, respectively.

The covariant derivative

$$
\not{D} = D_\mu \gamma^\mu = (\partial_\mu + i\frac{g}{2} A_\mu \tau_\alpha + i\frac{g^{'}}{2} Y B_\mu)\gamma^\mu, \tag{1.2}
$$

with
- Three differential operators $A^\mu = (A^\mu_1, A^\mu_2, A^\mu_3)$ are the three gauge bosons for the $SU(2)$ group,
- B_μ is the gauge boson associated with the generator Y of the group $U(1)$,
- g and $g^{'}$ are two independent coupling constants associated with group $SU(2)$ and $U(1)$,
- $\tau_\alpha = (\tau_1, \tau_2, \tau_3)$ are the three Pauli matrices.

Under local gauge invariance, field derivatives transform into [7]

$$
\begin{aligned}
\not{D} l_{\alpha R} &= (\partial_\mu - i g^{'} B_\mu) l_{\alpha R}, \\
\not{D} l_{\alpha L} &= (\partial_\mu + i\frac{g^{'}}{2} B_\mu - i\frac{g}{2}\tau_\alpha A_\mu) l_{\alpha L}.
\end{aligned}
\tag{1.3}
$$

The SM fermions show are lepton doublets $l_{\alpha L}$, the quark doublets $q_{\alpha L}$, the lepton singlets $l_{\alpha R}$ and quark singlets $u_{\alpha R}$, $d_{\alpha R}$ where subscripts L and R refer to left and right chirality, respectively.

The values of weak isospin I_3, hypercharge Y and electric charge Q of the fermion doublets and singlets are described in Table 1.1 [17].

Table 1.1: The eigenvalues of the weak isospin I, of the third component I_3, of the hypercharge Y and of the charge $Q = I_3 + Y/2$ of the fermion doublets of the 1^{st} generation and singlets of the 1^{st} family of the SM (e, u, d).

	I	I_3	Y	Q
Lepton doublet l_L $L = (\nu_{eL}\ e_L)^T$	1/2	$(1/2, -1/2)^T$	-1	$(0, -1)^T$
Lepton singlet l_R	0	0	-2	-1
Quark doublet q_L $q_L = (u_L\ d_L)^T$	1/2	$(1/2, -1/2)^T$	1/3	$(2/3, -1/3)^T$
Quark singlet u_R	0	0	4/3	2/3
Quark singlet d_R	0	0	-2/3	-1/3
Higgs doublet $\phi = (\phi^\dagger\ \phi)^T$	1/2	$(1/2, -1/2)^T$	1	$(1, 0)^T$

The charge of neutrino being 0 implies that neutrinos do not interact in the electromagnetic regime; they just participate in the weak interaction.

1.2 Neutrino and weak interaction

The SM Lagrangian of $SU(2) \times U(1)$ is re-written to be suitable considering the charge of lepton doublets and singlets as shown in Table 1.1 (we just consider one-generation case: e, u, d) [17]

$$\begin{aligned} \mathcal{L}_I = & -\frac{1}{2}\bar{l}_L(g\not{A}\tau - g^{'}\not{B})l_L - \frac{1}{2}\overline{q}_L(g\not{A}\tau + \frac{1}{3}g^{'}\not{B})q_L \\ & + g^{'}\bar{l}_R\not{B}l_R - \frac{2}{3}g^{'}\overline{u}_R\not{B}u_R + \frac{1}{3}g^{'}\overline{d}_R\not{B}d_R. \end{aligned} \tag{1.4}$$

The leptonic parts in Eq.(1.4) are contained in the first and third term (one-generation of leptons: e and ν_e)

$$\mathcal{L}_{I,l} = -\frac{1}{2}(\overline{\nu}_{eL}\ \overline{e}_L)\begin{pmatrix} g\not{A}_3 - g^{'}\not{B} & g(\not{A}_1 - i\not{A}_2) \\ g(\not{A}_1 + i\not{A}_2) & -g\not{A}_3 - g^{'}\not{B} \end{pmatrix}\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} + g^{'}\overline{e}_R\not{B}e_R. \tag{1.5}$$

In Eq.(1.5), the charged current (CC) and the neutral current (NC) parts can be separated explicitly into

$$\mathcal{L}_{I,l}^{NC} = -\frac{1}{2}\Big[\overline{\nu}_{eL}(g\not{A}_1 - g^{'}\not{B})\nu_{eL} - \overline{e}_L(g\not{A}_3 + g^{'}\not{B})e_L - 2g^{'}\overline{e}_R\not{B}e_R\Big], \tag{1.6}$$

$$\mathcal{L}_{I,l}^{CC} = -\frac{g}{2}\Big[\overline{\nu}_{eL}(\not{A}_1 - i\not{A}_2)e_L + \overline{e}_L(\not{A}_1 + i\not{A}_2)\nu_{eL}\Big]. \tag{1.7}$$

Charged current interaction
The charged vector boson field as shown here

$$W^\mu = \frac{A_1^\mu - iA_2^\mu}{\sqrt{2}}, \tag{1.8}$$

annihilates W^+ bosons and creates W^- bosons.

The charged current of lepton is given by the expression below

$$j_{W,l}^\mu = \overline{\nu}_e\gamma^\nu(1-\gamma^5)e = 2\ \overline{\nu}_e\gamma^\mu e_L. \tag{1.9}$$

As a consquence, we obtain

$$\mathcal{L}_{I,l}^{CC} = -\frac{g}{2\sqrt{2}}\ j_{W,l}^\mu W_\mu + hc. \tag{1.10}$$

Neutral current interaction
The Weinberg angle or weak mixing angle is a parameter in the Weinberg-Salam theory of electroweak interaction, and is denoted by θ_W.

$$\frac{g^{'}}{g} = \tan\theta_W. \tag{1.11}$$

The constants g and $g^{'}$ are related to the charge of electron through the expression

$$e = g\sin\theta_W = g^{'}\cos\theta_W. \tag{1.12}$$

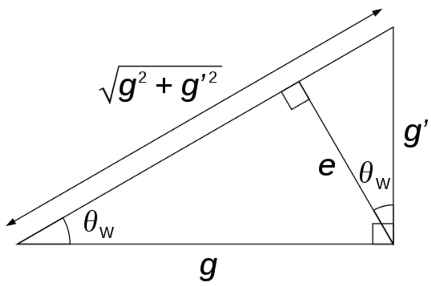

Figure 1.2: Weinberg angle θ_W and relation between couplings g, $g^{'}$ and e

The linear combination of A_3^μ and B^μ are given through θ_W

$$\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B^\mu \\ A_3^\mu \end{pmatrix}, \tag{1.13}$$

$$\begin{aligned} A^\mu &= \sin\theta_W A_3^\mu + \cos\theta_W B^\mu, \\ Z^\mu &= \cos\theta_W A_3^\mu - \sin\theta_W B^\mu, \end{aligned} \tag{1.14}$$

where Z^μ is a vector boson field for weak interactions.

The neutral currents for leptons are given by

$$\begin{aligned} j^\mu_{\gamma,l} &= -\bar{e}\gamma^\mu e, \\ j^\mu_{Z,l} &= 2\ g_L^\nu \bar{\nu}_{eL}\gamma^\mu \nu_{eL} + 2\ g_L^l \bar{e}_L \gamma^\mu e_L + 2\ g_R^l \bar{e}_R \gamma^\mu e_R. \end{aligned} \tag{1.15}$$

The neutral current Lagrangian is given as follow

$$\mathcal{L}^{NC}_{I,l} = \mathcal{L}^Z_{I,l} + \mathcal{L}^\gamma_{I,l} = -\bar{e}\gamma^\mu e - \frac{g}{2\cos\theta_W} j^\mu_{Z,l} Z_\mu. \tag{1.16}$$

From the above, the neutrino interactions in the SM occur via mediation by $W^\pm$ and Z bosons. The CC and NC interactions exchange $W^\pm$ and Z bosons, respectively. In the above section, we assume only the first generation fermions. To extend to the three flavor case, the CC interaction will have to be flavor dependent. Hence, interactions of this kind can be used to label the flavor of neutrinos [17].

1.3 Masslessness of neutrino in Standard Model

Under the SM, neutrinos are predicted to be massless. To explain that, François Englert and Peter Higgs introduced the Higgs mechanism which generates mass terms for massive fields and guarantees that $SU(2) \times U(1)$ local gauge invariance is unbroken. The complex Higgs fields forms an SU(2) × U(1) doublet [9]

$$\phi(x) = \begin{pmatrix} \phi^\dagger \\ \phi^o \end{pmatrix}, \tag{1.17}$$

where
- $\phi^\dagger(x)$ is a charged complex scalar field
- $\phi^o(x)$ is a neutral complex scalar field.

The Lagrangian of the field ϕ as given in Eq.(1.1)

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi), \tag{1.18}$$

where the potential is given by

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2, \tag{1.19}$$

where μ^2 and λ are positive constants.

We define

$$v = \sqrt{-\frac{\mu^2}{\lambda}}, \tag{1.20}$$

and neglecting the irrelevant constant term $v^4/4$, the Higgs potential could be re-written

$$V(\phi) = \lambda\Big(\phi^\dagger \phi - \frac{v^2}{2}\Big)^2. \tag{1.21}$$

The minimum of such a potential is

$$\phi^\dagger \phi = \frac{v^2}{2}. \tag{1.22}$$

There are two components of Higgs: one is neutral and the other is charged. Due to the vacuum being electrically neutral, its state can be chosen to have the following form

$$\phi_{vac} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1.23}$$

We have $I_3(\phi) \neq 0$, $Y(\phi) \neq 0$ and $Q(\phi) = 0$. That means when Higgs boson is at the minimum of the potential, $SU(2)_L \times U(1)_Y$ breaks to $U(1)_Q$. The excitation state of the scalar doublet ϕ around such minimum is given by

$$\phi_{vac} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{1.24}$$

Subsituting Eq.(1.24) into Eq.(1.1), giving us

$$\mathcal{L}_{h,l} = -\frac{v+h}{\sqrt{2}}\, \bar{l}'_{eL} Y'^l l'_{eR} + hc, \tag{1.25}$$

where $\bar{l}'_{eL} = V_L^{l\dagger} l_{eL}$ and $l'_{eR} = V_R^{l\dagger} l_{eR}$.

A complex matrix Y^l can be diagonalized by a bi-unitary transformation, we have

$$Y'^l = V_L^{l\dagger}\; Y^l\; V_R^l = y^l_\alpha \delta_{\alpha\beta}, \quad (\alpha, \beta = e, \mu, \tau) \tag{1.26}$$

where V_L and V_R are unitary matrices, Y^l is a diagonal matrix whose elements are positive.

We can re-write Eq.(1.25) as

$$\mathcal{L}_{h,l} = -\sum_{\alpha=e,\mu,\tau} \frac{y^l_\alpha}{\sqrt{2}}\, v\, \bar{l}_\alpha l_\alpha - \sum_{\alpha=e,\mu,\tau} \frac{y^l_\alpha}{\sqrt{2}}\, h\, \bar{l}_\alpha l_\alpha, \tag{1.27}$$

where $l_\alpha = l_{\alpha L} + l_{\alpha R}$ $(\alpha = e, \mu, \tau)$ are the fields of the charged leptons with defined masses

$$E_e \equiv e, E_\mu \equiv \mu, E_\tau \equiv \tau. \tag{1.28}$$

In conculsion, in the first term, there is only mass term of charged leptons given by

$$m_\alpha = \frac{y^l_\alpha v}{\sqrt{2}}. \tag{1.29}$$

And the second term describes the interaction of the massive neutrino and scalar Higgs bosons. The neutrinos are massless because there are no right-handed neutrinos, therefore the term in Eq.(1.25) vanish.

In the fact, the neutrino oscillation was discovered by the Super-Kamiokande Observatory and the Sudbury Neutrino Observatory and this discovery was recognized with the 2015 Nobel Prize in Physics. We have observed the change of the neutrino flavors through oscillations and it implies that neutrinos are massive. In this chapter, under SM, neutrino are massless. As a experimental consequence, the phenomenon of neutrino oscillations may signal new predictions beyond the SM and that would require modifications to the SM of particle physics.

Chapter 2

Neutrino mixing matrix

There exist three types of neutrinos: electron neutrinos ν_e, muon neutrinos ν_μ and tau neutrinos ν_τ. These classifications are referred to as a neutrinos's "flavors". The Standard Model states that neutrinos are massless and chargeless, and only undergo weak interactions. However, it has been observed that neutrinos can change their flavors during their travel. That is, a neutrino which was generated with a certain flavor might end up having a flavor different from its initial flavor after travelling some distance. For example, ν_e at the source may oscillate into ν_μ over a distance to the detector. A neutrino flavor is determined as a superposition of the mass eigenstates. In terms of neutrino flavors, mass eigenstates are mixed with each other and cannot be determined at the same time. This only happens when at least one mass eigenstates is non-zero. This phenomenon is called *neutrino oscillation*.

2.1 Unitarity of neutrino mixing matrix

2.1.1 Neutrino oscillation theory

Neutrino oscillations are a results of neutrino mixing via: the left-handed flavor neutrino fields. We denote mass eigenstates by Latin indices and flavor eigenstates by Greek indices. If the neutrinos are massive, the weak interaction (or flavor) $\nu_\alpha(\alpha = e, \mu, \tau, \cdots)$ and mass eigenstates $\nu_i(i = 1, 2, 3, \cdots)$ do not coincide, leading to the phenomenon of flavor transition. In general, the left-handed components of the neutrino flavor fields are superpositions of the left-handed components of the neutrino fields with defined mass m_i [12].

$$\nu_{\alpha L} = \sum_{i=1}^{n} V_{\alpha i}\ \nu_{iL}, \qquad (\alpha = e, \tau, \mu) \tag{2.1}$$

where V is a unitary mixing matrix which comes from the diagonalization of the neutrino mass matrix from Eq.(1.26).

The number of massive neutrino fields n can be greater than or equal to 3. If n is larger than 3, it means that there are sterile neutrinos that do not take part in the standard weak interactions. In this section, we do not to consider the sterile neutrinos.

We have the charged lepton current Eq.(1.9) and the neutral lepton current Eq.(1.15) in the mass basis [16]

$$CC : j_\mu^{CC} = 2\sum_{i=1}^{3} \overline{\nu}_{iL} \ \gamma_\mu (V_L^\dagger V_R)_{i\alpha} l_{\alpha L}, \tag{2.2}$$

$$NC : j_\mu^{NC} = \sum_{i=1}^{3} \overline{\nu}_{iL} \ \gamma_\mu \nu_{iL}, \tag{2.3}$$

where $V^\dagger$ and V are diagonalizations of the charged lepton mass matrix in Eq.(1.26). All the mixing matrix does is to define the charged lepton weak eigenstates in terms of the mass eigenstates. Thus, $V_L^\dagger V_R$ is the leptonic mixing matrix. Taking the unphysical phases in $V_L^\dagger V_R$ into account in the charged lepton fields, the mixing matrix is redefined. The charged current becomes

$$j_\mu^{CC} = \sum_{i=1}^{3} \overline{\nu}_{iL} \ \gamma_\mu U_{i\alpha} l_{\alpha L}, \tag{2.4}$$

where U is called the *Pontecorvo-Maki-Nakagawa-Sakata* (PMNS) mixing matrix.

The relationship between flavor and mass fields become

$$\nu_{\alpha L} = \sum_{i=1}^{3} U_{\alpha i} \nu_{iL}. \qquad (\alpha = e, \tau, \nu) \tag{2.5}$$

Due to small neutrino mass differences, the neutrino flavor eigenstate is represented by a coherent superposition of mass eigenstates [1]. Let us assume that the initial state at t = 0 can be expressed as follows

$$|\nu_\alpha(0)\rangle = \sum_{i=1}^{3} U_{\alpha i}^* |\nu_i(0)\rangle. \tag{2.6}$$

[1]By convention, a field operator creates anti-particles, a anti-field operator creates particles. The relation between the mass basis and the flavor basis is

* Field operator ν	$\nu_\alpha = U_{\alpha i}\nu_i$	$\overline{\nu}_\alpha = U_{\alpha i}^*\overline{\nu}_i$
* One-particle states $\|\nu\rangle$	$\|\nu_\alpha\rangle = U_{\alpha i}^*\|\nu_i\rangle$	$\|\overline{\nu}_\alpha\rangle = U_{\alpha i}\|\overline{\nu}_i\rangle$
* Wave functions $\nu(x) = \langle x\|\nu\rangle$	$\nu_\alpha(x) = U_{\alpha i}^*\nu_i(x)$	$\overline{\nu}_\alpha(x) = U_{\alpha i}^*\overline{\nu}_i(x)$

At some arbitrary time t, the flavor eigenstates will have the following form

$$|\nu_\alpha(t)\rangle = \sum_{i=1}^{3} U^*_{\alpha i}|\nu_i(t)\rangle. \tag{2.7}$$

In vacuum, the mass eigenstates are given by

$$|\nu_i(t)\rangle = T_i(t,T)|\nu_i(0)\rangle, \tag{2.8}$$

such that

$$|\nu_\alpha(t)\rangle = \sum_{i=1}^{3} U^*_{\alpha i}\ T_i(t,L)|\nu_i(t)\rangle, \tag{2.9}$$

where $T_i(t,L)$ gives the evolution of the mass eigenstate. The state ν_i has rest mass m_i and it obeys the Schrodinger equation

$$i\frac{\partial}{\partial \tau_i}|\nu_i(\tau_i)\rangle = m_i|\nu_i(\tau_i)\rangle \tag{2.10}$$

$$\Leftrightarrow |\nu_i(\tau_i)\rangle = e^{-im_i\tau_i}|\nu_i(0)\rangle, \tag{2.11}$$

where τ_i is the proper time for ν_i to travel from the neutrino source to the detector. From (2.8), we have

$$T_i(t,L) = e^{-im_i\tau_i}. \tag{2.12}$$

Note that the energy of neutrino state is not defined since each component have an energy $E = \sqrt{m_i^2 + p_i^2}$. Under Lorentz invariance, the phases $m_i\tau_i$ in the ν_i propagator $T_i(t,L)$ is given by the following form

$$m_i\tau_i = E_i t - p_i L. \tag{2.13}$$

We have $|\nu_i\rangle$ which is the state of neutrino with mass m_i, momentum p_i, and energy E propagating from the neutrino source to the detector

$$p_i = \sqrt{E^2 - m_i^2} = E\sqrt{1 - \frac{m_i^2}{E^2}} \cong E\Big(1 - \frac{1}{2}\frac{m_i^2}{E^2}\Big) \cong E - \frac{m_i^2}{2E}. \tag{2.14}$$

Substituting (2.13) into (2.12)

$$m_i\tau_i \cong E(t-L) + \frac{m_i^2}{2E}L. \tag{2.15}$$

In natural units, we have $L = ct = t$. Thus, the phases $E(t-L)$ equals 0. So,

$$T_i(t,L) = exp\Big(-i\frac{m_i^2}{2E}L\Big). \tag{2.16}$$

The probability amplitude of flavor transition $\nu_\alpha \rightarrow \nu_\beta$ after time t, distance L in vacuum and with energy E is given by

$$\begin{aligned}
A_{\nu_\alpha\rightarrow\nu_\beta} &= \langle\nu_\beta|\nu_\alpha(t,L)\rangle = \sum_{i,j}\Big(U_{\beta j}\langle\nu_j|\Big)\Big(T_i(t,L)U^*_{\alpha i}|\nu_i\rangle\Big)\\
&= \sum_{i,j} U_{\beta j}U^*_{\alpha i}\ T_i(t,L)\langle\nu_j|\nu_i\rangle\\
&= \sum_{i,j} U_{\beta j}U^*_{\alpha i}\ T_i(t,L)\delta_{ij}\\
&= \sum_{i} U_{\beta i}U^*_{\alpha i}\ e^{-i\frac{m_i^2}{2E}L},
\end{aligned} \tag{2.17}$$

where
- $U_{\alpha i}$ is the amplitude to find the neutrino mass eigenstate $|\nu_i\rangle$ in the state of the flavor neutrino $|\nu_\alpha\rangle$
- $U_{\beta i}$ is the amplitude to find the neutrino flavor eigenstate $|\nu_\beta\rangle$ in the mass state of neutrino $|\nu_i\rangle$.
Hence, the probability amptitude $\langle\nu_i(0)|\nu_i(\tau_i)\rangle$ it refers to the probability of finding the initial state $|\nu_i(0)\rangle$ in the state $|\nu_i(\tau_i)\rangle$.

The oscillation probability is obtained squaring the probability amplitude for $\nu_\alpha \rightarrow \nu_\beta$ (Derivation of this probability is given in Appendix B.1)

$$\begin{aligned}
P_{\nu_\alpha\rightarrow\nu_\beta} &= |A_{\nu_\alpha\rightarrow\nu_\beta}|^2\\
&= \delta_{\alpha\beta} - 4\sum_{j>i} Re(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2\Big(\frac{\Delta m^2_{ij}L}{4E}\Big)\\
&\qquad - 2\sum_{j>i} Im(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\Big(\frac{\Delta m^2_{ij}L}{2E}\Big),
\end{aligned} \tag{2.18}$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$.

Anti-neutrino
The anti-neutrino flavor states $|\bar{\nu}_\alpha\rangle$ are described by the anti-neutrino mass eigenstates $|\bar{\nu}_i\rangle$

$$|\overline{\nu}_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i}|\overline{\nu}_i\rangle. \tag{2.19}$$

Notice that the probability amplitude for anti-neutrino differs from the corresponding amplitude for neutrino, we have to exchange $U \rightarrow U^*$ (taking the complex conjugate of the product matrix). The anti-neutrino transition probability in vacuum [16]

$$P_{\nu_\alpha\rightarrow\nu_\beta}(U) = P_{\nu_\alpha\rightarrow\nu_\beta}(U^*). \tag{2.20}$$

The probability amplitude of oscillation $\overline{\nu}_\alpha \to \overline{\nu}_\beta$ can be expressed as

$$P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} = \delta_{\alpha\beta} - 4 \sum_{j>i} Re(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin^2\left(\frac{\Delta m^2_{ij} L}{4E}\right) + 2 \sum_{j>i} Im(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}) \sin\left(\frac{\Delta m^2_{ij} L}{2E}\right). \quad (2.21)$$

It can be seen from the such expression that if neutrinos are massless, all the mass squared splittings Δm^2_{ij} will vanish, then $P_{\nu_\alpha \to \nu_\beta} = \delta_{\alpha\beta}$ ($P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} = \delta_{\alpha\beta}$). Hence, the oscillation in vacuum of ν_α into a different flavor ν_β implies that neutrinos have non-zero masses. Note that at least two neutrino flavors have to be non-degenerated so that $P_{\nu_\alpha \to \nu_\beta} \neq \delta_{\alpha\beta}$ ($P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} \neq \delta_{\alpha\beta}$). Finally, the probability of flavor change in vacuum is a periodic function of L/E, so the phenomenon of flavor change became known as "neutrino oscillations".

The unitarity of U implies that the following probability conservation relation

$$P_{\nu_\alpha \to \nu_\alpha} = 1 - \sum_{\beta \neq \alpha} P_{\nu_\alpha \to \nu_\beta}. \quad (2.22)$$

The probabilities (2.18) and (2.21) are called the *transition probability* and the *survival probability* respectively, and they can be expressed in the following way

$$P_{\nu_\alpha \to \nu_\alpha} = P_{\overline{\nu}_\alpha \to \overline{\nu}_\alpha} = 1 - 4 \sum_{j>i} \left[|U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2 (\frac{\Delta m^2_{ij} L}{4E}) \right]. \quad (2.23)$$

Converting to natural units, we have $1eV^{-1}$ of length $= 1.97 \times 10^{-7} m$. The conversion factor in such equation can be expressed for practical purposes

$$\begin{aligned} \frac{\Delta m^2_{ij}[eV^2]\ L[eV^{-1}]}{4E[eV]} &= \frac{\Delta m^2_{ij}[eV^2]\ L[m]}{4 \times 1.97 \times 10^{-7} E[eV]} \\ &= 1.269 \times 10^6 \frac{\Delta m^2_{ij}[eV^2]\ L[m]}{E[eV]} \\ &= 1.269 \frac{\Delta m^2_{ij}[eV^2]\ L[km]}{E[GeV]}. \end{aligned} \quad (2.24)$$

Two flavor case

One important special case is the one in which we consider only two flavors, due to experiments being analysed under this simple assumption [12]. In this case, we have only two mass eigenstates (ν_1, ν_2) and two mass eigenvalues (m_1, m_2). The mixing matrix is simply given by

$$\begin{pmatrix} |\nu_\alpha\rangle \\ |\nu_\beta\rangle \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}. \quad (2.25)$$

We denote $c = \cos\theta$ and $s = \sin\theta$, and θ is the mixing angle. At time $t = 0$, the initial state with momentum p_i is given by

$$|\nu(t=0)\rangle = |\nu_\alpha\rangle = c|\nu_1\rangle + s|\nu_2\rangle. \tag{2.26}$$

the two mass components of such neutrinos have energies E_1 and E_2, as given by

$$E_i = \sqrt{p^2 + m_i^2} \cong \ p + \frac{m_i^2}{2p} \cong \ E + \frac{m_i^2}{2E}. \tag{2.27}$$

After time t, the neutrino state becomes

$$|\nu(t)\rangle = ce^{-iE_1 t}|\nu_1\rangle + se^{-iE_2 t}|\nu_2\rangle. \tag{2.28}$$

The probability of finding the neutrino $|\nu_\beta\rangle$ $(\alpha \neq \beta)$ is

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = \left|(-s\langle\nu_1| + c\langle\nu_2|)\ |\nu_\alpha(t)\rangle\right|^2 \\
&= \left|(-s\langle\nu_1| + c\langle\nu_2|)\ (ce^{-iE_1 t}|\nu_1\rangle + se^{-iE_2 t}|\nu_2\rangle)\right|^2 \\
&= \left| - sce^{-iE_1 t}\langle\nu_1|\nu_1\rangle + cse^{-iE_2 t}\langle\nu_2|\nu_2\rangle\right|^2 \\
&= c^2 s^2 \left|e^{-iE_2 t} - e^{-iE_1 t}\right|^2 \\
&= \cos^2\theta \sin^2\theta (e^{-iE_2 t} - e^{-iE_1 t})(e^{iE_2 t} - e^{iE_1 t}) \\
&= \cos^2\theta \sin^2\theta (2 - e^{-i(E_1 - E_2)t} - e^{i(E_1 - E_2)t}) \\
&= \cos^2\theta \sin^2\theta \Big(2 - \cos[(E_1 - E_2)t] + i\sin[(E_1 - E_2)t] \\
&\qquad\qquad - \cos[(E_1 - E_2)t] - i\sin[(E_1 - E_2)t]\Big) \\
&= 2\cos^2\theta \sin^2\theta \Big[1 - \cos[(E_2 - E_1)t]\Big] \\
&= \frac{\sin^2 2\theta}{2}\Big[1 - \cos\big(\frac{m_2^2 - m_1^2}{2E}t\big)\Big] = \frac{\sin^2 2\theta}{2}\Big[1 - \cos\big(\frac{\Delta m^2}{2E}t\big)\Big] \\
&= \frac{\sin^2 2\theta}{2}\ 2\sin^2\Big(\frac{\Delta m^2}{4E}t\Big) \\
&= \sin^2 2\theta\ \ \sin^2\Big(\frac{\Delta m^2}{4E}t\Big).
\end{aligned} \tag{2.29}$$

For relativistic neutrinos, one can also approximate $L \sim t$

$$P(\nu_\alpha \to \nu_\beta) = sin^2 2\theta\ \ \sin^2\Big(\frac{\Delta m^2}{4E}L\Big). \tag{2.30}$$

The transition probability is determined by the factor L/E. The amplitude of oscillation depends on mixing angle θ and the frequency depends upon the mass-squared difference Δm^2.

Note that, if $\theta = 0$ (no mixing) or $\Delta m^2 = 0$ (same or zero mass), neutrinos will not oscillate. Also, flipping the sign of Δm^2 does not affect the probability. We cannot distinguish the ordering of two mass states by measuring the oscillation probability [17].

2.1.2 The PMNS matrix

Up-to-date data on the parameters of neutrino oscillations have been collected assuming 3 flavor neutrino mixing in vacuum. The currently available data on neutrino oscillations come from sources, such as the solar (ν_e), atmospheric (ν_μ and $\overline{\nu}_\mu$), reactor ($\overline{\nu}_e$), accelerator (ν_μ and $\overline{\nu}_\mu$) neutrinos. In the case of having three standard neutrino flavor families, the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix can be parameterized by three mixing angles (θ_{12}, θ_{23}, and θ_{13}), and one Dirac CP phase (δ_{CP}) or two Majorana phases (α_1 and α_2) depending on whether massive neutrinos are Dirac or Majorana particles as well as two independent mass-squared splitting, for example Δm^2_{21} and Δm^2_{31}.

The form of the matrix U can be obtained using sequential rotations around the axes spanned by massive neutrino states m_1, m_2, m_3 [8].

$$U = R_{23}R_{13}R_{12},$$

with R_{ij} describing rotations in the i-j plane by an angle of θ_{ij} with additional phases [23].

$$\begin{aligned}
U_{PMNS} &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix}}_{\text{Atmospheric}} \underbrace{\begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & 1 \end{bmatrix}}_{\text{Reactor}} \underbrace{\begin{bmatrix} c_{12} & s_2 & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Solar}} \underbrace{\begin{bmatrix} e^{i\phi_1/2} & 0 & 0 \\ 0 & e^{i\phi_2/2} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Majorana phases}} \\
&= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{bmatrix} \times diag\ (1,\ e^{i\frac{\phi_1}{2}},\ e^{i\frac{\phi_2}{2}})
\end{aligned} \tag{2.31}$$

where
- $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$,
- θ_{ij} is mixing angle which θ_{12}, θ_{13} and θ_{23} are the solar, reactor and atmospheric mixing angles, respectively,
- δ, ϕ_1, ϕ_2 are CP-violating phases. In the experiments, we only detect CP violation through δ since Majorana phases is not sensitive and plays no role in oscillation physics.

For two flavor case, it is clear that there is no CP violation because the mixing matrix has no δ_{CP} - CP violating phases.

2.1.3 Unitary triangle representation of PMNS matrix

The neutrino oscillation probabilites can be generally formulated as follows

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{2.32}$$

U_{PMNS} is square matrix, thus it has the following property

$$U^{\dagger}U = \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.33}$$

Similarly,

$$UU^{\dagger} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} U_{e1}^* & U_{\mu 1}^* & U_{\tau 1}^* \\ U_{e2}^* & U_{\mu 2}^* & U_{\tau 2}^* \\ U_{e3}^* & U_{\mu 3}^* & U_{\tau 3}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.34}$$

So, PMNS leptonic mixing matrix $U \in C^{m \times m}$ is unitary (orthogonal in real case)

$$U^{\dagger}U = UU^{\dagger} = 1 \tag{2.35}$$

$$\rightarrow \; U_{ji}^* U_{jk} = U_{ij} U_{kj}^* = \delta_{ik}. \tag{2.36}$$

The unitarity of mixing matrices ensures that probabilities sum to 1. The probability of the oscillating neutrino having electron, muon or tau flavor should be equal to 1. Eq.(2.33) gives the normalization of elements matrix, as follows

$$\begin{aligned} &U_{e1}U_{e1}^* + U_{e2}U_{e2}^* + U_{e3}U_{e3}^* = 1 \\ &U_{\mu 1}U_{\mu 1}^* + U_{\mu 2}U_{\mu 2}^* + U_{\mu 3}U_{\mu 3}^* = 1 \\ &U_{\tau 1}U_{\tau 1}^* + U_{\tau 2}U_{\tau 2}^* + U_{\tau 3}U_{\tau 3}^* = 1. \end{aligned} \tag{2.37}$$

Eq.(2.34) and Eq.(2.33) yield six unitary triangles, respectively
$*$ Three Dirac triangles are governed by the orthogonality relations [19]

$$\begin{aligned} \Delta_e :\; & U_{\mu 1}U_{\tau 1}^* + U_{\mu 2}U_{\tau 2}^* + U_{\mu 3}U_{\tau 3}^* = 0 \\ \Delta_\mu :\; & U_{\tau 1}U_{e1}^* + U_{\tau 2}U_{e2}^* + U_{\tau 3}U_{e3}^* = 0 \\ \Delta_3 :\; & U_{e1}U_{\mu 1}^* + U_{e2}U_{\mu 2}^* + U_{e3}U_{\mu 3}^* = 0. \end{aligned} \tag{2.38}$$

$*$ Three Majorana triangles are dictated by the orthogonality relations [19]

$$\begin{aligned} \Delta_1 :\; & U_{e2}^*U_{e3} + U_{\mu 2}^*U_{\mu 3} + U_{\tau 2}^*U_{\tau 3} = 0 \\ \Delta_2 :\; & U_{e1}^*U_{e3} + U_{\mu 1}^*U_{\mu 3} + U_{\tau 1}^*U_{\tau 3} = 0 \\ \Delta_3 :\; & U_{e1}^*U_{e2} + U_{\mu 1}^*U_{\mu 2} + U_{\tau 1}^*U_{\tau 2} = 0. \end{aligned} \tag{2.39}$$

Unitarity requires

$$U_{e1}^*U_{e2} + U_{\mu 1}^*U_{\mu 2} + U_{\tau 1}^*U_{\tau 2} = 0 \tag{2.40}$$
$$\rightarrow \quad \vec{a} \quad + \quad \vec{b} \quad + \quad \vec{c} \qquad = 0.$$

Such three vectors can be used to define a triangle in two dimension coordinates

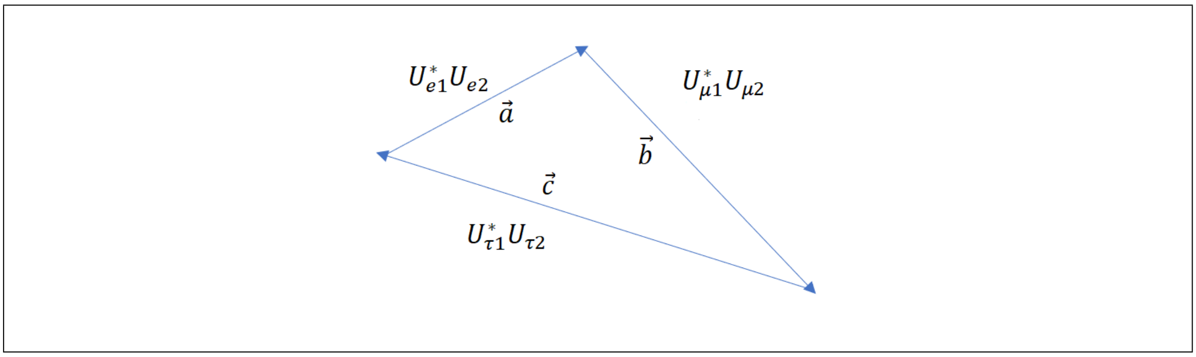

Figure 2.1: The triangle is used to represent Eq.(2.40).

The sides can be defined as follows

$$(a, b, c) = (|U^*_{e1}U_{e2}|, |U^*_{\mu1}U_{\mu2}|, |U^*_{\tau1}U_{\tau2}|). \tag{2.41}$$

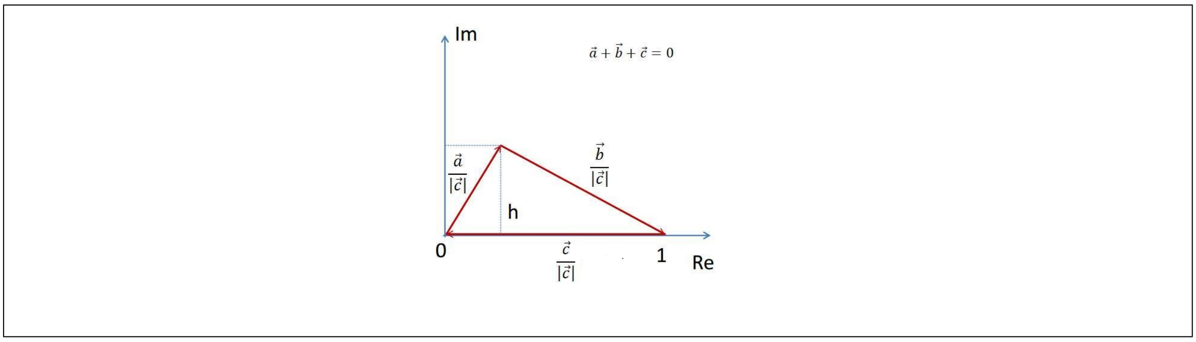

Figure 2.2: The normalized unitary triangle.

We normalize the area of unitary triangle to get the relation

$$\begin{aligned} &\vec{a}+\vec{b}+\vec{c}=0 \\ \Leftrightarrow &\frac{\vec{a}}{|\vec{c}|}+\frac{\vec{b}}{|\vec{c}|}+\vec{1}=0. \end{aligned} \tag{2.42}$$

The height of the triangle

$$h = Im\left[\frac{a}{c}\right] = Im\left[\frac{b}{c}\right]. \tag{2.43}$$

The non-normalized area

$$S = \frac{1}{2}.Im\left[\frac{a}{c}\right].|c|^2. \tag{2.44}$$

Utilizing the following property given complex number z

$$Im(z) = \frac{z-\bar{z}}{2i}. \tag{2.45}$$

So, the normalized area

$$S^{'} = \frac{1}{2}h.1 = \frac{1}{2}.\ Im\Big[\frac{a}{c}\Big].1 = \frac{1}{4i}\Big(\frac{a}{c} - \frac{a^*}{c^*}\Big). \tag{2.46}$$

We infer that

$$\begin{aligned} S = S^{'}|c|^2 &= \frac{1}{4i}\Big(\frac{a}{c} - \frac{a^*}{c^*}\Big)c^*c \\ &= \frac{1}{4i}(ac^* - a^*c) = \frac{1}{2}Im[ac^*]. \end{aligned} \tag{2.47}$$

Similarly, we have

$$S = \frac{1}{2}Im[ac^*] = \frac{1}{2}Im[cb^*] = \frac{1}{2}Im[ba^*]. \tag{2.48}$$

Combining (2.40) and (2.48)

$$\begin{aligned} S &= \frac{1}{2}Im\Big[U^*_{e1}U_{e2}U_{\tau 1}U^*_{\tau 2}\Big] \\ &= \frac{1}{2}Im\Big[U^*_{\tau 1}U_{\tau 2}U_{\mu 1}U^*_{\mu 2}\Big] \\ &= \frac{1}{2}Im\Big[U^*_{\mu 1}U_{\mu 2}U_{e1}U^*_{e2}\Big]. \end{aligned} \tag{2.49}$$

From (2.21) and (2.18), we have

$$P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} - P_{\nu_\alpha \to \nu_\beta} = 4\sum_{i>j} Im[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}] \sin\Big(\frac{\Delta m^2_{ij}L}{2E}\Big). \tag{2.50}$$

Jarlskog invariant is defined

$$J = Im\Big[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}\Big]. \qquad (\alpha \neq \beta, i \neq j) \tag{2.51}$$

Comparing with the area of the triangle, as we have done before

$$J = 2S. \tag{2.52}$$

In conclusion, these six unitary triangles have the same area which is equal to half of the Jarlskog's invariant. Eq.(2.40) could be compacted in the following form

$$\sum_k U_{ik}U^*_{jk} = 0. \tag{2.53}$$

With any fixed and different i and j corresponding to each k, these numbers form the sides of the triangle in the complex plane. There are six choices of i and j (three independent) and hence six such triangles, each of which is called *unitary triangle.*

Their shapes are very different but they all have the same area, which is related to CP violation phase. The area vanishes for the specific parameters in the Standard Model for which there would be no CP violation. In other words, to have CP violating effects, J must be non-zero . Since the three sides of the triangles are open to direct experiment, as are the three angles, a class of tests on the Standard Model is to check that the triangle closes.

The angles in triangle representation
For the flavor neutrino oscillations, we consider the Dirac triangles

$$U_{\alpha 1}U^*_{\beta 1} + U_{\alpha 2}U^*_{\beta 2} + U_{\alpha 3}U^*_{\beta 3} = 0. \tag{2.54}$$

Let α, β, γ be the three triangle angles, each angle is the relative phases between two adjacent sides [1]

$$\begin{aligned} \alpha &= arg\Big(- \frac{U_{\alpha 3}U^*_{\beta 3}}{U_{\alpha 2}U^*_{\beta 2}}\Big) \\ \beta &= arg\Big(- \frac{U_{\alpha 1}U^*_{\beta 1}}{U_{\alpha 3}U^*_{\beta 3}}\Big) \\ \gamma &= arg\Big(- \frac{U_{\alpha 2}U^*_{\beta 2}}{U_{\alpha 1}U^*_{\beta 1}}\Big). \end{aligned} \tag{2.55}$$

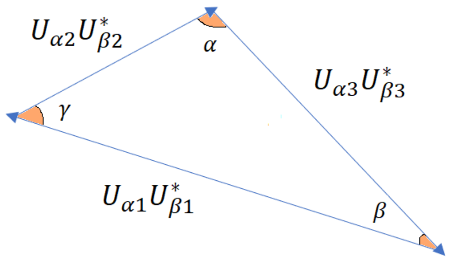

Figure 2.3: The unitary triangle denotes three angles.

2.2 Non-unitarity of neutrino mixing matrix

In this previous section, we describe the neutrino mixing using a unitary matrix and this basically describe our data well (with some exception), but there is still some place for a non-unitary mixing matrix in neutrino oscillation. The phenomenological impact of these new physics in neutrino oscillation measurement facilities are [6]
- The new mass scale is kinematically accessible in meson decays, the sterile states will be produced in the neutrino beam
- The extra neutrinos are too heavy to be produced, the PMNS matrix will become a non-unitarity matrix.

A generic feature of many Beyond the SM scenarios is the inclusion of one or more new massive fermionic singlets, uncharged under the Standard Model (SM) gauge group

$SU(3)_C \times SU(2)_L \times U(1)_Y$. If n extra right-handed neutrinos are added to the SM Lagrangian, the full mixing matrix is as shown [6]

$$U_{PMNS}^{extended} = \begin{pmatrix} N & \Theta \\ R & S \end{pmatrix}, \tag{2.56}$$

where
- N is the 3×3 active-light sub block (PMNS matrix)
- Θ shows $3 \times n$ sub-block consists of the mixing between active and heavy states
- R and S sub-block represent the mixing of sterile states including light and heavy states, respectively.

These new sterile states mix with the SM neutrinos and therefore, the true mixing matrix is enlarged from the 3×3 U_{PMNS} matrix to a $n \times n$ matrix [14]

$$U_{PMNS}^{Extended} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} & \cdots & U_{en} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & \cdots & U_{\mu n} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & \cdots & U_{\tau n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{s_n 1} & U_{s_n 2} & U_{s_n 3} & \cdots & U_{s_n n} \end{bmatrix} \tag{2.57}$$

2.2.1 The effective Lagrangian

In this section, we use the Einstein summation convention, in which we sum over the repeated indices.

The effective Lagrangian in the mass basis
The leptonic unitary matrix U_{PMNS} is replaced by a non-unitary one. Firstly, we analyze the effective low-energy Lagrangian in the mass basis [3]

$$\begin{aligned}\mathcal{L}_{eff} = \frac{1}{2}(\overline{\nu}_i i\partial\nu_i - \overline{\nu^c}_i m_i \nu_i + hc) - \frac{g}{2\sqrt{2}}(W_\mu^+ \bar{l}_\alpha \gamma_\mu (1-\gamma_5) N_{\alpha i}\nu_i + hc) \\ - \frac{g}{2\cos\theta_W}(Z_\mu \overline{\nu}_i \gamma^\mu (1-\gamma_5)(N^\dagger N)_{ij}\nu_j + hc) + \cdots ,\end{aligned} \tag{2.58}$$

where
- $N_{\alpha i}$ is a general non-unitary matrix,
- The first term is a kinetic term, the second one is neutrino mass term, the third one is charged current interaction and the last one is to modify the neutral current coupling [3].
Notice that a non-unitary matrix will include a Majorana mass term, albeit the analysed data would make no difference to consider neutrinos of the Dirac type.

For the non-unitary matrix, the mass eigenstates and flavor eigenstates are *not orthonormal*. Indeed, N connects the quantum fields in the mass basis with the flavor basis where the weak couplings are diagonal

$$\nu_\alpha = \sum_i N_{\alpha i}\nu_i. \tag{2.59}$$

The mass eigenstates become orthonormal

$$\langle \nu_i|\nu_j\rangle = \delta_{ij}. \tag{2.60}$$

The relation between mass state and flavor state is [3]

$$|\nu_\alpha\rangle = \frac{1}{\sqrt{(NN^\dagger)_{\alpha\alpha}}}\sum N^*_{\alpha i}|\nu_i\rangle = \sum_i \tilde{N}^*_{\alpha i}|\nu_i\rangle, \tag{2.61}$$

where the normalization factor has been absorbed into the definition of $\tilde{N}$ on the right-hand side. The flavor eigenstates are not orthogonal [2]

$$\langle \nu_\beta|\nu_\alpha\rangle = (\tilde{N}\tilde{N}^\dagger)_{\beta\alpha} \neq \delta_{\alpha\beta}. \tag{2.62}$$

The effective Lagrangian in the flavor basis

Re-writing the Lagrangian in Eq.(2.58) in the flavor basis using Eq.(2.59)

$$\begin{aligned}\nu_\alpha = N_{\alpha i}\nu_i \rightarrow \nu_i = (N_{\alpha i})^{-1}\nu_\alpha \rightarrow \overline{\nu}_i = (N^{-1}_{\alpha i})^*\overline{\nu}_\alpha,\\ \nu_\alpha = N_{\alpha i}\nu_i \rightarrow \overline{\nu}_\alpha = N^*_{\alpha i}\overline{\nu_i} \rightarrow \overline{\nu}_i = (N^*_{\alpha i})^{-1}\overline{\nu}_\alpha,\end{aligned}$$

It follows that [3]

$$\begin{aligned}\mathcal{L}_{eff} = &\frac{1}{2}\Big(i\overline{\nu}_\alpha\partial(NN^\dagger)^{-1}_{\alpha\beta}\nu_\beta - \overline{\nu^c}_\alpha[(N^{-1})^t m N^{-1}]_{\alpha\beta}\nu_\beta + hc\Big)\\ &- \frac{g}{2\sqrt{2}}[W^+_\mu \overline{l}_\alpha\gamma_\mu(1-\gamma_5)\nu_\alpha + hc]\\ &- \frac{g}{2\cos\theta_W}[Z_\mu\overline{\nu}_\alpha\gamma^\mu(1-\gamma_5)\nu_\alpha + hc] + \cdots,\end{aligned} \tag{2.63}$$

where $m = diag(m_1, m_2, m_3)$.

In this basis, the weak couplings (CC coupling and NC coupling) are diagonal. In contrast, both the kinetic (the first term) and neutrino mass term (the second term) are non-diagnonal, so they do not have the canonical form. To define the kinematic properties, we transform the Lagrangian to ensure that canonical kinetic and mass term are obtained in the neutrino fields. That means the neutrino mass term is diagonalized by a unitary transformation and the kinetic term is both diagonalized and normalized, to obtain canonical kinetic energies. At least two normalizations of neutrino fields differ, a non-unitary weak mixing matrix, connecting the quantum fields in the flavor basis with those in the mass basis in Eq.(2.59) [3].

[2]Notice that the flavor eigenstates are not orthonormal at low-energy. In the corresponding complete high energy - hypothetical theory, it is possible to be orthonormal flavor basis [3].

2.2.2 Oscillation probability

The free Hamiltonian H^{free} describes free neutrino propagation. *Time* evolution of the neutrino mass eigenstates is governed by Schrodinger equation [3]

$$i\frac{d}{dt}|\nu_i\rangle = H_{free}|\nu_i\rangle. \tag{2.64}$$

Because of the orthogonality of the mass basis, we have

$$\langle\nu_j|H_{free}|\nu_i\rangle = \delta_{ij}E_i, \tag{2.65}$$

where E_i are the eigenvalues.
Using the completeness relation in the mass basis $\sum_j |\nu_j\rangle\langle\nu_j| = 1$, we have

$$i\frac{d}{dt}|\nu_i\rangle = \sum_j |\nu_j\rangle\langle\nu_j|H_{free}|\nu_i\rangle = E_i|\nu_i\rangle, \tag{2.66}$$

which is the usual time propagation for free states.

Since the flavor basis is not orthonormal and have no completeness relation, as $\sum_a |\nu_\alpha\rangle\langle\nu_\alpha| \neq 1$. The time evolution reads

$$\begin{aligned} i\frac{d}{dt}|\nu_\alpha\rangle &= \sum_j |\nu_j\rangle\langle\nu_j|H_{free}|\nu_\alpha\rangle = \sum_{i,j} |\nu_j\rangle\langle\nu_j|H_{free}\tilde{N}^*_{\alpha i}|\nu_i\rangle = \sum_{i,j} \tilde{N}^*_{\alpha i}\delta_{ij}E_i|\nu_j\rangle \\ &= \sum_{i,j} \tilde{N}^*_{\alpha i}(\tilde{N}^*_{\beta j})^{-1}\delta_{ij}E_i|\nu_\beta\rangle = \sum_i \tilde{N}^*_{\alpha i}(\tilde{N}^*_{\beta i})^{-1}E_i|\nu_\beta\rangle \qquad (i=j) \\ &= \sum_\beta [\tilde{N}^*E(\tilde{N}^*)^{-1}]_{\alpha\beta}|\nu_\beta\rangle. \end{aligned} \tag{2.67}$$

where $E = diag(E_1, E_2, E_3)$.

The combination $(\tilde{N}^*E(\tilde{N}^*)^{-1})$ is not Hermitian, albeit the free Hamiltonian is Hermitian. It means the evolution of the flavor bra states $\langle\nu_\alpha|$ differs from the flavor kets $|\nu_\alpha\rangle$, but both have the same probability equation [3].

Neutrino flavor evolution in *space* is governed by Schrodinger equation in the following way [3]

$$\begin{aligned} i\frac{d}{dL}|\nu_i\rangle &= H_{free}|\nu_i\rangle \\ \to |\nu_i(L)\rangle &= e^{-i\phi}|\nu_i(0)\rangle. \end{aligned} \tag{2.68}$$

where L is the distance traveled by the neutrino.
We infer that

$$|\nu_\alpha(L)\rangle = \sum_i \tilde{N}^*_{\alpha i}|\nu_i(L)\rangle = \sum_i \tilde{N}^*_{\alpha i}e^{-i\phi}\ |\nu_i(0)\rangle \tag{2.69}$$

The probability amplitude of flavor transition $\nu_\alpha \to \nu_\beta$ after distance L in vacuum is given by

$$\begin{aligned} A_{\nu_\alpha \to \nu_\beta} = \langle \nu_\beta | \nu_\alpha(L) \rangle &= \sum_i \langle \nu_j | \tilde{N}_{\beta j} \tilde{N}^*_{\alpha i} e^{-i\phi} | \nu_i(0) \rangle \\ &= \sum_i \tilde{N}_{\beta i} e^{-i\phi} \tilde{N}^*_{\alpha i} \\ &= \frac{1}{\sqrt{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}}} \sum_i N^*_{\alpha i} e^{-i\phi} N_{\beta i}. \end{aligned} \tag{2.70}$$

The oscillation probability becomes

$$P_{\nu_\alpha \to \nu_\beta} = |A_{\nu_\alpha \to \nu_\beta}|^2 = \frac{1}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} \Big| \sum_i N^*_{\alpha i} e^{-i\phi} N_{\beta i} \Big|^2. \tag{2.71}$$

We consider this term as the oscillation probability in vacuum (derivation of this equation is given in Appendix B.2). Hence, we have the oscillation probabililty for a neutrino of initial flavor α and energy E_ν to transit into a neutrino of flavor β after a distance L as follows

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta} = \frac{1}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} \Big\{ &\Big| \sum_i N^*_{\alpha i} N_{\beta i} \Big|^2 \\ &- 4 \sum_{j>i} Re[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin^2(\frac{\Delta m^2_{ji} L}{4E}) \\ &- 2 \sum_{j>i} Im[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin(\frac{\Delta m^2_{ji} L}{2E}) \Big\}, \end{aligned} \tag{2.72}$$

where now, without assuming unitarity, the leading term is not a fuction of $\dfrac{\Delta m^2 L}{E}$.

Eq.(2.72) has the term $Im[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}]$ called *Jarlskog invariants* of CP violation. Non-unitarity have 9 different Jarlskog invariants, and unitary triangles deformed to become polygons [18].

The consequence of Eq.(2.72) is that the non-unitarity of N generates a "zero distance" effect. For instance, a flavor transition at the source before oscillations can take place. Indeed, for $L = 0$, it reads [13]

$$P_{\nu_\alpha \to \nu_\beta}(E, L = 0) = \frac{|(NN^\dagger)_{\alpha\beta}|^2}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} \neq 0. \tag{2.73}$$

So, an effect can be tested in near detectors (with unitarity of the U matrix, $P_{\nu_\alpha \to \nu_\beta} = \delta_{\alpha\beta}$ at $L = 0$). Nevertheless, because of non-unitarity, the oscillation probability does not sum up to a total probability of 100%. It also means that the total oscillation probability is not necessarily equal to 1 or 0 in neutrino disappearance and appearance experiments, respectively.

Chapter 3

Calculations and results

In this chapter, we will use the ROOT - data analysis framework to calculate the value of the unitary matrix and the draw unitary triangles with current data. We will perform the following:

- From four measuared parameters ($\sin\theta_{13}$, $\sin\theta_{23}$, $\sin\theta_{23}$ and δ_{CP}), we calculate the values of each element in the PMNS matrix. And then, using the ROOT framework to draw unitary triangle.
- In the unitary triangle, coordinate of C vertex might be changed due to uncertainties of the four parameters, especially δ_{CP}.
- Decreasing uncertainty of the δ_{CP} in T2K and DUNE experiments in the future, we re-draw unitary triangle and then consider how the colored-region change.

3.1 The unitary triangle with our current neutrino landscape

The current best-fit values from NuFit.org [22] are given in following table. The numbers in the second and third column are assuming Normal Ordering or Inverted Ordering (Appendix A) respectively and the numbers given are relative to the respective global minimums. The table 3.1 contains not only the best-fit values but also includes 1σ uncertainty.

Table 3.1: Three-flavor oscillation parameters from fit to global data as of November 2017.

	Normal Ordering	Inverted Ordering
	best fit $\pm 1\sigma$	best fit $\pm 1\sigma$
$\sin^2\theta_{12}$	$0.307^{+0.013}_{-0.012}$	$0.307^{+0.013}_{-0.012}$
$\sin^2\theta_{23}$	$0.538^{+0.038}_{-0.069}$	$0.554^{+0.023}_{-0.033}$
$\sin^2\theta_{13}$	$0.02206^{+0.00075}_{-0.00075}$	$0.02227^{+0.00074}_{-0.00074}$
$\delta_{CP}/^o$	234^{+43}_{-31}	278^{+26}_{-29}

We represent the current neutrino data with the unitary triangle. Each element of matrix in Eq.(2.32) corresponds to each element of matrix in Eq.(2.31)

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{3.1}$$

$$\begin{aligned} s_{12}^2 + c_{12}^2 &= \sin^2\theta_{12} + \cos^2\theta_{12} = 1 \\ s_{23}^2 + c_{23}^2 &= \sin^2\theta_{23} + \cos^2\theta_{23} = 1 \\ s_{13}^2 + c_{13}^2 &= \sin^2\theta_{13} + \cos^2\theta_{13} = 1 \end{aligned} \tag{3.2}$$

Both the normal hierarchy and the inverted hierarchy are calculated independently.

The values of PMNS matrix are calculated as

$$U_{normal} = \begin{pmatrix} 0.823 & 0.548 & -0.087 + 0.120 * i \\ -0.323 + 0.073 * i & 0.601 + 0.049 * i & 0.725 \\ 0.456 + 0.068 * i & -0.575 + 0.045 * i & 0.672 \end{pmatrix}$$

$$U_{inverted} = \begin{pmatrix} 0.823 & 0.548 & 0.0206 + 0.148 * i \\ -0.387 + 0.091 * i & 0.554 + 0.060 * i & 0.729 \\ 0.397 + 0.083 * i & -0.622 + 0.055 * i & 0.668 \end{pmatrix}$$

From PMNS matrix, there are six triangles having the same area, so that means we just choose one of six others

$$\Delta_2 : U_{e1}U_{e3}^* + U_{\mu 1}U_{\mu 3}^* + U_{\tau 1}U_{\tau 3}^* = 0 \tag{3.3}$$

Since the matrix U is unitary, any given pair of rows or columns can be used to define a triangle in the complex plane. In such a figure, we obtain the triangle corresponding to the unitarity conditions on the first and third columns. The unitary triangle for normal hierarchy and for inverted hierarchy are shown

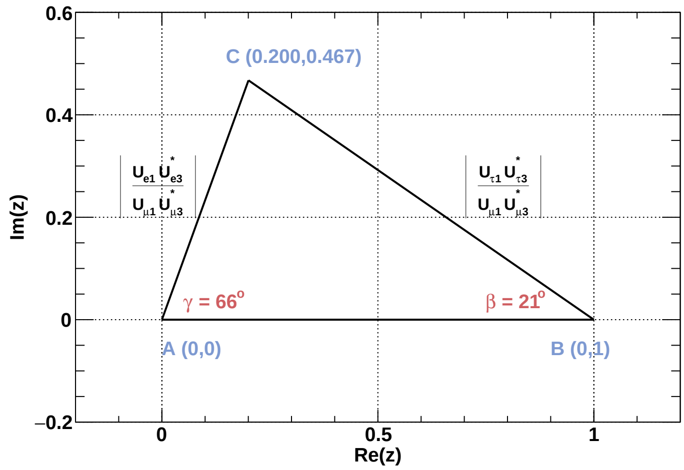

Figure 3.1: Normal hierarchy.

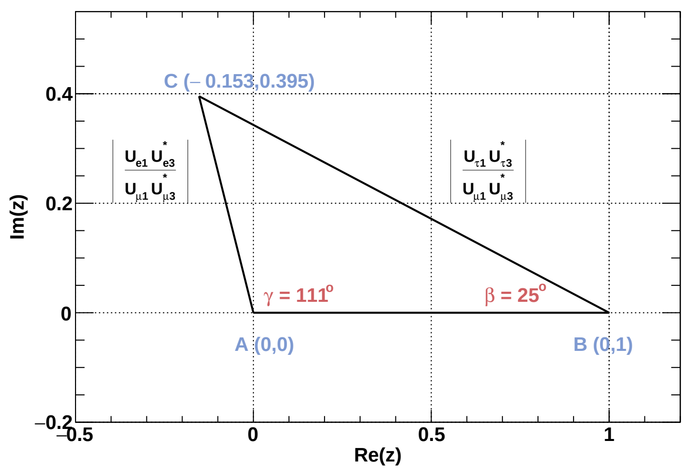

Figure 3.2: Inverted hierarchy.

3.2 Simple model for estimating uncertainty

The goal is calculate the confidence interal for the vertex C in each unitary triangle. Theoretically, it is possible to take into account the uncertainties of coordinates of the C vertex in the triangle. However, it is complicated when calculating the complex numbers from CP phase.

We keep two fixed points (A(0,0) and B(0,1)) and change C vertex with changing four parameters on the uncertainties scale. The simplest way to handle uncertainty is to assume Gaussian distribution for each parameter (a kind of Bayesian approach). Each variable (three angles and one CP phase) is given randomly and assumed to obey a Gaussian distribution.

The 1σ ranges of the magnitude of the elements of the three-flavor leptonic mixing matrix is defined under the assumption of the matrix U being unitary. Each value corresponds to a global minimum and global maximum.

$$|U| = \begin{pmatrix} 0.815 \to 0.831 & 0.537 \to 0.559 & 0.146 \to 0.151 \\ 0.320 \to 0.393 & 0.536 \to 0.662 & 0.678 \to 0.747 \\ 0.425 \to 0.456 & 0.552 \to 0.632 & 0.647 \to 0.721 \end{pmatrix} \tag{3.4}$$

After scaling and rotating the triangle, such that two of its vertices always coincide with (0,0) and (1,0), we plot the 1σ allowed regions of the third vertex. The contours for normal (above) and inverted (bottom) ordering are defined according to the common global minimum [22]. Figure (3.3) and Figure (3.4) show the unitary triangle representation with the uncertainty of the C vertex's coordinates [11].

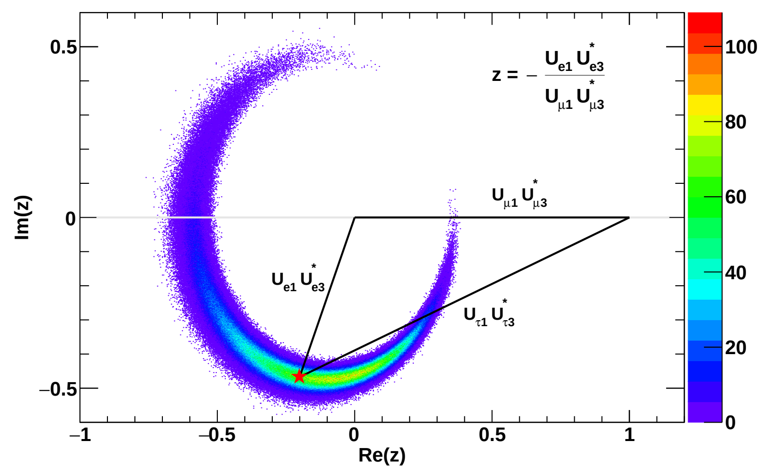

Figure 3.3: Normal hierarchy with 1σ uncertainties for 4 oscillation parameters.

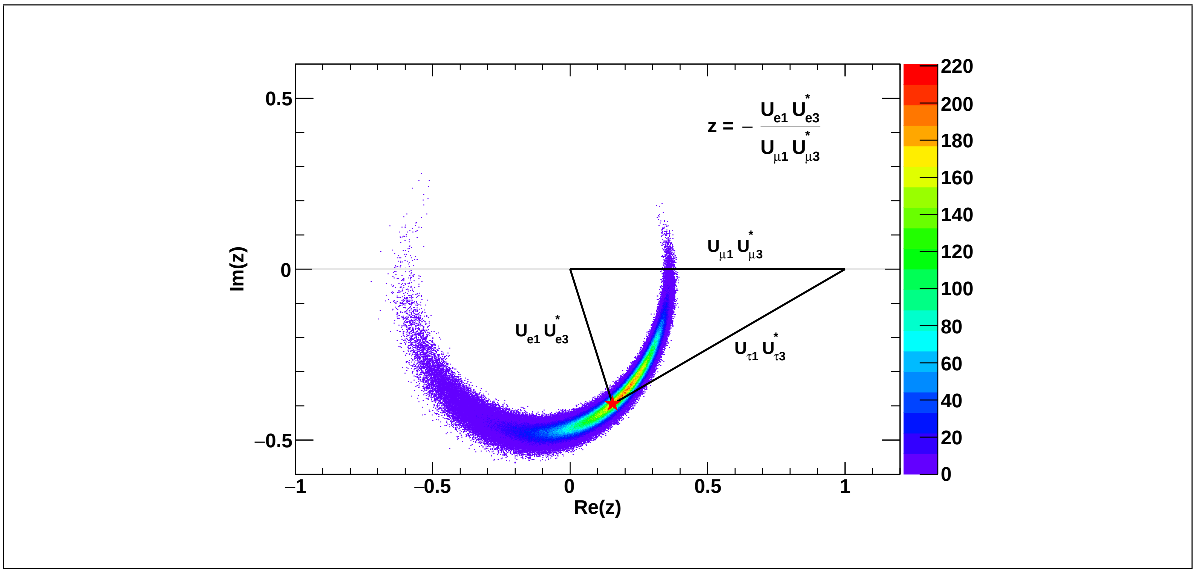

Figure 3.4: Inverted hierarchy with 1σ uncertainties for 4 oscillation parameters.

3.3 Unitary test with future neutrino experiments

Today, there are large-scale experiments that have been conducted to measure neutrinos: T2K, NOvA, MINOS and MINOS+. Furthermore, the status and prospects of future long-based neutrino experiments are promising, with experiments *Hyper-Kamiokande* and *DUNE*.

Hyper-Kamiokande experiment

The Hyper-Kamiokande (HK) detector is to be the third generation water Cherenkov detector which will be hosted in the Tochibora mine, about 295 km away from the J-PARC proton accelerator research complex in Tokai, Japan. It is designed to include two half megaton tanks equipped with ultra high sensitivity photosensors and is bigger than its predecessor, Super-Kamiokande (SK). The Hyper-Kamiokande detector is both a "microscope," used to observe elementary particles, and also a "telescope" for observing the Sun and supernovas, using neutrinos[21]. Hyper-Kamiokande will be able to measure the highest precision of the leptonic CP violation that could explain the baryon asymmetry in the Universe. The atmospheric neutrinos will allow us to determine the neutrino mass ordering and be able to precisely test the three-flavor neutrino oscillation paradigm and search for new phenomena.[17].

DUNE experiment

The Deep Underground Neutrino Experiment (DUNE) is a cutting-edge, international experiment for neutrino science and proton decay studies. DUNE will consist of two neutrino detectors placed in the path of an intense neutrino beam. One detector will record particle interactions near the source of the beam, at the Fermi National Accelerator Laboratory in Batavia, Illinois. A second, much larger, detector will be installed more than a kilometer underground at the Sanford Underground Research Laboratory in Lead, South Dakota - 1300 kilometers downstream of the source. These detectors will enable us to search for new subatomic phenomena and potentially transform our understanding of neutrinos and their role in the universe.[20].

In the future, the experiments will be done at Hyper-K and DUNE will be able to decrease uncertainties of δ_{CP}.

For the Hyper-K experiment, the uncertainty of δ_{CP} is decreased to $\pm 23^o$

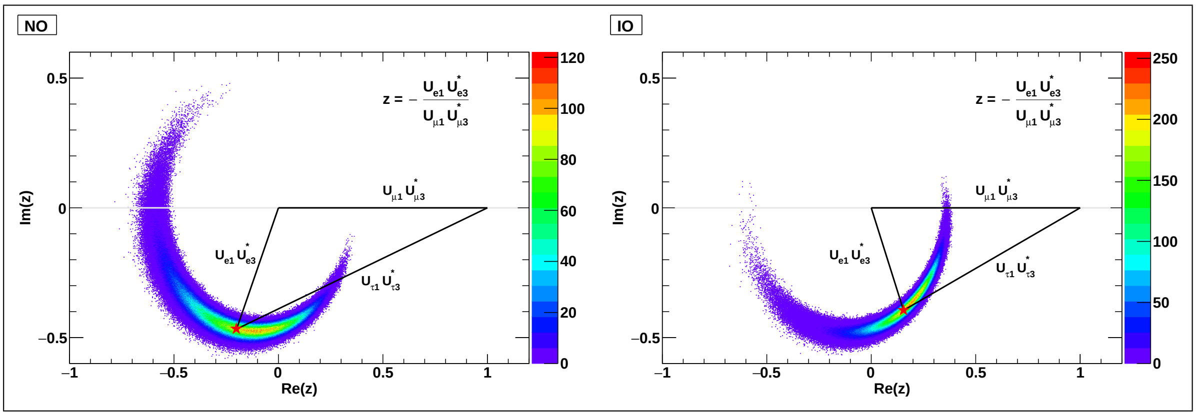

Figure 3.5: Leptonic unitarity triangle for uncertianty of $\delta_{CP} = \pm 23^o$

For combining expected data from Hyper-K and DUNE, the uncertainty value of the δ_{CP} becomes $\pm 15^o$

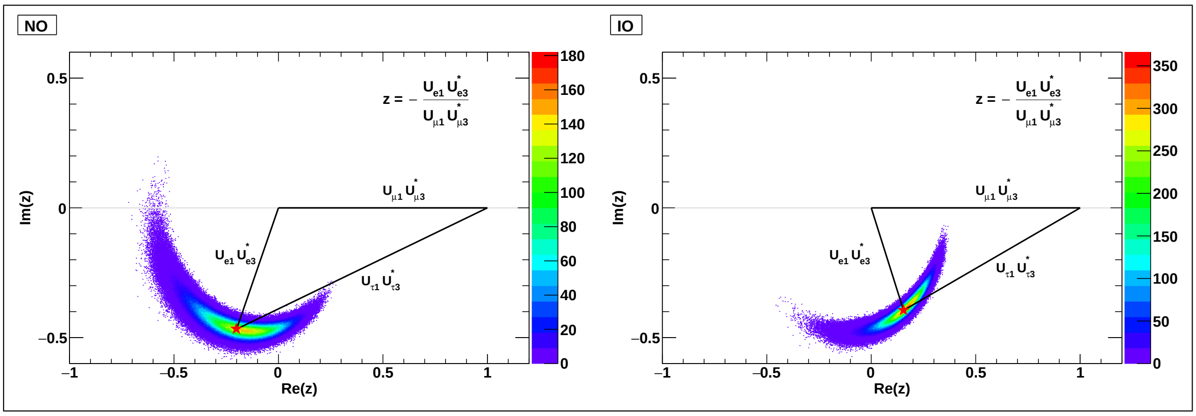

Figure 3.6: Leptonic unitarity triangle for uncertianty of $\delta_{CP} = \pm 15^o$

In Figure (3.5) and (3.6), when the uncertainty of δ_{CP} is reduced from 234^{+43}_{-31} and 278^{+26}_{-29} (in this Table 3.1) to $\pm 23^o$ and $\pm 15^o$ for NO and IO, respectively, we see that the area of colored-region is shrinked and is located near the peak of triangle, giving us the precise value of best fit.

Conclusion and Proposition

Conclusion

This thesis has studied basic problems of neutrino oscillation related to unitarity of the mixing matrix. First, we gave an overview of neutrinos in the SM. Neutrino interact via mediating $W^{\pm}$ and Z bosons. In the SM, neutrinos are not right-handed, so neutrinos are massless.

In addition, we also introduced the unitary mixing matrix in leptonic sector. Firstly, from neutrino oscillation theory, we derived probabilities for both oscillation and survival. If $P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} \neq P_{\nu_\alpha \to \nu_\beta}$, it means that neutrino oscillation shows evidence of existence of CP violation. Neutrino oscillation is described by the PMNS matrix including three mixing angles, one Dirac phase and two Majorana phases. To extract oscillation parameters from data, we built the unitary triangle. From the unitary PMNS matrix, we could derive six unitary triangles having the same area. The areas of unitary triangles are proven to be equal to half of Jarlskog's invariant. In other words, the area of the unitary triangle is related to the CP violation phase.

Moreover, we extended the PMNS matrix for sterile neutrinos and heavy neutrinos, from 3×3 matrix to $n \times n$ matrix. It is called a non-unitary matrix. We need to modify the Lagragian in the mass basis and flavor basis by adding in a non-unitary matrix instead of unitary one. Similar to the unitary matrix, we derived the oscillation probability.

Finally, we derived the results based on current data from global fit framework for ν oscillation [22]. The inputs for numerical calculation are put into ROOT, and outputs for the values of the PMNS matrix elements are derived. From the derived values, we can draw two unitary triangles corresponding to the normal and inverted hierarchies.

Each oscillation parameter has its own uncertainty, so for each one, we generate one million points which is varied around the best-fit value as a Gaussian distribution and then follow the same procedure as the above section. Each set of value of the oscillation parameter uncertainties gives a point C, the set of C points then form a colored region which has a cresent shape as we can see in the figure.

In the future, with more statistics, Hyper-K experiment will reduce the uncertainty of δ_{CP} down to 23^o, which will result in an uncertainty of 15^o when combined with the result of DUNE. The larger statistics will also make the colored region smaller and closer to the best-fit value.

Currently, all measurements of the unitarity triangle are consistent with the peak lying somewhere within the colored-region of uncertainties. By improving the current measurements and performing new ones, the size of this allowed region will be reduced to measure the position of the vertex ever more precisely. The absence of CP violation implies a flat triangle, which means $Im(z) = 0$.

Especially, a precise study of CP asymmetry in the lepton sector is one of the major goals of Hyper-K. The existence of CP violation is one of necessary conditions to explain the matter-antimatter asymmetry of the Universe.

Proposition

After studying some aspects of the unitarity of matrix and the unitary triangle, we realized that some further works should be done to have a better calculation and also to extend this subject.

First, this thesis implemented Bayesian approach in which the C vertex is Gaussian generated. However, this results in the best-fit value not lying at the center of the generated 3σ region. Thus, to cross-check the obtained result, we will try another method such as Chi-squared test.

Second, due to the time limit of the internship, we investigated only the effect of δ_{CP} on the uncertainty area. In future work, we will include the effect of mixing angles (s_{12}^2, s_{23}^2 and s_{13}^2) into the analysis. The neutrino oscillation experiments are measuring the mixing angles with better and better precision, so we hope that by including the newly measured values, the size of the uncertainty area will be reduced.

Finally, it would be interesting to study the correlation between δ_{CP} and each parameter ($\sin^2\theta_{13}$, $\sin^2\theta_{23}$ and $\sin^2\theta_{12}$) to show the effect of the parameters on δ_{CP}.

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[22] NuFit, `http://wwww.nu-fit.org`

[23] Particle Data Group, `http://pdg.lbl.gov`

Appendix A

Neutrino mass hierarchy

There are three neutrino types and until recently they were thought to be massless. Due to the discovery of neutrino oscillation, it is now known that not only do they have mass, but also that the masses of the three mass eigenstates (m_1, m_2, m_3) are different.

Experiments observing the oscillations of neutrinos produced in the sun have measured the squared difference of the masses m_1, m_2 and $\Delta m_{12}^2 = m_1^2 - m_2^2$, and the squared difference of the masses m_1 and m_3 (Δm_{13}^2) have been determined for oscillations of neutrinos in the Earth's atmosphere. The oscillation experiments just only probe the squared difference of the masses, the absolute values of m_1, m_2, and m_3 as well as the question of whether or not m_2 is heavier than m_3 remain unknown. The latter question is known as the "neutrino mass hierarchy problem." If m_2 is lighter than m_3 ($m_1 < m_2 < m_3$), the hierarchy is said to be "normal ordering" (NO). But if it is heavier ($m_3 < m_1 < m_2$), the hierarchy is called "inverted ordering" (IO) [21].

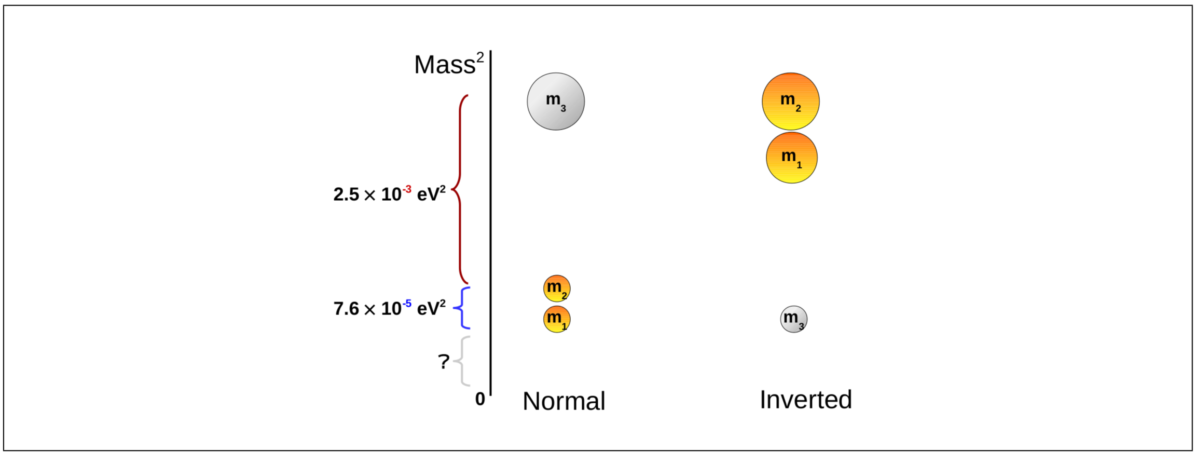

Figure A.1: Neutrino mass hierarchy. Though the value of the individual masses m_1, m_2 and m_3 are unknown, there are two possible orderings.

Appendix B

The oscillation probability formula

B.1 For unitary matrix of neutrino mixing

The flavor eigenstates are related to the mass eigenstates by the 3×3 unitary PMNS matrix

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{B.1}$$

The mass eigenstates can be defined via flavor eigenstates

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{e2} & U^*_{e3} \\ U^*_{\mu 1} & U^*_{\mu 2} & U^*_{\mu 3} \\ U^*_{\tau 1} & U^*_{\tau 2} & U^*_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \tag{B.2}$$

From Eq.(B.1), we have the wave function at time $t = 0$

$$|\Phi(0)\rangle = |\nu_\mu\rangle = U^*_{\mu_1}|\nu_1\rangle + U^*_{\mu_2}|\nu_2\rangle + U^*_{\mu_3}|\nu_3\rangle. \tag{B.3}$$

The time-dependent wave function

$$|\Phi(\vec{x}, t)\rangle = U^*_{\mu_1}|\nu_1\rangle e^{-i\phi_1} + U^*_{\mu_2}|\nu_2\rangle e^{-i\phi_2} + U^*_{\mu_3}|\nu_3\rangle e^{-i\phi_3}, \tag{B.4}$$

expressed in compact form becomes

$$|\nu_\alpha\rangle = \sum_i^3 U^*_{\alpha,i}|\nu_i\rangle e^{-i\phi_i}, \tag{B.5}$$

where $\phi_i = m_i \tau_i = \dfrac{m_i^2}{2E} L$.
From Eq.(B.2) we have

$$\begin{aligned} |\nu_1\rangle &= U_{e1}|\nu_{e1}\rangle + U_{\mu 1}|\nu_{\mu 1}\rangle + U_{\tau 1}|\nu_{\tau 1}\rangle \\ |\nu_2\rangle &= U_{e2}|\nu_{e2}\rangle + U_{\mu 2}|\nu_{\mu 2}\rangle + U_{\tau 2}|\nu_{\tau 2}\rangle \\ |\nu_3\rangle &= U_{e3}|\nu_{e3}\rangle + U_{\mu 3}|\nu_{\mu 3}\rangle + U_{\tau 3}|\nu_{\tau 3}\rangle. \end{aligned}$$

Eq.(B.4) can be written as

$$\begin{aligned}
|\Phi(\vec{x},t)\rangle = &\ U^*_{\mu 1}\Big(U_{e1}|\nu_e\rangle + U_{\mu 1}|\nu_\mu\rangle + U_{\tau 1}|\nu_\tau\rangle\Big)e^{-i\phi_1} \\
&+ U^*_{\mu 2}\Big(U_{e2}|\nu_e\rangle + U_{\mu 2}|\nu_\mu\rangle + U_{\tau 2}|\nu_\tau\rangle\Big)e^{-i\phi_2} \\
&+ U^*_{\mu 3}\Big(U_{e3}|\nu_e\rangle + U_{\mu 3}|\nu_\mu\rangle + U_{\tau 3}|\nu_\tau\rangle\Big)e^{-i\phi_3} \\
= &\ \Big(U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3}\Big)|\nu_e\rangle \\
&+ \Big(U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3}\Big)|\nu_\mu\rangle \\
&+ \Big(U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3}\Big)|\nu_\tau\rangle \\
= &\ c_e|\nu_e\rangle + c_\mu|\nu_\mu\rangle + c_\tau|\nu_\tau\rangle. \qquad \text{(B.6)}
\end{aligned}$$

In compact form

$$|\Phi(\vec{x},t)\rangle = \sum_i U^*_{\alpha i}U_{\beta i}e^{-i\phi_i}|\nu_\beta\rangle. \qquad \text{(B.7)}$$

The oscillation probability from muon neutrino to electron neutrino is

$$\begin{aligned}
P_{\nu_\mu\to\nu_e} &= |\langle\nu_e|\Phi(\vec{x},t)|^2 = c_e c_e^* \\
&= |U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3}|^2. \qquad \text{(B.8)}
\end{aligned}$$

In general, oscillation probability is given by

$$P_{\nu_\alpha\to\nu_\beta} = \Big|\sum_i U^*_{\alpha i}U_{\beta i}e^{-i\phi_i}\Big|^2. \qquad \text{(B.9)}$$

With three neutrino mixing, using the following property of complex number

$$|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2Re|z_1 z_2^* + z_1 z_3^* + z_2 z_3^*|. \qquad \text{(B.10)}$$

Eq.(B.8) can be re-expressed as

$$\begin{aligned}
P_{\nu_\mu\to\nu_e} &= U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3}|^2 \\
&= |U^*_{\mu 1}U_{e1}|^2 + |U^*_{\mu 2}U_{e2}|^2 + |U^*_{\mu 3}U_{e3}|^2 \\
&+ 2Re[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}e^{i(\phi_2-\phi_1)}] \\
&+ 2Re[U^*_{\mu 1}U_{e1}U_{\mu 3}U^*_{e3}e^{i(\phi_3-\phi_1)}] \\
&+ 2Re[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}e^{i(\phi_3-\phi_2)}]. \qquad \text{(B.11)}
\end{aligned}$$

We generlize that to

$$P_{\nu_\alpha\to\nu_\beta} = \sum_i |U^*_{\alpha i}U_{\beta i}|^2 + 2\sum_{j>i} Re[U^*_{\alpha i}U_{\beta i}U^*_{\alpha j}U_{\beta j}e^{i(\phi_j-\phi_i)}]. \qquad \text{(B.12)}$$

The unitary condition

$$\begin{aligned}
&|U^*_{\mu 1}U_{e1} + U^*_{\mu 2}U_{e2} + U^*_{\mu 3}U_{e3}|^2 = 0 \\
\Leftrightarrow &|U^*_{\mu 1}U_{e1}|^2 + |U^*_{\mu 2}U_{e2}|^2 + |U^*_{\mu 3}U_{e3}|^2 + 2Re[U^*_{\mu 1}U_{e1}U^*_{\mu 2}U_{e2} + U^*_{\mu 1}U_{e1}U^*_{\mu 2}U_{e2} + U^*_{\mu 1}U_{e1}U^*_{\mu 2}U_{e2}] = 0 \\
\Rightarrow &\sum_i |U^*_{\alpha i}U_{\beta i}|^2 + 2\sum_{j>i} Re[U^*_{\alpha i}U_{\beta i}U^*_{\alpha j}U_{\beta j}] = \sigma_{\alpha\beta}. \qquad \text{(B.13)}
\end{aligned}$$

This follows from Eq.(B.11) and Eq.(B.13)

$$\begin{aligned} P_{\nu_\mu \to \nu_e} &= 2Re[U^*_{\mu 1} U_{e1} U_{\mu 2} U^*_{e2} (e^{i(\phi_2 - \phi_1)} - 1)] \\ &+ 2Re[U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3} (e^{i(\phi_2 - \phi_1)} - 1)] \\ &+ 2Re[U^*_{\mu 1} U_{e1} U_{\mu 2} U^*_{e2} (e^{i(\phi_2 - \phi_1)} - 1)]. \end{aligned} \tag{B.14}$$

In short,

$$P_{\nu_\alpha \to \nu_\beta} = \sigma_{\alpha\beta} + 2 \sum_{j>i} Re[U^*_{\alpha i} U_{\beta i} U^*_{\alpha j} U_{\beta j} (e^{i(\phi_j - \phi_i)} - 1)]. \tag{B.15}$$

We have

$$\begin{aligned} & Re\Big[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} (e^{i(\phi_j - \phi_i)} - 1)\Big] \\ &= Re\Big[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i} (\cos(\phi_j - \phi_i) - 1 + i \sin(\phi_j - \phi_i))\Big] \\ &= Re\Big\{\Big(Re[U^*_{\alpha i} U_{\beta i} U_{\alpha i} N^*_{\beta i}] + i Im[U^*_{\alpha i} U_{\beta i} U_{\alpha i} N^*_{\beta i}]\Big)\Big(-2\sin^2(\frac{\phi_j - \phi_i}{2}) + i \sin(\phi_j - \phi_i)\Big)\Big\} \\ &= -2Re[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}] \sin^2(\frac{\phi_j - \phi_i}{2}) - Im[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}] \sin(\phi_j - \phi_i). \end{aligned} \tag{B.16}$$

We infer the oscillation probability

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta} = \sigma_{\alpha\beta} &- 4 \sum_{j>i} Re[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}] \sin^2(\frac{\phi_j - \phi_i}{2}) \\ &- 2 \sum_{j>i} Im[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}] \sin(\phi_j - \phi_i). \end{aligned} \tag{B.17}$$

From Eq.(2.12) and Eq.(2.16), we substitute $\phi_j - \phi_i = \dfrac{\Delta m^2_{ij} L}{2E}$

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta} = \sigma_{\alpha\beta} &- 4 \sum_{j>i} Re[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}] \sin^2(\frac{\Delta m^2_{ij} L}{4E}) \\ &- 2 \sum_{j>i} Im[U^*_{\alpha i} U_{\beta i} U_{\alpha i} U^*_{\beta i}] \sin(\frac{\Delta m^2_{ij} L}{2E}). \end{aligned} \tag{B.18}$$

B.2 For non-unitary matrix of neutrino mixing

From Eq.(2.70), we infer the oscillation probability

$$P_{\nu_\alpha \to \nu_\beta} = |A_{\nu_\alpha \to \nu_\beta}|^2 = \frac{1}{(NN^\dagger)_{\alpha\alpha} (NN^\dagger)_{\beta\beta}} \Big| \sum_i N^*_{\alpha i} e^{-i\phi} N_{\beta i} \Big|^2. \tag{B.19}$$

We consider this term as the oscillation probability in unitary matrix

$$
\begin{aligned}
P_{\nu_\mu \to \nu_e} &= \frac{1}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} |N^*_{\mu 1} N_{e1} e^{-i\phi_1} + N^*_{\mu 2} N_{e2} e^{-i\phi_1} + N^*_{\mu 3} N_{e3} e^{-i\phi_1}|^2 \\
&= \frac{1}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} \Big\{ |N^*_{\mu 1} N_{e1}|^2 + |N^*_{\mu 2} N_{e2}|^2 + |N^*_{\mu 3} N_{e3}|^2 \\
&\qquad + 2Re\Big[N^*_{\mu 1} N_{e1} N_{\mu 2} N^*_{e2} \ e^{i(\phi_2 - \phi_1)} \Big] \\
&\qquad + 2Re\Big[N^*_{\mu 1} N_{e1} N_{\mu 3} N^*_{e3} \ e^{i(\phi_3 - \phi_1)} \Big] \\
&\qquad + 2Re\Big[N^*_{\mu 2} N_{e2} N_{\mu 3} N^*_{e3} \ e^{i(\phi_3 - \phi_2)} \Big] \Big\}.
\end{aligned} \tag{B.20}
$$

Omitting this term $((NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta})^{-1}$

$$
\Big| \sum_i N^*_{\alpha i} e^{-i\phi} N_{\beta i} \Big|^2 = \sum_i |N^*_{\alpha i} N_{\beta i}|^2 + 2 \sum_{j>i} Re\Big[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i} e^{i(\phi_j - \phi_i)} \Big]. \tag{B.21}
$$

From the non-unitary condition for three neutrinos, we can derive

$$
\Big| \sum_i N^*_{\alpha i} N_{\beta i} \Big|^2 = \sum_i |N^*_{\alpha i} N_{\beta i}|^2 + 2 \sum_{j>i} Re\Big[N^*_{\alpha i} N_{\beta i} N_{\alpha j} N^*_{\beta j} \Big]. \tag{B.22}
$$

Combining the two equations, Eq.(B.21) and Eq.(B.22)

$$
\begin{aligned}
\Big| \sum_i N^*_{\alpha i} e^{-i\phi} N_{\beta i} \Big|^2 &= \Big| \sum_i N^*_{\alpha i} N_{\beta i} \Big|^2 - 2 \sum_{j>i} Re\Big[N^*_{\alpha i} N_{\beta i} N_{\alpha j} N^*_{\beta j} \Big] \\
&\qquad + 2 \sum_{j>i} Re\Big[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i} e^{i(\phi_j - \phi_i)} \Big] \\
&= \Big| \sum_i N^*_{\alpha i} N_{\beta i} \Big|^2 + 2 \sum_{j>i} Re\Big[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i} (e^{i(\phi_j - \phi_i)} - 1) \Big].
\end{aligned} \tag{B.23}
$$

We have

$$
\begin{aligned}
& Re\Big[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i} (e^{i(\phi_j - \phi_i)} - 1) \Big] \\
&= Re\Big[\Big(N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i} \Big) \Big(\cos(\phi_j - \phi_i) - 1 + i \sin(\phi_j - \phi_i) \Big) \Big] \\
&= Re\Big\{ \Big(Re[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] + i Im[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \Big) \\
&\qquad \Big(-2 \sin^2(\frac{\phi_j - \phi_i}{2}) + i \sin(\phi_j - \phi_i) \Big) \Big\} \\
&= -2Re[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin^2(\frac{\phi_j - \phi_i}{2}) - Im[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin(\phi_j - \phi_i).
\end{aligned} \tag{B.24}
$$

The oscillation probability is given by

$$P_{\nu_\alpha \to \nu_\beta} = \frac{1}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} \Big\{ \Big| \sum_i N^*_{\alpha i} N_{\beta i} \Big|^2 - 4 \sum_{j>i} Re[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin^2(\frac{\phi_j - \phi_i}{2}) - 2 \sum_{j>i} Im[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin(\phi_j - \phi_i) \Big\}. \quad \text{(B.25)}$$

After time t and distance L, the difference in phase of the mass eigenstates are written as

$$\phi_j - \phi_i = m_j \tau_j - m_i \tau_i = \frac{m_j^2}{2E} L - \frac{m_i^2}{2E} L = \frac{\Delta m^2_{ji}}{2E} L. \quad \text{(B.26)}$$

We infer the oscillation probability

$$P_{\nu_\alpha \to \nu_\beta} = \frac{1}{(NN^\dagger)_{\alpha\alpha}(NN^\dagger)_{\beta\beta}} \Big\{ \Big| \sum_i N^*_{\alpha i} N_{\beta i} \Big|^2 - 4 \sum_{j>i} Re[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin^2(\frac{\Delta m^2_{ji} L}{4E}) - 2 \sum_{j>i} Im[N^*_{\alpha i} N_{\beta i} N_{\alpha i} N^*_{\beta i}] \sin(\frac{\Delta m^2_{ji} L}{2E}) \Big\}. \quad \text{(B.27)}$$

Appendix C

Inputs for numerical calculation

C.1 The unitary triangle code

```
bool isNormalHierarchy = true;

double sinsq12 = 0.307;
double sinsq23N = 0.538;
double sinsq23I = 0.544;
double sinsq23 = isNormalHierarchy?sinsq23N:sinsq23I;
double sinsq13N = 0.02206;
double sinsq13I = 0.02227;
double sinsq13 = isNormalHierarchy?sinsq13N:sinsq13I;

double deltaN = 1.3*TMath::Pi();
double deltaI = 1.544*TMath::Pi();
double delta = isNormalHierarchy?deltaN:deltaI;

cout<<"The global analysis input" <<endl;
cout<<"sinsq12: "<<sinsq12<<endl;
cout<<"sinsq23: Normal "<<sinsq23N<<" Inverted "<<sinsq23I<<endl;
cout<<"sinsq13: Normal "<<sinsq13N<<" Inverted "<<sinsq13I<<endl;
cout<<"deltacp: Normal "<<deltaN<<" Inverted "<<deltaI<<endl;

double s12=TMath::Sqrt(sinsq12);
double c12=TMath::Sqrt(1-sinsq12);
double s23=TMath::Sqrt(sinsq23);
double c23=TMath::Sqrt(1-sinsq23);
double s13=TMath::Sqrt(sinsq13);
double c13=TMath::Sqrt(1-sinsq13);

double scp=TMath::Sin(delta);
```

```
double ccp=TMath::Cos(delta);
cpPhasePos = TComplex(ccp,scp);
cpPhaseNeg = TComplex(ccp,-scp);

if(isNormalHierarchy)
{
    cout<<"Normal hierarchy" <<endl;
}
else
    cout<<"Inverted hierarchy" <<endl;
{
cout<<"s12: "<<s12<<" c12 "<<c12<<endl;
cout<<"s23: "<<s23<<" c23 "<<c23<<endl;
cout<<"s13: "<<s13<<" c13 "<<c13<<endl;
cout<<"scp: "<<cpPhasePos.Re()<<" ccp "<<cpPhasePos.Im()<<endl;

double Ue1 = c12*c13;
double Ue2 = s12*c13;
TComplex Ue3 = cpPhaseNeg*s13;

TComplex Umu1 = -s12*c23 - c12*s23*s13*cpPhasePos;
TComplex Umu2 = c12*c23 - s12*s23*s13*cpPhasePos;
double Umu3 = s23*c13;

TComplex Utau1 = s12*s23 - c12*c23*s13*cpPhasePos;
TComplex Utau2 = -c12*s23 - s12*c23*s13*cpPhasePos;
double Utau3 = c23*c13;

cout<<"Ue1 "<<Ue1<<endl;
cout<<"Ue2 "<<Ue2<<endl;
cout<<"Ue3 Re "<<Ue3.Re() <<" Im "<<Ue3.Im()<<endl;
cout<<"Umu1 Re "<<Umu1.Re() <<" Im "<<Umu1.Im()<<endl;
cout<<"Umu2 Re "<<Umu2.Re()<<" Im "<<Umu2.Im()<<endl;
cout<<"Umu3  "<<Umu3<<endl;
cout<<"Utau1 Re "<<Utau1.Re()<<" Im "<<Utau1.Im()<<endl;
cout<<"Utau2 Re "<<Utau2.Re()<<" Im "<<Utau2.Im()<<endl;
cout<<"Utau3  "<<Utau3<<endl;

double norm_e = Ue1**2 + Ue2**2 + Ue3.Rho2();
double norm_mu = Umu1.Rho2() + Umu2.Rho2() + Umu3**2;
double norm_tau = Utau1.Rho2() + Utau2.Rho2() + Utau3**2;
cout<<"norm e "<<norm_e<<endl;
cout<<"norm mu "<<norm_mu<<endl;
cout<<"norm tau "<<norm_tau<<endl;
```

```
TComplex *Ue1Ue3 = new TComplex(Ue1*(Ue3.Re()),-Ue1*Ue3.Im());
TComplex *Umu1Umu3 = new TComplex((Umu1.Re())*Umu3, (Umu1.Im())*Umu3);
TComplex *Utau1Utau3 = new TComplex((Utau1.Re())*Utau3,
                      (Utau1.Im())*Utau3);

cout<<"Ue1Ue3 Re "<<Ue1Ue3->Re()<<" Im "<<Ue1Ue3->Im()
    <<" Amp "<<Ue1Ue3->Rho()<<endl;
cout<<"Umu1Umu3 Re "<<Umu1Umu3->Re()<<" Im "<<Umu1Umu3->Im()
    <<" Amp "<<Umu1Umu3->Rho()<<endl;
cout<<"Utau1Utau3 Re "<<Utau1Utau3->Re()<<" Im "<<Utau1Utau3->Im()
    <<" Amp "<<Utau1Utau3->Rho()<<endl;

TComplex *sideB = new TComplex((Ue1Ue3->Re()*Umu1Umu3->Re()
                  + Ue1Ue3->Im()*Umu1Umu3->Im())/Umu1Umu3->Rho2(),
                  (-Ue1Ue3->Re()*Umu1Umu3->Im()
                  + Ue1Ue3->Im()*Umu1Umu3->Re())/Umu1Umu3->Rho2());

TComplex *sideA = new TComplex((Utau1Utau3->Re()*Umu1Umu3->Re()
                  + Utau1Utau3->Im()*Umu1Umu3->Im())/Umu1Umu3->Rho2(),
                  (-Utau1Utau3->Re()*Umu1Umu3->Im()
                  + Utau1Utau3->Im()*Umu1Umu3->Re())/Umu1Umu3->Rho2());

cout<<"sideA Re "<<sideA->Re()<<" Im "<<sideA->Im()
    <<" Amp "<<sideA->Rho()<<" angle beta "<<sideA->Theta()<<endl;
cout<<"sideB Re "<<sideB->Re()<<" Im "<<sideB->Im()
    <<" Amp "<<sideB->Rho()<<" angle gamma "<<sideB->Theta()<<endl;

c = new TCanvas("c");
gROOT->SetStyle("Plain");
gStyle->SetOptStat(0); //tat bang thong ke
c->Range(-1.0,-1.0,1.0,1.0);
gPad->SetGrid();
gPad->SetTickx(1);
gPad->SetTicky(1);

if(isNormalHierarchy)
{
    TH2D *h2 = new TH2D("h2","",200,-0.2,1.2,200, -0.2,0.6);
}
else
{
    TH2D *h2 = new TH2D("h2","",200,-0.5,1.2,200, -0.2,0.55);
}
h2->GetXaxis()->SetTitle("Re(z)");
h2->GetYaxis()->SetTitle("Im(z)");
```

```
h2->GetYaxis()->SetTitleOffset(1.2);
h2->GetXaxis()->CenterTitle();
h2->GetYaxis()->CenterTitle();
h2->GetXaxis()->SetNdivisions(505);
h2->GetYaxis()->SetNdivisions(505);
h2->Draw("AXIS");

TLine *linec = new TLine(0,0,1,0);
TLine *linea = new TLine(1,0,-1-sideA->Re(), -sideA->Im());
TLine *lineb = new TLine(0,0,sideB->Re(), sideB->Im());
linec->SetLineWidth(2);
linea->SetLineWidth(2);
lineb->SetLineWidth(2);
linec->Draw("same");
linea->Draw("same");
lineb->Draw("same");

if(isNormalHierarchy)
{
    TLatex vertexA(0,-0.07,"A (0,0)");  //dinh
    TLatex vertexB(0.9,-0.07,"B (0,1)");
    TLatex vertexC(0.15,0.5,"C (0.200,0.467)");

    TLatex texsideB(-0.1,0.25,"#left| #frac{U_{e1} U_{e3}^{*}}
                   {U_{#mu1} U_{#mu3}^{*}} #right|");
    TLatex texsideA(0.7,0.25,"#left| #frac{U_{#tau1} U_{#tau3}^{*}}
                   {U_{#mu1} U_{#mu3}^{*}} #right|");

    TLatex angleA(0.05,0.02,"#gamma = 66^{o}");
    TLatex angleB(0.75,0.02,"#beta = 21^{o}");
}
else
{
    TLatex vertexA(-0.05,-0.07,"A (0,0)");  //dinh
    TLatex vertexB(0.9,-0.07,"B (0,1)");
    TLatex vertexC(-0.25,0.41,"C (#minus 0.153,0.395)");

    TLatex texsideB(-0.4,0.25,"#left| #frac{U_{e1} U_{e3}^{*}}
                   {U_{#mu1} U_{#mu3}^{*}} #right|");
    TLatex texsideA(0.55,0.25,"#left| #frac{U_{#tau1} U_{#tau3}^{*}}
                   {U_{#mu1} U_{#mu3}^{*}} #right|");

    TLatex angleA(0.03,0.02,"#gamma = 111^{o}");
    TLatex angleB(0.65,0.02,"#beta = 25^{o}");
}
```

```
texsideA.SetTextSize(0.035);
texsideB.SetTextSize(0.035);
vertexA.SetTextSize(0.04);
vertexB.SetTextSize(0.04);
vertexC.SetTextSize(0.04);
angleA.SetTextSize(0.04);
angleB.SetTextSize(0.04);

angleA.SetTextColor(46);
angleB.SetTextColor(46);
vertexA.SetTextColor(38);
vertexB.SetTextColor(38);
vertexC.SetTextColor(38);

vertexA.Draw("same");
vertexB.Draw("same");
vertexC.Draw("same");
texsideA.Draw("same");
texsideB.Draw("same");
angleA.Draw("same");
angleB.Draw("same");

if(isNormalHierarchy)
   c->Print("triangle_NO.pdf")
else
   c->Print("triangle_IO.pdf");
```

C.2 The uncertainties code

```
bool isNormalHierarchy = true;

ce = new TCanvas("ce");
gROOT->SetStyle("Plain");
gStyle->SetOptStat(0);
ce->Range(-1.0,-1.0,1.0,1.0);
gPad->SetTickx(1);
gPad->SetTicky(1);
gStyle->SetPalette(1);

TH2F *h2 = new TH2F("h2","",1000,-1,1.2,1000, -0.6,0.6);
h2->GetXaxis()->SetTitle("Re(z)");
h2->GetYaxis()->SetTitle("Im(z)");
```

```
h2->GetYaxis()->SetTitleOffset(1.2);
h2->GetXaxis()->CenterTitle();
h2->GetYaxis()->CenterTitle();
h2->GetXaxis()->SetNdivisions(505);
h2->GetYaxis()->SetNdivisions(505);
h2->Sumw2();

double gsinsq12 = 0.307;
double gsinsq12va = 0.012;
double gsinsq23 = isNormalHierarchy? 0.538:0.554;
double gsinsq23va = isNormalHierarchy? 0.033:0.023;
double gsinsq13 = isNormalHierarchy? 0.02206:0.02227;
double gsinsq13va = isNormalHierarchy? 0.00075:0.00074;
double gcp = isNormalHierarchy? 1.3:1.544;
double gcpva = isNormalHierarchy? 0.083:0.083;

//The Uncertainties
const int n = 1000000;
double x,y;
for (int i=0; i<n; i++)
{
double sinsq12 = gRandom->Gaus(gsinsq12,gsinsq12va) ;
    double sinsq23 = gRandom->Gaus(gsinsq23,gsinsq23va) ;
    double sinsq13 = gRandom->Gaus(gsinsq13,gsinsq13va) ;
    double delta = gRandom->Gaus(gcp,gcpva) ;

    double s12 = TMath::Sqrt(sinsq12);
    double c12 = TMath::Sqrt(1-sinsq12);
    double s23 = TMath::Sqrt(sinsq23);
    double c23 = TMath::Sqrt(1-sinsq23);
    double s13 = TMath::Sqrt(sinsq13);
    double c13 = TMath::Sqrt(1-sinsq13);

    double scp = TMath::Sin(delta*TMath::Pi());
    double ccp = TMath::Cos(delta*TMath::Pi());
    cpPhasePos = TComplex(ccp,scp);
    cpPhaseNeg = TComplex(ccp,-scp);


    double Ue1 = c12*c13;
    double Ue2 = s12*c13;
    TComplex *Ue3 = new TComplex(cpPhaseNeg.Re()*s13,cpPhaseNeg.Im()*s13);
    TComplex *Umu1 = new TComplex(-s12*c23-c12*s23*s13*(cpPhasePos.Re()),
                    -c12*s23*s13*(cpPhasePos.Im()));
```

```
    TComplex *Umu2 = new TComplex(c12*c23-s12*s23*s13*(cpPhasePos.Re()),
                    -s12*s23*s13*(cpPhasePos.Im()));
    double Umu3 = s23*c13;

    TComplex *Utau1 = new TComplex(s12*s23-c12*c23*s13*(cpPhasePos.Re()),
                                   -c12*c23*s13*(cpPhasePos.Im()));
    TComplex *Utau2 = new TComplex(-c12*s23-s12*c23*s13*(cpPhasePos.Re()),
                                   -s12*c23*s13*(cpPhasePos.Im()));
    double Utau3 = c23*c13;

    TComplex *Ue1Ue3 = new TComplex(Ue1*(Ue3->Re()), -Ue1*Ue3->Im());
    TComplex *Umu1Umu3 = new TComplex((Umu1->Re())*Umu3, (Umu1->Im())*Umu3);
    TComplex *Utau1Utau3 = new TComplex((Utau1->Re())*Utau3,
                                         (Utau1->Im())*Utau3);

    TComplex *sideB = new TComplex(-(Ue1Ue3->Re()*Umu1Umu3->Re()
                      + Ue1Ue3->Im()*Umu1Umu3->Im())/Umu1Umu3->Rho2(),
                      -(-Ue1Ue3->Re()*Umu1Umu3->Im()
                      + Ue1Ue3->Im()*Umu1Umu3->Re())/Umu1Umu3->Rho2());

    TComplex *sideA = new TComplex(-(Utau1Utau3->Re()*Umu1Umu3->Re()
                      + Utau1Utau3->Im()*Umu1Umu3->Im())/Umu1Umu3->Rho2(),
                      -(-Utau1Utau3->Re()*Umu1Umu3->Im()
                      + Utau1Utau3->Im()*Umu1Umu3->Re())/Umu1Umu3->Rho2());

        x = sideB->Re();
        y = sideB->Im();
        h2->Fill(x,y);
     }

TH2F *hc = new TH2F("hc","",1000,-1,1,1000, -1,1);
hc->Sumw2();

// Only vertex C at best fit value
double sinsq12f = 0.307;
double sinsq23f = isNormalHierarchy? 0.538:0.554;
double sinsq13f = isNormalHierarchy? 0.02206:0.02227;
double deltaf = isNormalHierarchy? 1.3:1.544;

double s12f = TMath::Sqrt(sinsq12f);
double c12f = TMath::Sqrt(1-sinsq12f);
double s23f = TMath::Sqrt(sinsq23f);
double c23f = TMath::Sqrt(1-sinsq23f);
double c13f = TMath::Sqrt(1-sinsq13f);
```

```
double scpf = TMath::Sin(deltaf*TMath::Pi());
double ccpf = TMath::Cos(deltaf*TMath::Pi());
TComplex cpPhasePosf = TComplex(ccpf,scpf);
TComplex cpPhaseNegf = TComplex(ccpf,-scpf);

double Ue1f = c12f*c13f;
double Ue2f = s12f*c13f;
TComplex Ue3f = cpPhaseNegf*s13f;

TComplex Umu1f = -s12f*c23f - c12f*s23f*s13f*cpPhasePosf;
TComplex Umu2f = c12f*c23f - s12f*s23f*s13f*cpPhasePosf;
double Umu3f = s23f*c13f;

TComplex Utau1f = s12f*s23f - c12f*c23f*s13f*cpPhasePosf;
TComplex Utau2f = -c12*s23f - s12f*c23f*s13f*cpPhasePosf;
double Utau3f = c23f*c13f;

cout<<"Ue1f "<<Ue1f<<endl;
cout<<"Ue2f "<<Ue2f<<endl;
cout<<"Ue3f "<<Ue3f<<endl;
cout<<"Umu1f "<<Umu1f<<endl;
cout<<"Umu2f "<<Umu2f<<endl;
cout<<"Umu3f "<<Umu3f<<endl;
cout<<"Utau1f "<<Utau1f<<endl;
cout<<"Utau2f "<<Utau2f<<endl;
cout<<"Utau3f "<<Utau3f<<endl;

TComplex *Ue1fUe3f = new TComplex(Ue1f*Ue3f.Re(), -Ue1f*Ue3f.Im());
TComplex *Umu1fUmu3f = new TComplex((Umu1f.Re())*Umu3f,
                                    (Umu1f.Im())*Umu3f);
TComplex *Utau1fUtau3f = new TComplex((Utau1f.Re())*Utau3f,
                                      (Utau1f.Im())*Utau3f);

TComplex *sideBf = new TComplex(-(Ue1fUe3f->Re()*Umu1fUmu3f->Re()
                   + Ue1fUe3f->Im()*Umu1fUmu3f->Im())/Umu1fUmu3f->Rho2(),
                   -(-Ue1fUe3f->Re()*Umu1fUmu3f->Im()
                   + Ue1fUe3f->Im()*Umu1fUmu3f->Re())/Umu1fUmu3f->Rho2());

TComplex *sideAf = new TComplex(-(Utau1fUtau3f->Re()*Umu1fUmu3f->Re()
                   + Utau1fUtau3f->Im()*Umu1fUmu3f->Im())/Umu1fUmu3f->Rho2(),
                   -(-Utau1fUtau3f->Re()*Umu1fUmu3f->Im()
                   + Utau1fUtau3f->Im()*Umu1fUmu3f->Re())/Umu1fUmu3f->Rho2());

cout<<"sideAf Re "<<sideAf->Re()<<" Im "<<sideAf->Im()
    <<" Amp "<<sideAf->Rho()<<endl;
```

```
cout<<"sideBf Re "<<sideBf->Re()<<" Im "<<sideBf->Im()
    <<" Amp "<<sideBf->Rho()<<endl;

const int m = 1;
double a,b;
for (int i=0; i<m ;i++)
{
    a[i] = sideBf->Re();
    b[i] = sideBf->Im();
    cout<<"a: "<<a[i]<<endl;
    cout<<"b: "<<b[i]<<endl;
    hc->Fill(a,b);
}

h2->Draw("colz");

TLine *linei = new TLine(-0.9399,0,1.1455,0);
linei->SetLineWidth(2);
linei->SetLineColor(18);
linei->Draw("same");

TLine *linec= new TLine(0,0,1,0);
TLine *linea= new TLine(1,0,1-sideAf->Re(),-sideAf->Im());
TLine *lineb= new TLine(0,0,sideBf->Re(),sideBf->Im());
linec->SetLineWidth(2);
linea->SetLineWidth(2);
lineb->SetLineWidth(2);
linec->Draw("same");
linea->Draw("same");
lineb->Draw("same");

hc->SetMarkerColor(2);
hc->SetMarkerStyle(29);
hc->SetMarkerSize(2);
hc->Draw("same");

if(isNormalHierarchy)
{
TLatex texsideA(0.4,-0.3,"U_{#tau1} U^{*}_{#tau3}");
    TLatex texsideB(-0.3,-0.2,"U_{e1} U^{*}_{e3}");
    TLatex texsideC(0.5,0.05,"U_{#mu1} U^{*}_{#mu3}");
    TLatex texsideD(0.5,0.4,"z = #minus #frac{U_{e1} U^{*}_{e3}}
                            {U_{#mu1} U^{*}_{#mu3}}");
}
else
```

```
{
    TLatex texsideA(0.6,-0.25,"U_{#tau1} U^{*}_{#tau3}");
    TLatex texsideB(-0.2,-0.2,"U_{e1} U^{*}_{e3}");
    TLatex texsideC(0.5,0.05,"U_{#mu1} U^{*}_{#mu3}");
    TLatex texsideD(0.5,0.4,"z = #minus #frac{U_{e1} U^{*}_{e3}}
                             {U_{#mu1} U^{*}_{#mu3}}");
}

texsideA.SetTextSize(0.037);
texsideB.SetTextSize(0.037);
texsideC.SetTextSize(0.037);
texsideD.SetTextSize(0.04);

texsideA.Draw("same");
texsideB.Draw("same");
texsideC.Draw("same");
texsideD.Draw("same");

if(isNormalHierarchy)
ce->Print("NO.pdf");
else
ce->Print("IO.pdf");
```