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Project:

SYNERGIES AND PROSPECTS FOR NEUTRINO MASS MEASUREMENTS

Supervisor: Dr. Cao Van Son (IFIRSE)

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Abstract

Neutrino mass, fundamental parameter in particle physics, remains one of the most intriguing puzzles in our understanding of the universe. Recent advancements in experimental techniques and theoretical frameworks have led to unprecedented opportunities for precise measurements of neutrino masses. In this study, I explored the synergies between various experimental approaches, including tritium beta decay experiments, neutrinoless double-beta decay searches, cosmological observations, and neutrino oscillation experiments, in constraining the absolute neutrino mass scale. We discuss the current status of neutrino mass measurements and highlight the prospects offered by upcoming experiments, such as the KATRIN, and cosmological surveys like the Cosmic Microwave Background (CMB) and large-scale structure (LSS) observations. Furthermore, we address the challenges and uncertainties associated with these measurements and outline the potential impact of neutrino mass determination on our understanding of particle physics, astrophysics, and cosmology. This abstract aims to provide a comprehensive overview of the state-of-the-art in neutrino mass measurements and the promising avenues for future research in this field.

Keywords: neutrino mass, effective Majorana mass, neutrino mass sum rules, flavor symmetries

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1 Introduction

The discovery of neutrino oscillations has profoundly impacted our understanding of particle physics, revealing that neutrinos possess mass. This revelation challenges the Standard Model (SM), which originally postulated neutrinos as massless particles. Determining the absolute scale of neutrino masses is a major challenge in modern physics, with significant implications for cosmology, astrophysics, and fundamental physics.

Neutrinos, being elusive particles that interact only via the weak nuclear force and gravity, make direct mass measurements exceptionally difficult. However, recent advances in experimental techniques and theoretical models offer new opportunities to explore neutrino masses with unprecedented precision. A comprehensive strategy combining beta decay experiments, cosmological observations, and neutrino oscillation studies provides a pathway to constrain and potentially measure the absolute neutrino mass scale.

Beta decay experiments, such as Karlsruhe Tritium Neutrino Experiment (KATRIN), aim to measure the endpoint spectrum of tritium decay to directly determine the electron neutrino mass. Complementarily, neutrinoless double-beta decay searches probe the Majorana nature of neutrinos, offering insights into their mass and providing a potential path to detecting lepton number violation. Cosmological observations, particularly those involving the Cosmic Microwave Background (CMB) and Large-scale Structure (LSS) surveys, place stringent upper limits on the sum of neutrino masses by examining their impact on the evolution of the universe.

The goal of this thesis is to explore the synergistic potential of combining data from these diverse approaches to enhance our understanding of neutrino masses. By integrating results from laboratory experiments and cosmological measurements, this work aims to provide a coherent picture of current constraints and future prospects for neutrino mass determination. Additionally, the implications of potential discoveries for the SM and beyond will be discussed, underscoring the interdisciplinary nature of this quest.

Furthermore, this thesis delves into the testing of neutrino mass sum rules within the context of flavor symmetry models. These models predict specific relationships among neutrino masses and mixing parameters, offering a framework to interpret current data and guide future experimental efforts. The exploration of these sum rules provides critical insights into the underlying symmetries governing neutrino physics and paves the way for new theoretical developments.

This introductory chapter sets the stage by outlining the fundamental principles of neutrino physics, the significance of neutrino mass measurements, and the various methodologies employed in this pursuit. Section 2 contains basic background, I will provide an explanation of the Pontecorvo–Maki–Nakagawa–Sakata matrix (PMNS matrix), constraints on neutrino mass, and how flavor symmetries relate to neutrino mass sum rules. Then section 3 describes in details how the Monte Carlo simulation work and the counting method applied to obtain the allow region of effective Majorana mass. In section 4, I present the result plots and explaination for each sum rules. I will discuss my findings in section 5, my work in this study and future prospects will be then summarised in section 6.

2 Background

In this section, I will provide an overview of the key concepts related to the discovery and implications of neutrino mass. Neutrinos, once believed to be massless, have been shown through experimental evidence to possess a non-zero mass. This discovery indicates that the SM is incomplete and requires extensions. The following subsections explore the PMNS matrix, which describes neutrino mixing, and the various experimental and observational constraints on neutrino masses.

2.1 PMNS matrix

The PMNS matrix encapsulates the mixing between neutrino flavor and mass eigenstates. It can be parameterized by three mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$) and CP-violating phases (δ_{CP}, along with two Majorana phases ρ_1 and ρ_2 if neutrinos are Majorana particles). The PMNS matrix is typically expressed as:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\text{CP}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{23}c_{13} \end{pmatrix} \text{Diag}(e^{i\rho_1}, e^{i\rho_2}, 1),$$

where $c_{ij} = \cos(\theta_{ij})$ and $s_{ij} = \sin(\theta_{ij})$. The mixing angles are directly related to the probabilities of transitions between different neutrino flavors as they propagate.

2.2 Constraints on Neutrino Mass

2.2.1 From Neutrino Oscillation Experiments

Neutrino oscillation experiments have provided precise measurements of the differences in squared masses between the three neutrino mass eigenstates. As shown in Jimenez et al. [2022], the two independent mass-squared differences are given by:

$$\Delta m^2_{21} = m^2_2 - m^2_1 \approx 7.42\,(\pm 0.21) \times 10^{-5}\,\text{eV}^2 \quad \text{(at 68.4\% C.L.)}$$

$$\Delta m^2_{3l} = m^2_3 - m^2_l = \begin{cases} 2.510\,(\pm 0.027) \times 10^{-3}\,\text{eV}^2 & \text{(NH)} \\ -0.249\,(\pm 0.027) \times 10^{-3}\,\text{eV}^2 & \text{(IH)} \end{cases} \quad \text{(at 68.4\% C.L.)}$$

Matter effects within the Sun have definitively established the positive sign of the solar mass splitting, Δm^2_{21}. In contrast, the atmospheric mass splitting, Δm^2_{31}, determined primarily through neutrino oscillations in a vacuum, is sensitive only to its absolute value. Consequently, the neutrino mass ordering remains an open question, with two possible scenarios: normal ordering, where Δm^2_{31} is positive, and inverted ordering, where Δm^2_{31} is negative.

2.2.2 From Cosmology

Cosmology offers a unique avenue to study neutrino masses through their subtle yet significant effects on the universe's LSS and the CMB. Although neutrinos are incredibly light, their sheer abundance means they contribute to the total energy density of the universe, affecting the rate of cosmic expansion and the growth of structures over time. By analyzing the CMB, which provides a snapshot of the early universe, and the distribution of galaxies and galaxy clusters in the LSS, cosmologists can infer the sum of the neutrino masses.

Neutrinos suppress the formation of structures on small scales, leading to a characteristic "neutrino fingerprint" in the power spectrum of the LSS. This suppression, combined with precise measurements of the CMB, allows for the extraction of constraints on the total neutrino mass. The most stringent current limits come from the Planck satellite, whose observations, when combined with Baryon Acoustic Oscillations (BAO) data from galaxy surveys, set an upper limit on the sum of the neutrino masses to be less than 0.12 eV at 90% confidence level Aghanim et al. [2020]. This result is particularly remarkable because it not only limits the absolute scale of neutrino masses but also complements the mass-squared differences measured in oscillation experiments, providing a more complete picture of the neutrino mass hierarchy.

2.2.3 From Beta Decay

Beta decay experiments provide the most direct method to measure the absolute mass of neutrinos. In beta decay, a neutron decays into a proton, an electron, and an antineutrino. The energy released in this process is shared between the electron and the antineutrino, with the electron's energy spectrum extending up to a maximum endpoint that depends on the mass of the neutrino:

$$(A, Z) \rightarrow (A, Z+1) + e^{-} + \bar{\nu}_i$$

The precise measurement of the electron's energy spectrum near this endpoint allows scientists to infer the effective mass of the neutrino. This effective mass, denoted as $\langle m_\beta \rangle$, is a weighted sum of the neutrino masses squared, modulated by the elements of the PMNS matrix:

$$\langle m_\beta \rangle = \sqrt{\sum_i |U_{ei}|^2 m_{\nu_i}^2}$$

The KATRIN experiment, located in Germany, is the most advanced project dedicated to this measurement. KATRIN uses a high-resolution spectrometer to measure the endpoint region of the tritium beta decay spectrum with unprecedented precision. Recent results from KATRIN have set an upper limit on the effective neutrino mass of less than 0.8 eV at 90% confidence level Aker et al. [2021]. The experiment aims to reach a sensitivity of 0.2 eV, which would provide a critical test of the neutrino mass scale and potentially reveal the true mass of the lightest neutrino. This direct measurement is essential because it complements the indirect constraints from cosmology and oscillation experiments, offering a more comprehensive understanding of neutrino properties.

2.2.4 From Neutrinoless Double Beta Decay

Neutrinoless double beta decay is a theoretical process that would have profound implications for our understanding of neutrino physics, particularly the nature of neutrinos themselves. Unlike standard double beta decay, where two neutrons in a nucleus decay into two protons, two electrons, and two antineutrinos, neutrinoless double beta decay involves no emission of neutrinos:

$$(A, Z) \to (A, Z+2) + 2e^-$$

The absence of neutrinos in this decay implies that neutrinos are Majorana particles, meaning they are their own antiparticles. This process violates lepton number conservation, a fundamental symmetry in the SM, and its observation would be a clear signal of new physics beyond the SM.

The rate of neutrinoless double beta decay depends on the effective Majorana mass of the neutrino, $\langle m_{\beta\beta}^{0\nu} \rangle$, which is a combination of the neutrino masses and the elements of the PMNS matrix:

$$\langle m_{\beta\beta}^{0\nu} \rangle = \left| m_{\nu_1} \cos^2\theta_{13} \cos^2\theta_{12} + m_{\nu_2} \cos^2\theta_{13} \sin^2\theta_{12} e^{i\rho_1} + m_{\nu_3} \sin^2\theta_{13} e^{i\rho_2} \right|$$

Several experiments around the world are dedicated to searching for this rare decay, including Kamioka Liquid scintillator Anti-Neutrino Detector - Zero neutrino double beta decay experiment (KamLAND-Zen) in Japan, GERDA in Italy, and EXO-200 in the United States. These experiments utilize different isotopes and detection techniques to achieve the highest possible sensitivity to the decay. The current best limits on the effective Majorana mass come from KamLAND-Zen, which has placed constraints in the range of 28-122 meV at 90% confidence level, depending on the nuclear matrix element calculations used Abe et al. [2024]. The continued search for neutrinoless double beta decay is crucial not only for understanding the nature of neutrinos but also for uncovering the possible violation of lepton number conservation, which could provide clues about the origin of the matter-antimatter asymmetry in the universe.

2.3 Emergence of Neutrino Mass Sum Rules from Flavor Symmetries

The study of neutrino mass sum rules within the context of flavor symmetries provides a robust framework for understanding the patterns observed in neutrino oscillations. In models that do not include right-handed neutrinos, small neutrino masses can emerge from higher-dimensional operators that break lepton number conservation. The most notable of these is the dimension-five Weinberg operator, which contributes to the neutrino mass matrix after the Higgs field obtains its vacuum expectation value (VEV).

The neutrino mass matrix M_ν is typically a complex symmetric matrix, parameterized by six independent complex parameters. This matrix can be expressed as:

$$M_\nu = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix},$$

where a, b, c, d, e, and f represent complex parameters that govern the interactions between different neutrino flavors. When this matrix is diagonalized, it yields three complex mass eigenvalues $\tilde{m}_1$, $\tilde{m}_2$, and $\tilde{m}_3$. These eigenvalues, in general, are uncorrelated unless a symmetry is imposed.

When a flavor symmetry is assumed, the neutrino mass matrix can be significantly constrained. For example, in models with an A_4 family symmetry, the effective neutrino mass matrix that arises after electroweak symmetry breaking may take a form such as:

$$M_\nu = \begin{pmatrix} 2\alpha_s + \alpha_0 & -\alpha_s & -\alpha_s \\ -\alpha_s & 2\alpha_s & -\alpha_s + \alpha_0 \\ -\alpha_s & -\alpha_s + \alpha_0 & 2\alpha_s \end{pmatrix} \cdot \frac{v^2}{\Lambda_L},$$

where α_s and α_0 are parameters determined by the symmetry, v is the Higgs field VEV, and Λ_L is the cutoff scale of the symmetry. This form is always diagonalizable by a specific unitary transformation, resulting in mass eigenvalues that satisfy a specific sum rule:

$$\tilde{m}_1 - \tilde{m}_3 = 2\tilde{m}_2.$$

This relation among the neutrino masses is a direct consequence of the symmetry and is a testable prediction of the model. More generally, these sum rules offer a way to connect the model parameters with observable quantities in neutrino experiments, thereby providing a means to validate or refute the underlying flavor symmetries.

3 Scientific methods and materials

3.1 Monte Carlo and Counting Method

1. Experimental constraints

$\downarrow$

2. Generate mixing/mass parameter sets with MC

$\downarrow$

3. Apply sum rules to fill out the satisfied parameter sets

$\downarrow$

4. Compute the allowed region on observable parameters

Figure 1: Flowchart of the computational steps from input data to final plots

The input data for this analysis include constraints on neutrino masses derived from multiple sources, such as neutrino oscillation experiments, cosmological observations, beta decay experiments, and neutrinoless double-beta decay experiments. These constraints guide the exploration of the parameter space of neutrino masses, with a particular focus on testing the validity of various neutrino mass sum rules.

In the analysis, a Monte Carlo method was employed with 10,000,000 random values for m_1 within the range of 0.001 eV to 0.06 eV. This range was selected to encompass both theoretical and experimental bounds on neutrino masses. Depending on whether normal Mass Ordering (MO) or inverted MO is assumed, the values of m_2 and m_3 were determined as follows:

For normal MO, m_1 is randomly selected within the specified range. The other masses are then calculated using the following relations:

$$m_2 = \sqrt{\Delta m_{21}^2 + m_1^2}$$

$$m_3 = \sqrt{\Delta m_{31}^2 + m_1^2}$$

Here, m_1 is the lightest mass eigenstate.

For inverted MO: m_3 is randomly selected within the same range. The other masses are then calculated using:

$$m_1 = \sqrt{m_3^2 - \Delta m_{31}^2}$$

$$m_2 = \sqrt{\Delta m_{21}^2 + m_1^2}$$

Here, m_3 is the lightest mass eigenstate.

The Monte Carlo method thus allows for the exploration of the parameter space under both ordering scenarios, ensuring that the results reflect the full range of possible neutrino mass hierarchies.

The calculation begins by representing the neutrino mass eigenstates in polar form:

$$\tilde{m_i} = m_i e^{i\phi_i}$$

Here, m_i represents the physical mass eigenstates, and ϕ_i are the associated phases. To test the validity of a particular sum rule, the following relation is assumed:

$$\tilde{m_1} + \tilde{m_2} = \tilde{m_3}$$

Expanding this expression leads to:

$$m_1 e^{i\phi_1} + m_2 e^{i\phi_2} = m_3 e^{i\phi_3}$$

Further simplification gives:

$$m_1 + m_2 e^{i(\phi_2 - \phi_1)} = m_3 e^{i(\phi_3 - \phi_1)}$$

Using Euler's identity, this expression can be separated into its real and imaginary components:

- **Real Part:**
$$m_1 + m_2 \cos(\alpha_{21}) = m_3 \cos(\alpha_{31})$$
- **Imaginary Part:**
$$m_2 \sin(\alpha_{21}) = m_3 \sin(\alpha_{31})$$

Here, α_{21} and α_{31} represent the phase differences between the corresponding mass eigenstates. These two relations must hold for the sum rule to be valid under the assumed conditions. In general form, ϕ_i is from 0 to 2π, but α_{21} and α_{32} is from 0 to π folllowing the reference de Gouvêa and Jenkins [2008].

From the above relations, m_2 and m_3 can be expressed as functions of the lightest neutrino mass, α_{21}, and α_{31}:

$$m_2 = \frac{m_1}{\frac{\sin(\alpha)}{\tan(\beta)} - \cos(\alpha)}, m_3 = \frac{m_1 \frac{\sin(\alpha)}{\sin(\beta)}}{\frac{\sin(\alpha)}{\tan(\beta)} - \cos(\alpha)} \tag{1}$$

To account for experimental uncertainties, the analysis introduces a 5% uncertainty in the calculations. Special attention is given to avoid issues arising from very small values of $\sin(\alpha)$

or $\sin(\beta)$, which could lead to significant errors due to the small magnitude of the neutrino mass. A function is employed to calculate m_3 from the lightest mass eigenstate, α, and β, ensuring that the computed m_3 adheres to the sum rule within the allowed uncertainty. This function applies specific criteria to prevent errors, such as division by values too close to zero.

Following the application of the Monte Carlo method and the testing of the neutrino mass sum rules, the results were visualized to show the correlations between the sum of neutrino masses (Σm_i) and the effective mass observed in neutrinoless double-beta decay ($m_{0\nu\beta\beta}$). These correlations are illustrated in Figure 1, which includes the 1-sigma and 3-sigma confidence regions and experimental constraints. This visual representation provides a comprehensive comparison between theoretical predictions and observational data.

Subsequent plots illustrate the 1-sigma and 3-sigma confidence regions, combined with experimental constraints, providing a visual comparison between theoretical predictions and observational data.

3.2 Neutrino mass sum rules

The following table presents sum rules for complex light neutrino mass eigenvalues ($\tilde{m}_i$), derived from combinations of neutrino mass generation mechanisms and discrete flavor symmetries. The table is adapted from Chauhan et al. [2024].

Sum Rule	Group	Seesaw Type
$\tilde{m}_1 + \tilde{m}_2 = \tilde{m}_3$	A_4[158, 247–251]; S_4[252]; A_5[75]	Weinberg
$\tilde{m}_1 + \tilde{m}_2 = \tilde{m}_3$	$\Delta(54)$[253]; S_4[254]	Type II
$\tilde{m}_1 + 2\tilde{m}_2 = \tilde{m}_3$	S_4[255]	Type II
$2\tilde{m}_2 + \tilde{m}_3 = \tilde{m}_1$	A_4 [117, 118, 158, 237, 248–251, 256–262]	Weinberg
	S_4[133, 263]; T'[156, 246, 264–267]; T_7[268]	
$2\tilde{m}_2 + \tilde{m}_3 = \tilde{m}_1$	A_4[269]	Type II
$\tilde{m}_1 + \tilde{m}_2 = 2\tilde{m}_3$	S_4[270]	Dirac
$\tilde{m}_1 + \tilde{m}_2 = 2\tilde{m}_3$	$L_e - L_\mu - L_\tau$[271]	Type II
$\tilde{m}_1 + \frac{\sqrt{3}+1}{2}\tilde{m}_3 = \frac{\sqrt{3}-1}{2}\tilde{m}_2$	A_5'[272]	Weinberg
$\tilde{m}_1^{-1} + \tilde{m}_2^{-1} = \tilde{m}_3^{-1}$	A_4[158]; S_4[247, 254]; A_5[78, 166]	Type I
$\tilde{m}_1^{-1} + \tilde{m}_2^{-1} = \tilde{m}_3^{-1}$	S_4[254]	Type III
$2\tilde{m}_2^{-1} + \tilde{m}_3^{-1} = \tilde{m}_1^{-1}$	A_4[118, 158, 235, 237, 238, 245, 273–281]; T'[246]	Type I
$\tilde{m}_1^{-1} + \tilde{m}_3^{-1} = 2\tilde{m}_2^{-1}$	A_4[282–284]; T'[285]	Type I
$\tilde{m}_3^{-1} \pm 2i\tilde{m}_2^{-1} = \tilde{m}_1^{-1}$	$\Delta(96)$[286]	Type I
$\tilde{m}_1^{1/2} - \tilde{m}_3^{1/2} = 2\tilde{m}_2^{1/2}$	A_4[236]	Type I
$\tilde{m}_1^{1/2} + \tilde{m}_3^{1/2} = 2\tilde{m}_2^{1/2}$	A_4[287]	Scotogenic
$\tilde{m}_1^{-1/2} + \tilde{m}_2^{-1/2} = 2\tilde{m}_3^{-1/2}$	S_4[240]	Inverse

Figure 2: List of sum rules applied

4 Results

In this section, we present the final histograms for the sum rules. Figure 3 shows the allowed region for the sum of neutrino masses $\sum m_{\nu_i}$ and the effective Majorana mass $\langle m_{0\nu\beta\beta}\rangle$. Using neutrino oscillation data, the distributions of these observable quantities can be derived by applying constraints on the neutrino mass-squared differences Δm^2_{21} and Δm^2_{31}, along with the leptonic mixing parameters. We also consider current and upcoming constraints from beta-decay experiments, searches for neutrinoless double-beta decay, and cosmology.

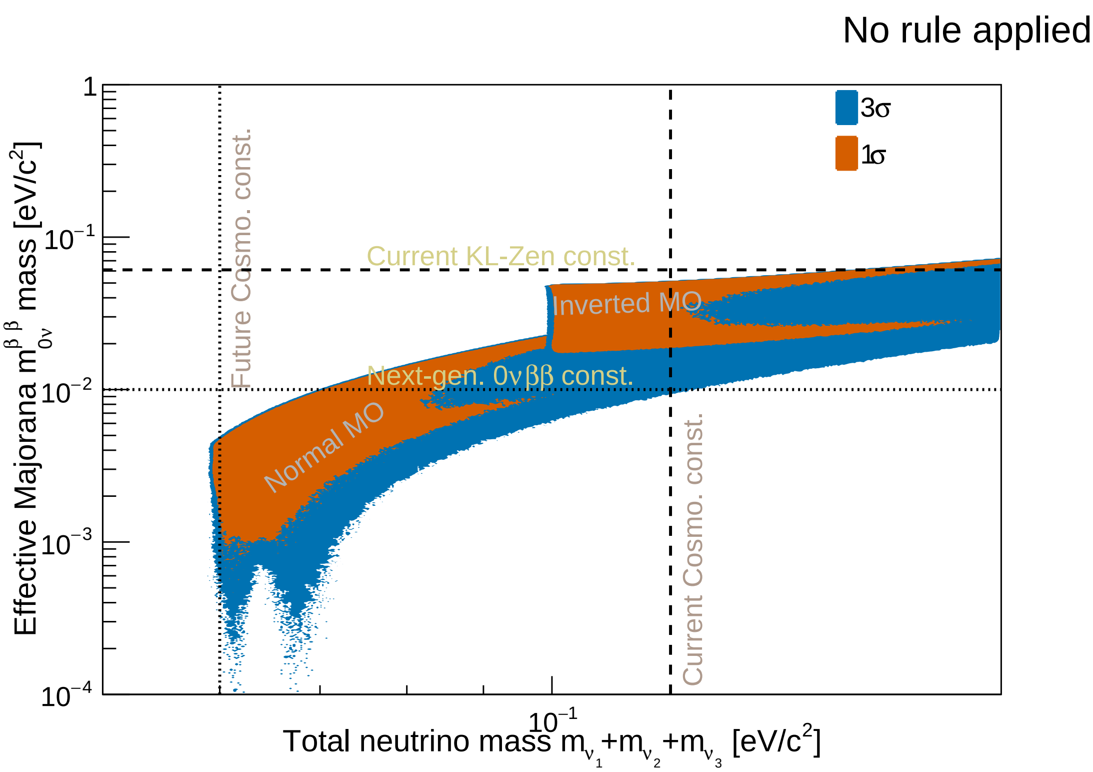


Figure 3: Allowed region for effective Majorana mass (searchable with neutrinoless double beta decay experiments) and total neutrino mass (measurable with cosmology). Here, only constraints from neutrino oscillation parameters are used. Both normal ordering (NO) and inverted ordering (IO) are allowed. The KamLAND-Zen constraint is approaching the normal mass ordering (MO) region.

As discussed in Section 2, the masses m_2 and m_3 can be expressed as functions of m_1, α_{21}, and α_{31}. The solutions for m_2 and m_3 for each rule are as follows:

$$\text{Rule 2:}\quad m_2 = 0.5\frac{m_1}{\frac{\sin\alpha}{\tan\beta} - \cos\alpha}, \qquad m_3 = \frac{m_1\frac{\sin\alpha}{\sin\beta}}{\frac{\sin\alpha}{\tan\beta} - \cos\alpha} \tag{2}$$

$$\text{Rule 3:}\quad m_2 = 0.5\frac{m_1}{\frac{\sin\alpha}{\tan\beta} - \cos\alpha}, \qquad m_3 = \frac{m_1\frac{\sin\alpha}{\sin\beta}}{\cos\alpha - \frac{\sin\alpha}{\tan\beta}} \tag{3}$$

$$\text{Rule 4:}\quad m_2 = \frac{m_1}{\frac{\sin\alpha}{\tan\beta} - \cos\alpha}, \qquad m_3 = 0.5\frac{m_1\frac{\sin\alpha}{\sin\beta}}{\frac{\sin\alpha}{\tan\beta} - \cos\alpha} \tag{4}$$

$$\text{Rule 5:}\quad m_2 = \frac{2m_1}{(\sqrt{3}-1)(\cos\alpha - \frac{\sin\alpha}{\tan\beta})}, \qquad m_3 = \frac{\sqrt{3}-1}{\sqrt{3}+1}m_2\frac{\sin\alpha}{\sin\beta} \tag{5}$$

$$\text{Rule 6:}\quad m_2 = m_1(\frac{\sin\alpha}{\tan\beta} - \cos\alpha), \qquad m_3 = \frac{\sin\beta}{\sin\alpha}m_1(\frac{\sin\alpha}{\tan\beta} - \cos\alpha) \tag{6}$$

$$\text{Rule 7:}\quad m_2 = 2m_1(\cos\alpha - \frac{\sin\alpha}{\tan\beta}), \qquad m_3 = -m_1\frac{\sin\beta}{\sin\alpha}(\cos\alpha - \frac{\sin\alpha}{\tan\beta}) \tag{7}$$

$$\text{Rule 8:}\quad m_2 = 2m_1(\cos\alpha - \frac{\sin\alpha}{\tan\beta}), \qquad m_3 = m_1\frac{\sin\beta}{\sin\alpha}(\cos\alpha - \frac{\sin\alpha}{\tan\beta}) \tag{8}$$

$$\text{Rule 9:}\quad m_2 = 2m_1(\sin\alpha + \frac{\cos\alpha}{\tan\beta}), \qquad m_3 = m_1\frac{\sin\beta}{\cos\alpha}(\sin\alpha + \frac{\cos\alpha}{\tan\beta}) \tag{9}$$

$$\text{Rule 10:}\quad m_2 = -2m_1(\sin\alpha + \frac{\cos\alpha}{\tan\beta}), \qquad m_3 = m_1\frac{\sin\beta}{\cos\alpha}(\sin\alpha + \frac{\cos\alpha}{\tan\beta}) \tag{10}$$

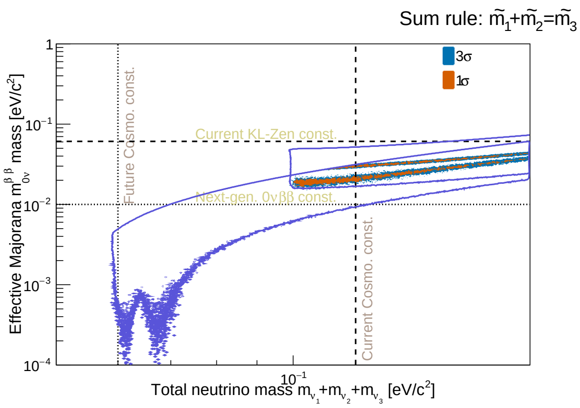
$$\text{Rule 11:}\quad m_2 = 0.25m_1(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}}{\tan\frac{\beta}{2}})^{-2}, \qquad m_3 = m_1\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}}{\tan\frac{\beta}{2}})^{-2} \tag{11}$$

$$\text{Rule 12:}\quad m_2 = 0.25m_1(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}}{\tan\frac{\beta}{2}})^{-2}, \qquad m_3 = m_1\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}(\cos\frac{\alpha}{2} - \frac{\sin\frac{\alpha}{2}}{\tan\frac{\beta}{2}})^{-2} \tag{12}$$

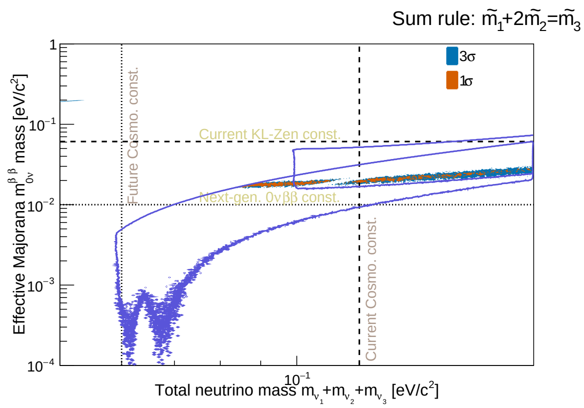
$$\text{Rule 13:}\quad m_2 = m_1(\frac{\sin\frac{\alpha}{2}}{\tan\frac{\beta}{2}} - \cos\frac{\alpha}{2})^2, \qquad m_3 = 4m_1(\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha}{2}})^2(\frac{\sin\frac{\alpha}{2}}{\tan\frac{\beta}{2}} - \cos\frac{\alpha}{2})^2 \tag{13}$$

The allowed regions for each sum rule are illustrated in Figures 4 and 5.

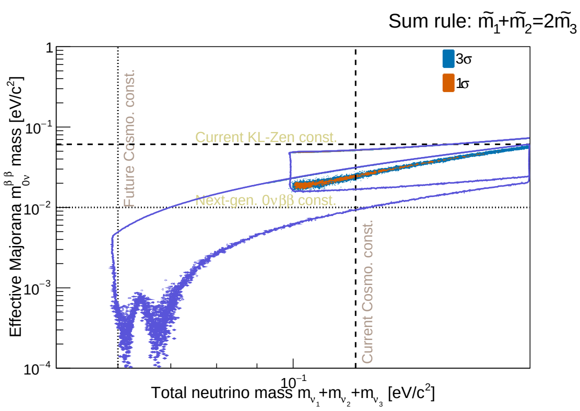
Figures 4a, 4b, and 4c illustrate that the allowed parameter space for these sum rules is in close proximity to current cosmological constraints. This proximity indicates that as future measurements become more precise, these models could encounter stricter constraints. The near overlap with cosmological limits highlights the potential for these models to be tested more rigorously with upcoming observational data, potentially leading to tighter bounds or even the exclusion of these models depending on the direction of future results.



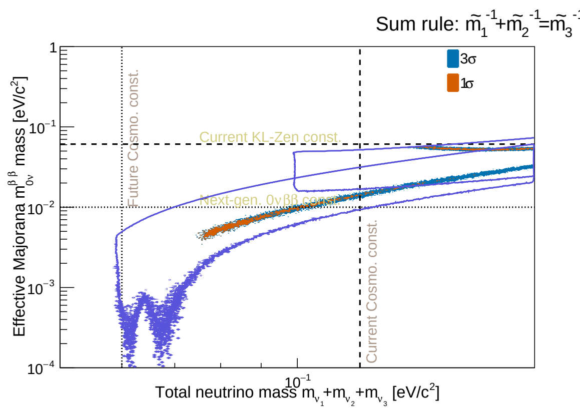
(a) Sum rule 1



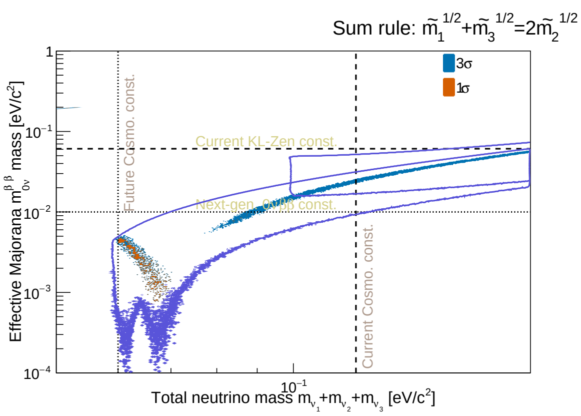
(b) Sum rule 2



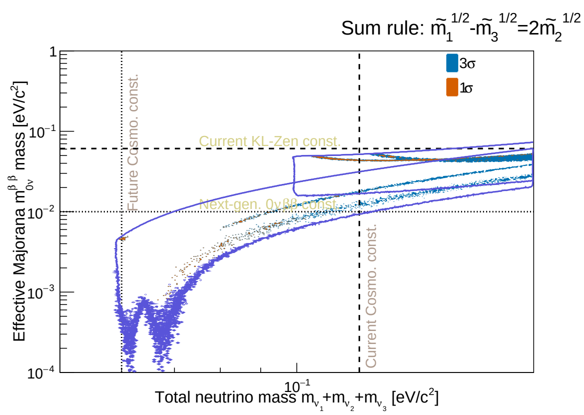
(c) Sum rule 4



(d) Sum rule 6

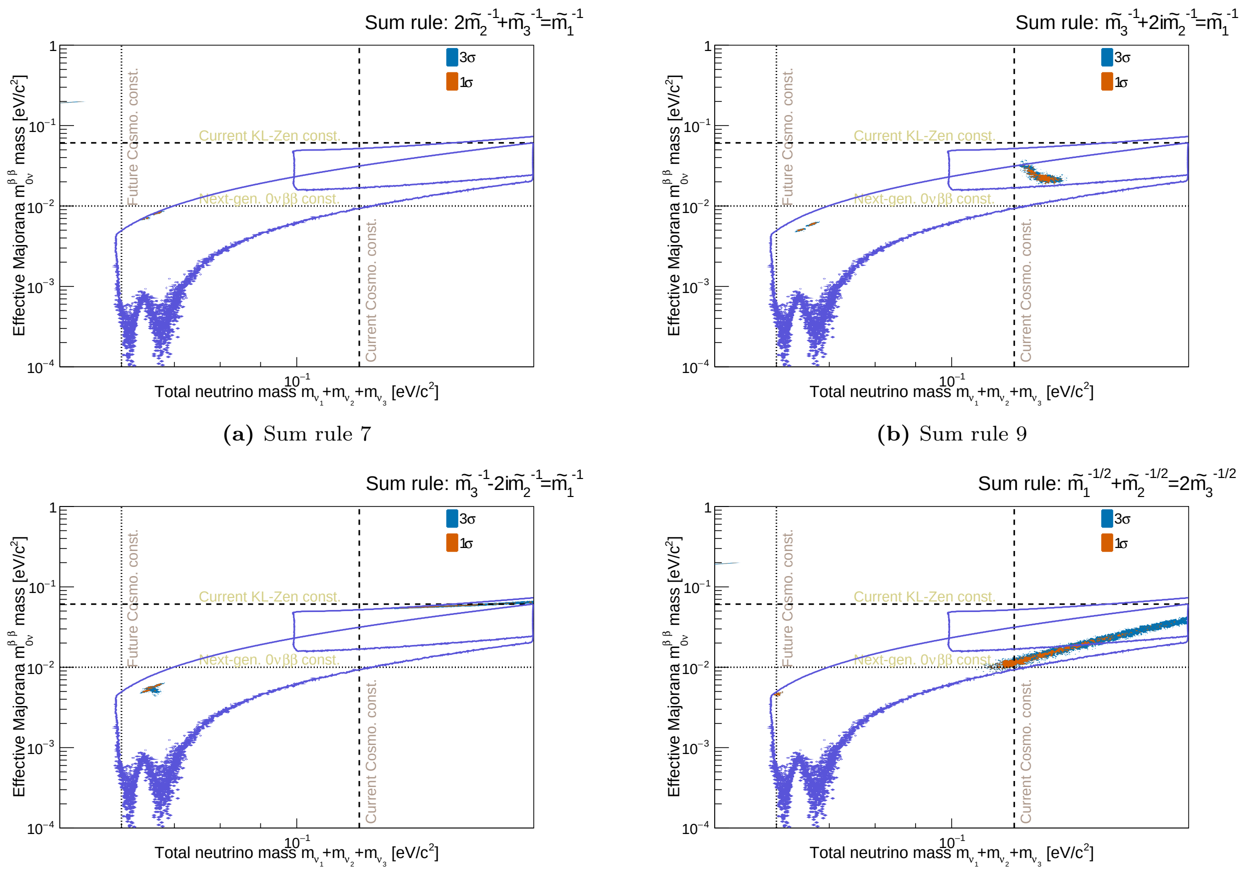


(e) Sum rule 12



(f) Sum rule 11

Figure 4: Allowed regions for sum rules 1, 2, 4, 6, 11, and 12.



(a) Sum rule 7

(b) Sum rule 9

(c) Sum rule 10

(d) Sum rule 13

Figure 5: Allowed regions for sum rules 7, 9, 10, and 13.

In contrast, Figures 5a, 5b, 5c, and 5d reveal that the allowed parameter spaces for sum rules 7, 9, 10, and 13 are relatively narrow. This narrowness suggests that these models are more constrained by current data and may offer less flexibility in the parameter space. The restrictive nature of these parameter spaces implies that any deviation from the predicted values would be more readily detectable. Conversely, the plots for sum rules 3 and 5 are not included in this discussion as these rules are excluded by more than 3-sigma from current measurements, indicating that they are inconsistent with the latest observational data and therefore not considered viable within the current experimental limits.

5 Discussion

The understanding of the rules of neutrino mass sum is intricately linked to the precision of various experimental measurements. As these measurements become increasingly accurate, they impose tighter constraints on the parameter space of possible neutrino masses, allowing us to either validate or rule out specific sum rules derived from theoretical models. This process is crucial in refining our understanding of the mechanisms behind neutrino mass generation and the flavor symmetries that might be at play.

One of the most significant measurements in this context is the effective Majorana mass ($m_{\beta\beta}$), which is probed by neutrinoless double beta decay experiments. The current generation of these experiments has already set an upper limit on ($m_{\beta\beta}$) around 0.1 eV. However, the next-generation experiments, such as those planned by the KamLAND-Zen, aim to reach sensitivities down to 0.01 eV or even lower. If these future experiments determine ($m_{\beta\beta}$) to be less than 0.01 eV, many neutrino mass sum rules predict a higher ($m_{\beta\beta}$) will be ruled out. On the other hand, if a positive signal is detected within this lower sensitivity range, it will lend strong support to sum rules that predict a relatively larger effective Majorana mass. This outcome would significantly narrow the range of viable theoretical models, thereby deepening our understanding of the underlying physics governing neutrino masses.

Cosmological observations, particularly those from the CMB and LSS surveys, provide another critical source of constraints on neutrino masses. The Planck satellite, in conjunction with other cosmological data, has established an upper limit on the sum of the neutrino masses $\sum m_\nu$ at less than 0.12 eV. Looking ahead, future experiments like the CMB-S4 are expected to tighten this constraint even further, potentially down to $\sum m_\nu \approx 0.0024$ eV according to Di Valentino [2010]. Should future cosmological measurements confirm this lower bound, sum rules that predict a higher total neutrino mass would be excluded. Contrary, if the cosmological constraint remains around 0.12 eV, sum rules that allow for slightly higher total masses would still be viable. This cross-verification between cosmological and laboratory-based measurements is essential, as it enables a robust validation of the sum rules across different experimental frameworks.

In this work, we have analyzed various sum rules for neutrino masses and their allowed regions based on current and future constraints. The results presented provide insights into the potential for probing these models with upcoming experimental and observational advancements. Further studies and more precise measurements will be essential to refine these constraints and enhance our understanding of neutrino mass ordering. As mentioned above that the Majorana phases are in range from 0 to π, we are about to run the simulation for those in range from 0 to 2π to compare the difference between them.

6 Conclusion

In this study, we explore the present landscape of leptonic mixing parameters and neutrino mass from different perspective: neutrino oscillation, cosmology and neutrino-less double beta decay. Specifically, we focus on the constrained neutrino mass spectra, and its implication to the neutrino mass sum rules which are emerged from the hypothesized "flavor" symmetry driving the leptonic mixing and neutrino mass matrix. We find that some rules are already ruled out, while some are still largely consistent. The result is shown in the Table 6 below. The future of neutrino oscillation measurement, (particularly neutrino mass ordering, CP violation) will help us to further constraint the allowed region of effective Majorana mass and total neutrino mass. The measurement of neutrino less double beta decay is possibile in near future and will strongly impact rule 1, 2, 4, 6, 11, 12. The cosmology measure the total neutrino mass and will rule out the rules 1, 2, 4 in case the constraints from cosmology is lowered to be less than 0.1 eV. Here we just look at the sum rules in their most basic form, without any correction.

Sum Rule	Group	Seesaw Type	
$\tilde{m}_1 + \tilde{m}_2 = \tilde{m}_3$	A_4[158, 247–251]; S_4[252]; A_5[75]	Weinberg	Largely allowed
$\tilde{m}_1 + \tilde{m}_2 = \tilde{m}_3$	$\Delta(54)$[253]; S_4[254]	Type II	
$\tilde{m}_1 + 2\tilde{m}_2 = \tilde{m}_3$	S_4[255]	Type II	Largely allowed
$2\tilde{m}_2 + \tilde{m}_3 = \tilde{m}_1$	A_4 [117, 118, 158, 237, 248–251, 256–262]	Weinberg	Disfavored
	S_4[133, 263]; T'[156, 246, 264–267]; T_7[268]		
$2\tilde{m}_2 + \tilde{m}_3 = \tilde{m}_1$	A_4[269]	Type II	
$\tilde{m}_1 + \tilde{m}_2 = 2\tilde{m}_3$	S_4[270]	Dirac	Largely allowed
$\tilde{m}_1 + \tilde{m}_2 = 2\tilde{m}_3$	$L_e - L_\mu - L_\tau$[271]	Type II	
$\tilde{m}_1 + \frac{\sqrt{3}+1}{2}\tilde{m}_3 = \frac{\sqrt{3}-1}{2}\tilde{m}_2$	A'_5[272]	Weinberg	Disfavored
$\tilde{m}_1^{-1} + \tilde{m}_2^{-1} = \tilde{m}_3^{-1}$	A_4[158]; S_4[247, 254]; A_5[78, 166]	Type I	Largely allowed
$\tilde{m}_1^{-1} + \tilde{m}_2^{-1} = \tilde{m}_3^{-1}$	S_4[254]	Type III	Largely allowed
$2\tilde{m}_2^{-1} + \tilde{m}_3^{-1} = \tilde{m}_1^{-1}$	A_4[118, 158, 235, 237, 238, 245, 273–281]; T'[246]	Type I	Marginally allowed
$\tilde{m}_1^{-1} + \tilde{m}_3^{-1} = 2\tilde{m}_2^{-1}$	A_4[282–284]; T'[285]	Type I	
$\tilde{m}_3^{-1} \pm 2i\tilde{m}_2^{-1} = \tilde{m}_1^{-1}$	$\Delta(96)$[286]	Type I	Marginally allowed
$\tilde{m}_1^{1/2} - \tilde{m}_3^{1/2} = 2\tilde{m}_2^{1/2}$	A_4[236]	Type I	Largely allowed
$\tilde{m}_1^{1/2} + \tilde{m}_3^{1/2} = 2\tilde{m}_2^{1/2}$	A_4[287]	Scotogenic	Largely allowed
$\tilde{m}_1^{-1/2} + \tilde{m}_2^{-1/2} = 2\tilde{m}_3^{-1/2}$	S_4[240]	Inverse	Marginally allowed

Figure 6: Conclusion table

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