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BACHELOR THESIS

Light-by-light scattering in Quantum Electrodynamics

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August 27, 2021

Acknowledgements

Let me borrow the words to express my sayings *Thank you!* to all people who have helped me on the way to complete this thesis. They are my parents, my teachers and my friends. As the quote "Energy cannot be created or destroyed, it can only be changed from one form to another", I believe that there's a flow of energy inside me that keeps me going on every day and it is from their loves and passions.

As the one supports me most on the thesis, he is Dr. Le Duc Ninh. I've learnt a lot from him. His lessons and instructions taught me the knowledge from its essence but in an intuitive way. That would help me a lot as I am in the very first stage of studying physics. Beside that, I've learnt from him the way to encouter a problem that seems frustrating at the first sight. That way is coming back to the very basic of the problem, so it is simplified and therefore, requiring the patience. Sometimes, while doing the thesis, I fell in a problem but it was his calm presence made me feel that it was not too complicated as I thought.

Thanks for many inspring lessons from the great teachers at the Theoretical Physics department. I want to say here how patient they were to let us make mistakes and do it again that helps us grow up and solidify the knowledge. Every lesson comes to us not only as an intellectual process, but also the view of life with the seemingly knowns and the unknowns, people and stories, challenges and fruits,.... I know, there will be some moments in my life, your words would always brought to me as the guidance and encouragement. My special thanks come to teacher Dang Ngoc Chau, Dr. Vu Quang Tuyen, Dr. Nguyen Huu Nha and Dr. Vo Quoc Phong.

I want to express my gratitude and admiration for the dedication of Prof. Jean Tran Thanh Van to the being of ICISE, where I have received the chance to do an internship during the thesis work. That's where I was facilitated in the great environment to acquire the guidance from Dr. Le Duc Ninh and Dr. Dao Thi Nhung in IFIRSE. That's also where I and other students be welcomed with the help from the member staffs of ICISE.

I would also like to thank Dr. Le Tho Hue as the refree of my thesis. He has given me considerate comments for a better thesis and more suggestions to develop it in the future works. That has helped me a lot at the last stage to examine the thesis.

My friends, Le Van Cuong, Nguyen Huy Hien, thank you for being together with me during this time. Thank you, my classmates of the 17^{th} Theoretical Physics class. There's no way I forget our interesting discussions during the lessons and so many things I've learnt from all of you. I would also like to thank the bachelor students Nguyen Hoang Duy Thanh, Le Truong My Hau, and Ho Quoc Trung for their sharing and support during the internship. Together we have joy and beautiful moments at ICISE.

And, saying thank you is never enough but that is how lucky I am to be the daughter of you, my parents. You're my teachers, my friends and my power. For wherever I have travelled and then come back home, you restore me and renew me.

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List of Abbreviations

LbyL scattering	**L**ight by **L**ight scattering
LO	**L**eading **O**rder
RHS	**R**ight **H**and **S**ide
LHS	**L**eft **H**and **S**ide
QED	**Q**uantum **E**lectro**D**ynamics
EM	**E**lectro**M**agnetic
DR	**D**imensional **R**egularization
C.o.M. system	**C**enter **o**f **M**ass system

Abstract

The thesis gives a consice overview of the light-by-light scattering at leading order in *Quantum Electrodynamics*. The obtained amplitudes as one- loop integrals are proven to be ultra-violet finite by the aid of *dimensional regularization*. From these finite anylitical expressions, *FORM*, a symbolic manipulation system, is ultilized to transform the amplitudes in terms of tensor integrals, which can be calculated by *COLLIER library*. Finally, the numerical results of cross sections are considered in various perspectives, so as to exhibit the influence of polarization, and how important each fermion contributes to this scattering process. As for further investigation, we expect to study the light-by-light scattering in more general theories, thus to feature the contributions from new particles beyond the Standard Model.

Chapter 0

Introduction

What phenomena could happen when two beams of light meet each other? We are all familiar to the processes result from the linear superposition of electromagnetic waves like interference, diffraction, or standing waves. But in the view of quantum mechanics, the quanta of light, photon, in the virtue of a particle, can interact with the others (e.g. Compton scattering) and themselves. The latter raises the question that is it possible for the scattering between two zero-charged, zero-mass particles like photons? While this known as light-by-light scattering is forbidden in classical EM theory, which bases on Maxwell's linear equations, it was early predicted by QED theory. In this thesis, we will derive how QED explains LbyL process $\lambda\lambda \to \lambda\lambda$ at LO. At this order, the photon scatters off one another through the virtual fermion- anti fermion pair creation and annihilation. It is also explained why that $\mathcal{O}(\alpha^4 = 3\text{x}10^{-9})$ process seldom occurs, make it difficult to test experimentally. Untill 2015, LHC used the ultra- peripheral heavy ion collision in ATLAS detector to study LbyL scattering. Doing that way increases the chance for this rare process. Once again, QED is proved to be a simple but effective theory to describe electromagnetic interaction.

Throughout this work, the QED Lagrangian for LbyL scattering is applied within the framework of perturbation theory. Then Wick's theorem helps to obtain the LO terms in the perturbation series. They are 1-loop integrals which are UV- divergent seperately. But when the sum of all possible processes is taken, it must be finite. With the help of the Dimensional regularization method, we can see clearly how these divergent terms compensate each other and vanish. For numerical calculations, FORM (Ref. [14]) was then used symbolically to transform the analytic expression to terms that can be processed by COLLIER's library (Ref. [6]). The final results were obtained for all leptons and quarks, each fermion behaves similarly, but differs by how important is its contribution to the LbyL scattering
The structure of the thesis is presented as follows:

- **CHAPTER 1: "Theoretical Background"** covers the necessary theoretical aspects for our problem. Firstly, we do a little review on *QED*. Then, we need to discuss about the *Perturbation theory*, which identifies the range of our problem. We also derive how *Wick's theorem* helps to find the LbyL's amplitude at our expected order of perturbation. The last point is about the *Dimensional Regularization method* to solve the UV divergences, which is often encoutered at loop orders.

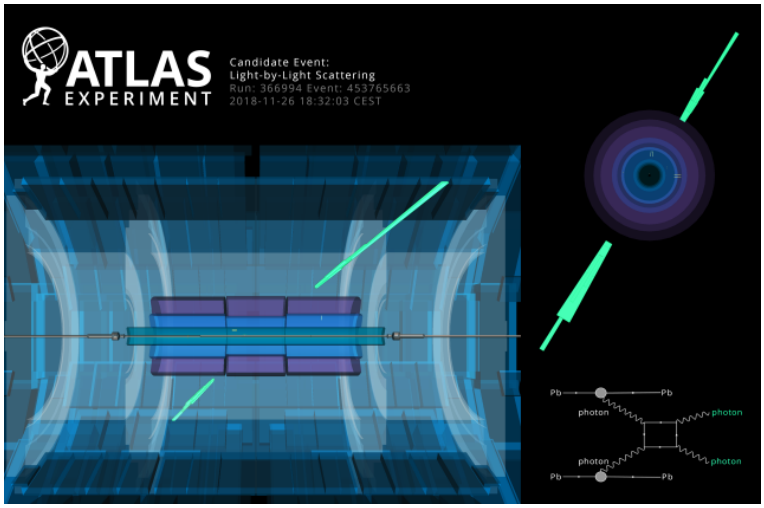


Figure 1: The first direct evidence of high-energy LbyL scattering

In this experiment, bunches of lead ions are accelerated to very high energy, pass close by each other but at a distance more than twice the radius of lead ions. At this distance, EM interaction is dominant while the strong interaction is bounded to the radius of a single proton. The electromagnetic field is then enhanced up to $10^{25}(V/m)$ and can be advantageously treated as quasi- real photons (see Ref. [2]).
Source: ATLAS Collaboration/CERN (2018).

- **CHAPTER 2: "LbyL scattering in QED"** is the application of those theories to our particular LbyL scattering. For simplicity, we first restrict our calculation by considering electron (positron) as the only fermion (anti- fermion). The probability amplitudes and the corresponding diagrams for all possible processes as four-leg loops are shown. One attribute of LbyL scattering is that it is independent of UV-divergences when the sum of terms is considered. It can be proved in the light of the Dimensional regularization. The convention to express the the amplitude in terms of the tensor integrals is also introduced.

- **CHAPTER 3: "Numerical methods"** introduces the tools to obtain the final numerical results. First, *FORM* is used to transform the analytical expressions of transition amplitudes (as obtained from Wick's theorem) to the forms as written in terms of Passarino-Veltman loop integrals (A, B, C, D, ...*functions*). These loop integrals can be calculated by COLLIER's library. The numerical results of LbyL scattering's amplitude is considered at different views: in each polarized case of photons, in unpolarized case, in different cases of fermions, ... for comparisions.

Chapter 1

Theoretical Background

1.1 Overview of the Quantum Electrodynamics

1.1.1 Electromagnetic field

Electromagnetic (EM) field is described through the electric field $\vec{E}(x)$ and the magnetic field $\vec{B}(x)$, which satisfy Maxwell's equations[1]:

$$\begin{aligned}
&\vec{\nabla}.\vec{E} = \rho, \\
&\vec{\nabla}.\vec{B} = 0, \\
&\vec{\nabla}\mathrm{x}\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \\
&\vec{\nabla}\mathrm{x}\vec{B} = \left(\vec{J} + \frac{\partial \vec{E}}{\partial t}\right),
\end{aligned} \tag{1.1}$$

in the presence of charge density ρ and current density $\vec{J}$. Now, we define an anti-symmetric 4-tensor $F_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ such that:

$$\begin{aligned}
E_i &= F_{0i}; \\
B_i &= \epsilon_{ijk} F^{jk},
\end{aligned} \tag{1.2}$$

where ϵ_{ijk} is the Levi- Civita tensor $(i, j, k = 1, 2, 3)$.
We also introduce the 4-vector $A_\mu(x)$ so that:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.3}$$

while $A_\mu(x)$ is real. Hence, Maxwell's equations are now expressed much more neatly as:

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = J_\nu, \tag{1.4}$$

where we have defined a 4-current $J_\mu = (\rho, \vec{J})$. Since then, the EM field is simply described by the vector- field $A_\mu(x)$. Applying the gauge transformation on the field $A_\mu(x)$ with an arbitrary function $\chi(x)$:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \chi(x), \tag{1.5}$$

[1] All the formulas are hence represented in natural units (see Appendix A)

we obtain as if nothing really happens to the EM field:

$$\begin{aligned} & F'_{\mu\nu} = F_{\mu\nu}, \\ \text{and} \qquad & \Box A'_\nu - \partial^\mu \partial_\nu A'_\mu = J'_\nu, \end{aligned} \tag{1.6}$$

where the field is free when $J_\nu = 0$ or else interacting with a current $J_\nu \neq 0$.
So we say ***the free EM field is gauge invariant*** ($J_\nu = 0$).

Lagrangian of the free EM field

We should remark that the two most important physical constraints (or symmetries) on the EM field are the *Lorentz invariance* and the *gauge invariance*. Thus, the Lagrangian density (or Lagrangian) of the free EM field is built so that those two symmetries are preserved. We may choose the preferred Lagrangian according to those requirements as[2]:

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.7}$$

The equation of motion for the free EM field is then derived to be:

$$\Box A_\nu - \partial_\nu \partial^\mu A_\mu = 0, \tag{1.8}$$

i.e. the Maxwell's equation Eq. (1.4) in the absence of charge $J_\nu = 0$.

Plane wave solution of the EM field

Special solutions for $A^\mu(x)$ from the equation of motion Eq. (1.8) are the plane waves:

$$A^\mu(x) = w^\mu(\vec{k}) e^{\pm i(\omega t - \vec{k}.\vec{r})}, \tag{1.9}$$

where w^μ is a 4- vector, which represents the wave amplitude and $\vec{k} = (k_1, k_2, k_3)$ is an arbitrary 3-wave vector so that $\omega = \left|\vec{k}\right|$. The vector w^μ lives in an space spanned from four basis vectors ε^μ_η ($\eta = 0, 1, 2, 3$), which will be considered as polarization vectors of photon. We would see later that the number of basis vectors for the EM field no more equals to four. Therefore, we obtain the general solution of Eq. (1.8) as the linear superposition of Eq. (1.9) as:

$$A^\mu(x) = \sum_\eta \int d^3\vec{k} \left[a^\eta_{\vec{k}} \varepsilon^\mu_\eta(\vec{k}) e^{-i(\omega t - \vec{k}\vec{r})} + a^{*\eta}_{\vec{k}} \varepsilon^{*\mu}_\eta(\vec{k}) e^{i(\omega t - \vec{k}\vec{r})} \right], \tag{1.10}$$

so that the requirement of real $A_\mu(x)$ is obtained.
Also remind the De- Broglie hypothesis of wave- particle duality:

$$(\omega, \vec{k}) \longrightarrow (E, \vec{p}), \tag{1.11}$$

[2]The Lagrangian of the EM field can also derived from the Lagrangian of a vector field, in which EM field is the *massless* case ($m = 0$) because photon is massless.

so that the EM field is also the field of its quanta- *photons*. Thus, we can rewrite Eq. (1.10) as:

$$A^{\mu}(x) = \sum_{\eta} \int d^3\vec{p} \left[a_{\vec{p}} \varepsilon^{\mu}_{\eta}(\vec{p}) e^{-ipx} + a^{*}_{\vec{p}} \varepsilon^{*\mu}_{\eta}(\vec{p}) e^{ipx} \right], \tag{1.12}$$

where $p = (E, \vec{p})$ and $x = (t, \vec{x})$, then $E = |\vec{p}|$ or $p^2 = 0$.

Quantization of the EM field

The conjugate- momentum of the free EM field is derived from its Lagrangian as:

$$\Pi^{\mu}(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}(x)} = F^{\mu 0}. \tag{1.13}$$

From the definition of $F^{\mu\nu}$ (Eq. (1.3)), we obtain:

$$\Pi^0 = 0 \qquad \text{and} \qquad \Pi^i = E^i. \tag{1.14}$$

For quantisation, we might expect the following commutation relations:

$$\begin{cases} \left[A_{\mu}(t, \vec{x}), \Pi_{\nu}(t, \vec{x'}) \right] = i\delta_{\mu\nu}\delta^3(\vec{x} - \vec{x'}), \\ \left[A_{\mu}(t, \vec{x}), A_{\nu}(t, \vec{x'}) \right] = \left[\Pi_{\mu}(t, \vec{x}), \Pi_{\nu}(t, \vec{x'}) \right] = 0. \end{cases} \tag{1.15}$$

While the last two relations of Eq. (1.15) are automatically satisfied, there's a problem with the first relation due to:

$$\left[A_0(t, \vec{x}), \Pi_0(t, \vec{x'}) \right] = 0. \tag{1.16}$$

That problem comes from the commutation relations themselves when applying to such a special case as EM field, and from the gauge invariance, which puts some more constraints on the field $A_{\mu}(x)$. To solve this problem, we use the gauge symmetry Eq. (1.5) to impose the condition :

$$A_0 = 0; \quad \text{and} \quad \vec{\nabla}.\vec{A} = 0. \tag{1.17}$$

This gauge condition is the *Coulomb gauge*. The commutation relations for EM fields, which is then modified according to the Coulomb gauge, become:

$$\begin{cases} \left[A_i(t, \vec{x}) \, , \, \Pi_j(t, \vec{x'}) \right] = i \int d^3\vec{k} \left(\delta_{ij} - \dfrac{k_i k_j}{\left|\vec{k}\right|^2} \right) e^{i\vec{k}(\vec{x} - \vec{x'})}, \\ \left[A_i(t, \vec{x}) \, , \, A_j(t, \vec{x'}) \right] = \left[\Pi_i(t, \vec{x}) \, , \, \Pi_j(t, \vec{x'}) \right] = 0, \end{cases} \tag{1.18}$$

where $i, j = 1, 2, 3$.
From Eq. (1.12), for $A^0(x) = 0$, we would set $\varepsilon^0_{\eta} = 0$. And from the gauge condition Eq. (1.17), we obtain:

$$\vec{k}\vec{\varepsilon_{\eta}} = 0 \qquad (\eta = 1, 2). \tag{1.19}$$

Thus, for a specific 3-momentum $\vec{k}$, $\vec{\varepsilon_\eta}$ must exist in the plane orthogonal to $\vec{k}$. In other words, there just needs two basis vectors. The number of basis vectors is also the number of polarization states of photons, therefore is two.
By imposing the normalization condition on the basis vectors:

$$\vec{\varepsilon}^*_\eta(\vec{k})\vec{\varepsilon}_{\eta'}(\vec{k}) = \delta_{\eta\eta'}, \tag{1.20}$$

so that the basis vectors can be chosen to fulfill the polarization sum (see Ref. [12]):

$$\sum_{\eta=1}^{2} \varepsilon^{*\mu}_\eta \varepsilon^\nu_\eta := -g^{\mu\nu}, \tag{1.21}$$

indeed this is the consequence of gauge symmetry in essence.
The quantization then treats $a^\eta_{\vec{k}}$ and $a^{*\eta}_{\vec{k}}$ as operators:

$$a^\eta_{\vec{k}}\ ,\ a^{*\eta}_{\vec{k}} \rightarrow \widehat{a}^\eta_{\vec{k}}\ ,\ \widehat{a}^{\dagger\eta}_{\vec{k}}. \tag{1.22}$$

Now, we re- express the Eq. (1.12) as:

$$A^\mu(x) = \sum_{\eta=1}^{2} \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \left[\widehat{a}^\eta_{\vec{k}}\varepsilon^\mu_\eta(\vec{k})e^{-ikx} + \widehat{a}^{\dagger\eta}_{\vec{k}}\varepsilon^{*\mu}_\eta(\vec{k})e^{ikx}\right], \tag{1.23}$$

where $E_k = k_0 = \left|\vec{k}\right|$ and the factor in the denomiator just appear for normalization condition of the field. From that and Eq. (1.18), we obtain the commutation relations of those operators as:

$$\begin{cases} \left[\widehat{a}^\eta_{\vec{k}}\ ,\ \widehat{a}^{\dagger\eta'}_{\vec{k}'}\right] = (2\pi)^3\delta_{\eta\eta'}\delta^3(\vec{k}-\vec{k}'), \\ \left[\widehat{a}^\eta_{\vec{k}}\ ,\ \widehat{a}^{\eta'}_{\vec{k}'}\right] = \left[\widehat{a}^{\dagger\eta}_{\vec{k}}, \widehat{a}^{\dagger\eta'}_{\vec{k}'}\right] = 0. \end{cases} \tag{1.24}$$

We define the vacuum state to be the one that is annihilated by any operator $\widehat{a}$:

$$\widehat{a}^\eta_{\vec{k}}\left|0\right\rangle = 0 \quad \text{for any } \vec{k}, \eta \tag{1.25}$$

and the state of a photon with a definite momentum k and polarization η as:

$$\left|k;\eta\right\rangle = \sqrt{2E_k}\widehat{a}^{\dagger\eta}_{\vec{k}}\left|0\right\rangle. \tag{1.26}$$

So that we can use $\widehat{a}$ and $\widehat{a}^\dagger$ to build out all states of quantized EM field. They are correspondingly called the annihilation and creation operators.

Photon propagator

We choose to find the propagator of photon through the contraction of fields (see Eq. (1.96)).

$$iD_{F\alpha\beta}(x_1 - x_2) = \langle 0| A_\alpha(x_1)A_\beta(x_2) |0\rangle = \overbrace{A_\alpha(x_1)A_\beta(x_2)} = \left[A_\alpha^-(x_1), A_\beta^+(x_2)\right], \quad (1.27)$$

where we suppose $t_1 > t_2$, i.e. it relates to the amplitude for a photon propagating from x_2 to x_1. The A^+, A^- parts are denoted for the terms of $a^\dagger$ and a^- in Eq. (1.23) correspondingly. Thus we obtain:

$$iD_{F\alpha\beta}(x_1 - x_2) = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \int \frac{d^3\vec{k'}}{(2\pi)^3\sqrt{2E'_k}} \sum_{\eta,\eta'=1}^{2} \varepsilon_\eta^\alpha(\vec{k})\varepsilon_\eta^{*\beta}(\vec{k'}) \left[\widehat{a}_{\vec{k}}^{\eta}\ ,\ \widehat{a}_{\vec{k'}}^{\dagger\eta'}\right] e^{-ikx_1}e^{ik'x_2}. \quad (1.28)$$

Using the commutation Eq. (1.24) and the chosen spin sum Eq. (1.21), after some transformations, we have:

$$iD_{F\alpha\beta}(x_1 - x_2) = \int d^3\vec{k}\frac{1}{(2\pi)^3}(-g_{\alpha\beta}) \left[\frac{e^{-iE_k(t_1-t_2)}}{2E_k}\right] e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}. \quad (1.29)$$

Now we denote:

$$f(E) = \frac{e^{-iE(t_1-t_2)}}{E + E_k}. \quad (1.30)$$

Using Cauchy integral theorem for Eq. (1.30), we have:

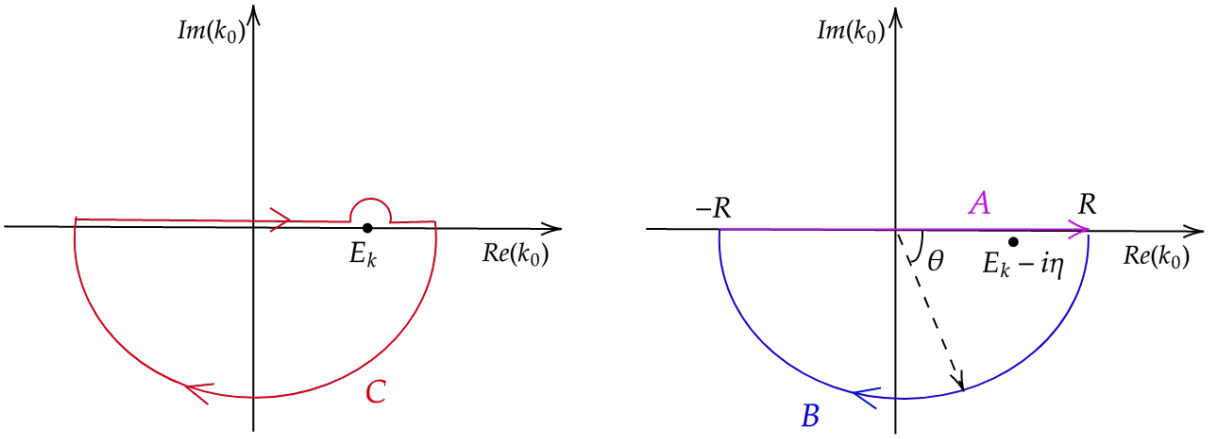
$$f(E) = \frac{1}{2\pi i}\oint \frac{f(k_0)}{k_0 - E_k}dk_0. \quad (1.31)$$

Thus, the propagator is now:

$$\begin{aligned} iD_{F\alpha\beta}(x_1 - x_2) &= \int d^3\vec{k}\frac{1}{(2\pi)^3}(-g_{\alpha\beta})f(E_k)e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)} \\ &= \int d^3\vec{k}\frac{1}{(2\pi)^3}(-g_{\alpha\beta})e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}\frac{1}{2\pi i}\oint dk_0\frac{e^{-ik_0(t_1-t_2)}}{(k_0 + E_k)(k_0 - E_k)}. \end{aligned} \quad (1.32)$$

Firstly, we consider the integral in k_0 in Eq. (1.32):

$$
\begin{aligned}
&\frac{1}{2\pi i}\oint dk_0 \frac{e^{-ik_0(t_1-t_2)}}{(k_0+E_k)(k_0-E_k)} \\
&\overset{E_k\to E_k-i\varsigma}{=} -\frac{1}{2\pi i}\oint_{\mathcal{C}} dk_0 \underbrace{\frac{e^{-ik_0(t_1-t_2)}}{(k_0+E_k-i\varsigma)(k_0-E_k+i\varsigma)}}_{=k_0^2-E_k^2+2i\varsigma E+\eta^2=k_0^2-E_k^2+i\varepsilon}\Big|_{\varsigma\to 0^+} \qquad \text{clock-wise} \\
&= -\frac{1}{2\pi i}\oint_{\mathcal{C}} dk_0 \frac{e^{-ik_0(t_1-t_2)}}{k_0^2-E_k^2+i\varepsilon}\Big|_{\varepsilon\to 0^+} \\
&= \frac{i}{2\pi}\left[\int_{\mathcal{A}} dk_0 \frac{e^{-ik_0(t_1-t_2)}}{k_0^2-E_k^2+i\varepsilon} + \int_{\mathcal{B}} dk_0 \frac{e^{-ik_0(t_1-t_2)}}{k_0^2-E_k^2+i\varepsilon}\right] \\
&= \frac{i}{2\pi}\lim_{R\to\infty}\left[\int_{-R}^{R} dk_0 \frac{e^{-ik_0(t_1-t_2)}}{k_0^2-E_k^2+i\varepsilon} + iR\int_0^{\pi} d\theta e^{i\theta} \underbrace{e^{-iR\cos\theta(t_1-t_2)}}_{\text{always finite}} \underbrace{\frac{e^{R\sin\theta(t_1-t_2)}}{R^2-E_k^2+i\varepsilon}}_{\to 0 \text{ when } R\to\infty}\right] \\
&= \frac{i}{2\pi}\int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0(t_1-t_2)}}{k_0^2-E_k^2+i\varepsilon}.
\end{aligned} \tag{1.33}
$$



Thus, Eq. (1.32) now becomes:

$$
\begin{aligned}
iD_{F\alpha\beta}(x_1-x_2) &= \int d^3\vec{k}\frac{1}{(2\pi)^3}(-g_{\alpha\beta})e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)}\frac{1}{2\pi i}\frac{i}{2\pi}\int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0(t_1-t_2)}}{k_0^2-E_k^2+i\varepsilon} \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\alpha\beta}}{k_0^2-E_k^2+i\varepsilon}e^{-ik(x_1-x_2)} \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\alpha\beta}}{k^2+i\varepsilon}e^{-ik(x_1-x_2)},
\end{aligned} \tag{1.34}
$$

The last equation of Eq. (1.34) happens because $E_k^2 = \left|\vec{k}\right|^2$. We define here a new 4-momentum $k = (k_0, \vec{k})$, which is no more on-shell, $k_0^2 - \left|\vec{k}\right|^2 \neq 0$. What is important here is the bringing of the positively infinitesimal ε so that the integrals are well-defined.
At last, we obtain the Feynman propagator of photon in momentum presentation as:

$$\boxed{iD_{F\alpha\beta}(k) = \frac{-ig_{\alpha\beta}}{k^2 + i\varepsilon}}, \tag{1.35}$$

with $\varepsilon \to 0^+$.

1.1.2 Dirac field

Lagrangian of free Dirac field

A spin 1/2- particle as a fermion (e.g.electron, positron,...) with mass m is described by a 4-component complex field , called *spinor*:

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}. \tag{1.36}$$

The dynamics of the free field is contained in Dirac Lagrangian:

$$\mathcal{L}_D = \overline{\Psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\Psi(x), \tag{1.37}$$

where

$$\overline{\Psi} = \Psi^\dagger \gamma^0 = (\Psi_1^*, \Psi_2^*, -\Psi_3^*, -\Psi_4^*), \tag{1.38}$$

and the Dirac matrices γ^μ ($\mu = 0, 1, 2, 3$), which are the 4 x 4 matrices, must satisfy the algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.\mathbb{1}. \tag{1.39}$$

We treat those matrices as the components of a matrix vector such that:

$$\gamma^\mu = \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3\right), \qquad \gamma_\mu = \left(\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\right). \tag{1.40}$$

We choose to express the Dirac matrices in the Dirac presentation from Pauli matrices $\sigma_{1,2,3}$ (see Appendix B) as the following way:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \tag{1.41}$$

The Lagrangian of Dirac field produces the Dirac equation as the equation of motion:

$$i\gamma^\mu \partial_\mu \Psi(x) = m\Psi(x), \tag{1.42}$$

and for the conjugate field:

$$i\partial_\mu \overline{\Psi}(x)\gamma^\mu = -m\overline{\Psi}(x). \tag{1.43}$$

Since now, we often denote $\not{A} = \gamma^\mu A_\mu$ for any 4-vector A_μ, thus $\not{\partial} = \gamma^\mu \partial_\mu$.

Plane wave solution of Dirac field

There're two special solutions of Dirac equation Eq. (1.42) as:

$$u(\vec{p})e^{-ipx}; \qquad\qquad v(\vec{p})e^{ipx} \tag{1.44}$$

corresponding to particle's and anti-particle's wave functions. Thus, the general solution of Dirac field is a linear combination of those wave functions as:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} \left[u_s(\vec{p}) c^s_{\vec{p}} e^{-ipx} + v_s(\vec{p}) d^{*s}_{\vec{p}} e^{ipx} \right], \tag{1.45}$$

$$\overline{\Psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} \left[\bar{u}_s(\vec{p}) c^{*s}_{\vec{p}} e^{ipx} + \bar{v}_s(\vec{p}) d^{s}_{\vec{p}} e^{-ipx} \right], \tag{1.46}$$

where s corresponds to two helicities (spin) of fermions; $p = (E_p, \vec{p})$ so that $p^2 = m^2$. Applying the solutions Eq. (1.46) to Eq. (1.42), we obtain Dirac equations in momentum space as following:

$$\begin{cases} (\not{p} - m)\, u_s(\vec{p}) = 0, \\ (\not{p} + m)\, v_s(\vec{p}) = 0 \end{cases}; \qquad \text{and} \qquad \begin{cases} \bar{u}_s(\vec{p})\, (\not{p} - m) = 0, \\ \bar{v}_s(\vec{p})\, (\not{p} + m) = 0. \end{cases} \tag{1.47}$$

The normalization conditions on the spinors are imposed so as:

$$\bar{u}_s(\vec{p}) u_r(\vec{p}) = 2m\delta_{rs}; \qquad\qquad \bar{v}_s(\vec{p}) v_r(\vec{p}) = -2m\delta_{rs}; \tag{1.48}$$

Quantization of Dirac field

The quantization of Dirac field is formulated from an important property of fermions, which says that no two fermions exists in the same state. So, the fermion field adopts the anti-commuation relations instead of the commutation ones of the photon fields:

$$\begin{cases} \left\{ \Psi_a(t, \vec{x}), \Psi^\dagger_b(t, \vec{x'}) \right\} = \delta^3(\vec{x} - \vec{x'})\delta_{ab}, \\ \left\{ \Psi_a(t, \vec{x}), \Psi_b(t, \vec{x'}) \right\} = \left\{ \Psi^\dagger_a(t, \vec{x}), \Psi^\dagger_b(t, \vec{x'}) \right\} = 0, \end{cases} \tag{1.49}$$

in which, $a, b = 1, 2, 3, 4$ is the component field index. Note that, in case of Dirac field, the momentum conjugate field $\Pi(x)$ to $\Psi(x)$ is its conjugate field $\overline{\Psi}(x)$.
The wave- function coefficients in Eq. (1.46) now become the operators:

$$\begin{aligned} c_{\vec{p}}\ ,\ c^*_{\vec{p}} &\to \widehat{c}_{\vec{p}}\ ,\ \widehat{c}^*_{\vec{p}}, \\ d_{\vec{p}}\ ,\ d^*_{\vec{p}} &\to \widehat{d}_{\vec{p}}\ ,\ \widehat{d}^*_{\vec{p}}. \end{aligned} \tag{1.50}$$

The vacuum state $|0\rangle$is defined to be annihilated by $c_{\vec{p}}$, $d_{\vec{p}}$:

$$\widehat{c}^s_{\vec{p}}\, |0\rangle = 0, \tag{1.51}$$

$$\widehat{d}^{s}_{\vec{p}}\,|0\rangle = 0, \tag{1.52}$$

for any $\vec{p}$, s. The state of a particle with a definite momentum p and spin s is built in the following way:

$$|p;s\rangle = \sqrt{2E_p}\widehat{c}^{\dagger s}_{\vec{p}}\,|0\rangle\,. \tag{1.53}$$

And for the one of anti-particle:

$$|p;s\rangle = \sqrt{2E_p}\widehat{d}^{\dagger s}_{\vec{p}}\,|0\rangle\,. \tag{1.54}$$

Fermion propagator

Similarly, the contraction of fermion fields gives us its propagator (see Eq. (1.96)):

$$iS_{Fab}(x_1 - x_2) = \overbrace{\Psi_a(x_1)\overline{\Psi}_b}(x_2) = \left\{\Psi^-_a(x_1), \overline{\Psi}^+_b(x_2)\right\}. \tag{1.55}$$

To obtain the propagator in momentum representation, we should follow the same procedure as in the case of photon propagator (see Eq. (1.1.1)). At last, we would receive:

$$\boxed{iS_{Fab}(p) = i\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}}, \tag{1.56}$$

with a positively infinitesimal ε.

1.1.3 Interacting fields- QED Lagrangian

QED describes the interactions between photons and fermions (charged particle). The essence of this theory is the requirement that it must be invariant under ***gauge transformation*** as following:

- For photon fields $A_\mu(x)$:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x), \tag{1.57}$$

  where e is the electron charge.

- For fermion fields $\Psi(x)$:

$$\begin{aligned}\Psi(x) &\to \Psi'(x) = e^{-i\alpha(x)}\Psi(x),\\ \overline{\Psi}(x) &\to \overline{\Psi'}(x) = e^{i\alpha(x)}\overline{\Psi}(x).\end{aligned} \tag{1.58}$$

Now let us first consider the effect of gauge transformation on the fermion Lagrangian Eq. (1.37). It is shown not to be gauge invariant, actually:

$$\mathcal{L}_D \to \mathcal{L}'_D = \mathcal{L}_D + \overline{\Psi}(x)\gamma^\mu\partial_\mu\alpha(x)\Psi(x), \tag{1.59}$$

with an arbitrary function $\alpha(x)$ to localize the gauge transformation. We see that the variant term is due to $\partial_\mu\alpha(x)$, which is indeed a vector. Hoping to eliminate this term, we

add a new *massless* vector field $\mathcal{A}_\mu(x)$, which transforms in the same way as the photon field $A_\mu(x)$ under gauge transformation Eq. (1.57) to Dirac Lagrangian as the following way:

$$\mathcal{L}_\mathcal{A} = \mathcal{L}_D - e\overline{\Psi}(x)\gamma^\mu \mathcal{A}_\mu(x)\Psi(x). \tag{1.60}$$

It's easy to see that the new Lagrangian is invariant under gauge transformation. The new Lagrangian term connects the spinor field $\Psi(x)$ to the gauge field $\mathcal{A}_\mu(x)$, thus describes the interaction between those fields. Since QED is for the interaction between photons and fermions, it should be right that the gauge field is the photon field $\mathcal{A}_\mu(x) = A_\mu(x)$. Then we should obtain the Lagrangian for QED as:

$$\boxed{\mathcal{L}_{QED} = i\overline{\Psi}(x)\gamma^\mu \left[\partial_\mu + ieA_\mu(x)\right]\Psi(x) - m\overline{\Psi}(x)\Psi(x) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}, \tag{1.61}$$

where we have brought the free Lagrangian of the free photon field Eq. (1.7) there, so it would fully describe the photon-fermion field system . We should check how the QED Lagrangian Eq. (1.61) be invariant under the gauge transformations of the fields (Eq. (1.57), Eq. (1.58)). It is often that we express the QED Lagrangian as:

$$\boxed{\mathcal{L}_{QED} = i\overline{\Psi}(x)\gamma^\mu D_\mu\Psi(x) - m\overline{\Psi}(x)\Psi(x) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}, \tag{1.62}$$

where we have introduced the covariant derivative $D_\mu = \partial_\mu + ieA_\mu(x)$.

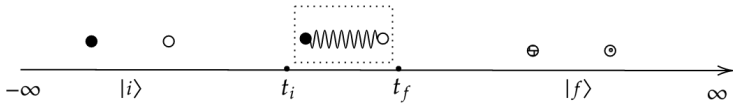
1.2 Overview of the Perturbation theory

The perturbation theory is used to solve the interacting problem. In free theories, the solution is determined by the planar wave expansion. In contrast, the reality becomes blurred when the particles interact with the other ones and then, disappear for something else to appear. Fortunately, physics does its best to simplify a complicated scattering by suitable assumptions. Firstly, we assume that the initial and the final states of the process are asymptotic free fields. Secondly, by the perturbation theory, we split the total Lagrangian (or Hamiltonian) into the free and the interaction parts. The latter, in cases when the interaction is sufficently weak, is treated as a perturbation. This makes QED to be a potential candidate because its dimensionless coupling constant is the fine structure constant $\alpha \approx 1/137$. Due to the assumptions above, it has been shown easier to represent the fields and the states in the interaction picture. Then the solution is expanded as a series of perturbed orders. We will see how to pick up one of these terms for a specific transition in question.

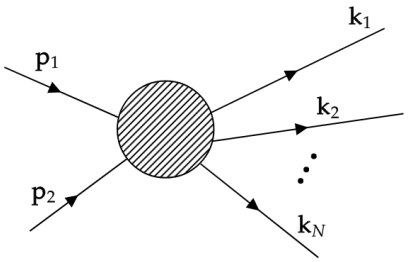
1.2.1 Asymptotic fields

Let's set up a system which changes over time t in such a way like this: Initally, we have a system of different particles that are far enough from each other so that they can move freely without any interaction. The system is now in a definite state $|i\rangle$. Nothing changes until t_i, the particles suddenly "feel" the others, i.e. begin to interact, and "perturb" the system for a little while. The system is no more in $|i\rangle$ but kind of some

complicated mixing state in a black box. As the time proceeds to t_f, after the interaction, new particles are born and moving far away from each other, the system turns to another definite, free state $|f\rangle$ as the final one. [3]



Come back to our process, where all we know is only the initial state, the final state and our interacting theory. The unknown is what's happening during the interaction.



In fact, the timescales over which the interaction occurs are extremely short. So it is close to the truth when we assume the fields in our interacting theory to behave asymtotically as that way (Ref. [10]). There,

$$\begin{aligned} \text{the initial state:} \quad |i\rangle &= |\mathbf{p_1}, \mathbf{p_2}; in\rangle = a^+_{in}(\mathbf{p_1}) a^+_{in}(\mathbf{p_2}) \,|0\rangle \text{ during } -\infty \leq t \leq t_i, \\ \text{and the final state:} \quad |f\rangle &= |\mathbf{k_1}, \mathbf{k_2}, \ldots, \mathbf{k_N}; out\rangle = a^+_{out}(\mathbf{k_1}) \ldots a^+_{out}(\mathbf{k_N}) \,|0\rangle \text{ during } t_f \leq t \leq \infty, \end{aligned}$$

are well-defined particle states that live in the free theories (this assumption suits well for the states in our scattering process, though it is no more reasonable for bound states). The interaction is adiabatically switched on and off in the interval of time $t_i < t < t_f$. Note that the free states are generated by creation operators a^+_{in} and a^+_{out} from the vaccum. And this vacuum is stable and *unique* (which differs from the vacuum state at general time t in the full interaction theory):

$$|0\rangle = |0; in\rangle = |0; out\rangle \,. \tag{1.63}$$

1.2.2 Interaction picture

The above discussion assumes that the particles behave free at asymtotic times $t \to \pm\infty$ and the interaction only exists at intermediate times. We have derived the field operators in the free theories, they are constructed in the Heisenberg picture. While there, only the operators evolve with time with the full Hamiltonian , and the states are unchanged. That leads to solving a non- linear equation for the fields as the interaction part is added. This solution for a general interaction seems to be impossible. Though different quantum

[3] The figures without source in the thesis are all created by a very helpful online tool- *Mathcha* *https://www.mathcha.io/*

pictures are physically equivalent, the interaction picture (I.P) (also called *Dirac picture*), as its name suggests, ultilizes all those features of the perturbation formalism. To be clear, let's have a brief summary of quantum pictures in the table below:

Evolution of:	Pictures		
	Schrodinger	Heisenberg	Dirac
Ket states	$\|\psi_S(t)\rangle = e^{-iH_S t}\|\psi_S(0)\rangle$	constant	$\|\psi_I(t)\rangle = e^{iH_{0,S}t}\|\psi_S(t)\rangle$
Operators	constant	$A_H(t) = e^{iH_S t}A_S e^{-iH_S t}$	$A_I(t) = e^{iH_{0,S}t}A_S e^{-iH_{0,S}t}$

Along the way, the full Hamiltonian H is divided into the free and the interaction parts:

$$H = H_0 + H_i, \tag{1.64}$$

where the lower indices S, H, I denoting for quantities in Schrodinger, Heisenberg and interaction picture, correspondingly. Due to the above identifications, we have $H_{0,S} = H_{0,I} = H_0$. And here's the thing, the time dependence is for both operators and states in I.P, but each is controlled by one part of the Hamiltonian. While the time evolution of the field operators just depend on the free Hamiltonian, only the interaction Hamiltonian affects the state vectors. From the Schrodinger equation for $|\Psi_S\rangle$, we can easily prove that:

$$i\frac{d}{dt}|\Psi_I(t)\rangle = H_{i,I}(t)\,|\Psi_I(t)\rangle\,. \tag{1.65}$$

Also notice that:

$$A_I(t) \to A_H(t) \text{ when } H = H_0 \text{ (no interaction).} \tag{1.66}$$

Thus, two most important characteristics of the I.P that make it serves best for the interacting theories are: *First*, the states in I.P are time- indepedent when the interaction is turned off; *second*, the equation of motion for the field operators in I.P are the same to the free ones in Heisenberg picture. And those are what we expect from the asymptotic assumption, despite any complication from the interaction. Accordingly, we can therefore retain the results of the free field theories, as we suppose to be known.

1.2.3 The S-matrix expansion

Until now, we are ready to build the S-matrix for a scattering process. Our process originates from a given initial state $|i\rangle$ at $t = -\infty$ with definite properties (number of particles, spin, momentums, ...). Thus we can redefine:

$$|i\rangle = |\Psi(-\infty)\rangle\,. \tag{1.67}$$

It evolves through the interactions and long after that, it finally reaches a free state $|\Psi(\infty)\rangle$ at $t = \infty$. We define ***S- matrix*** to be the one relates $|\Psi(-\infty)\rangle$ and $|\Psi(\infty)\rangle$ as:

$$|\Psi(\infty)\rangle = S\,|\Psi(-\infty)\rangle = S\,|i\rangle\,. \tag{1.68}$$

In other words, S- matrix contains all information of the transition process. If we want to measure the probability that the transition ends at a particular state $|f\rangle$, we can project the state $|\Psi(\infty)\rangle$ on the possible final eigenstate $|f\rangle$ by taking their dot product as:

$$\langle f|\Psi(\infty)\rangle = \langle f|\,S\,|\Psi(-\infty)\rangle = \langle f|\,S\,|i\rangle = S_{fi}. \tag{1.69}$$

Thus, the *S- matrix element* S_{fi} is defined as the probability amplitude for the transition from $|i\rangle$ to a specific state $|f\rangle$.
At the next step, we will derive the S- matrix, and then we will see the effect of the interaction exhibits on the perturbation series of the S- matrix.
We denote $|\Psi(t)\rangle$ as the state vector of the whole system at time t in our scattering process. All the quantities are hence presented in I.P, so the label "I" is omitted. From the equation of motion for $|\Psi(t)\rangle$ in Eq. (1.65) and the condition of the initial state $|i\rangle$, we can rewrite the equation in the integral form as:

$$|\Psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_i(t_1)\,|\Psi(t_1)\rangle\,. \tag{1.70}$$

That means our state at time t need the information of itself in the past ($t_1 < t$). This kind of equation can only be solved iteratively such that:

$$|\Psi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_i(t_1)\left(|i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 H_i(t_2)\,|\Psi(t_2)\rangle\right) \tag{1.71}$$

$$= |i\rangle + (-i)\int_{-\infty}^{t} dt_1 H_i(t_1)\,|i\rangle + (-i)^2\int_{-\infty}^{t} dt_1\int_{-\infty}^{t_1} dt_2 H_i(t_1)H_i(t_2)\,|\Psi(t_2)\rangle\,, \tag{1.72}$$

$$\tag{1.73}$$

and so on. In the limit $t \to \infty$, make a comparision with Eq. (1.68), the S-matrix:

$$S = \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n H_i(t_1)H_i(t_2)\ldots H_i(t_n). \tag{1.74}$$

To make more progress on the form of the S-matrix, we first try with a particular term, for example, with $n = 2$:

$$S^{(2)} = (-i)^2\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{t_1} dt_2 H_i(t_1)H_i(t_2) \tag{1.75}$$

$$= (-i)^2\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} dt_2\ \ \frac{1}{2}\Big[\theta(t_1-t_2)H_i(t_1)H_i(t_2) + \theta(t_2-t_1)H_i(t_1)H_i(t_2)\Big] \tag{1.76}$$

$$= \frac{(-i)^2}{2}\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} dt_2 \mathcal{T}\left\{H_i(t_1)H_i(t_2)\right\}. \tag{1.77}$$

For the Eq. (1.76) to happen, we have used the property of the theta function and the symmetry of the function $F(t_1, t_2, \ldots, t_n) = H_i(t_1)H_i(t_2)\ldots H_i(t_n)$ for any t_i's permutation (that requires H_i to contain an even number of fermion factors, as in QED case). For there's totally $n!$ ways of time ordering n functions , then we must divide a factor $n!$ for each term. For the Eq. (1.77), we define here the ***time- ordered product*** $\mathcal{T}$:

$$\mathcal{T}\{\phi(t_1)\phi(t_2)\} = \begin{cases}\phi(t_1)\phi(t_2) & t_1 > t_2,\\ \phi(t_2)\phi(t_1) & t_1 < t_2\end{cases} \tag{1.78}$$

$$\equiv \theta(t_1-t_2)\phi(t_1)\phi(t_2) + \theta(t_2-t_1)\phi(t_2)\phi(t_1). \tag{1.79}$$

We can then generalize the definition of $\mathcal{T}$-product of n functions $\mathcal{T}\{\phi(t_1)\phi(t_2)\ldots\phi(t_n)\}$

so that the function of later times stands to the left of the one of earlier times. Finally, we obtain the S- matrix in a more formal form, where all the functions are treated equally:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_n \mathcal{T}\{H_i(t_1)H_i(t_2)\ldots H_i(t_n)\}. \tag{1.80}$$

We can rewrite Eq. (1.80) in terms of the Hamiltonian density $\mathcal{H}_i(x)$ to obtain an explicitly covariant expression:

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \ldots \int d^4x_1 d^4x_2 \ldots d^4x_n \mathcal{T}\{\mathcal{H}_i(x_1)\mathcal{H}_i(x_2)\ldots\mathcal{H}_i(x_n)\}}. \tag{1.81}$$

Now the integration is taken over all space- time. This is so called ***Dyson expansion of the S- matrix***.
We can express that series in the exponential form to see it as a unitary transformation:

$$S = \mathcal{T}\exp\left\{-i \int d^4x \mathcal{H}_i(x)\right\}. \tag{1.82}$$

The first term of S-matrix ($n = 0$) is $S^{(0)} = \mathbb{1}$ means no interaction at all. So we define the T-matrix such that:

$$S = 1 + iT, \tag{1.83}$$

and their corresponding elements:

$$S_{fi} = \delta_{fi} + iT_{fi}, \tag{1.84}$$

then we just need to consider the T- matrix only for what really happens in our interaction. The T-matrix element is preferred to be expressed through the *Feynman amplitude* $\mathcal{M}$:

$$T_{fi} = (2\pi)^4 \delta^4(P_i - P_f)\mathcal{M}_{fi}, \tag{1.85}$$

where P_i, P_f are the total four- momenta of the intial and final states. The seperation of T-matrix element into a constant factor and the amplitude $\mathcal{M}$ makes an approriate distinction. Where the preceding factor is no more than the energy-momentum conservation for our process. The remaining factor is what specializes the transition. We can see from Eq. (1.85) that the Feynman amplitude can be found direcly from the S- matrix expansion together with the Wick's theorem (discussed next). But there is a more intuitive and simpler way to write the Feynman amplitude term by term from its corresponding graph, that is the *Feynman's rules* (see Appendix C in case of QED).

1.2.4 Wick's Theorem

We now need one more step to transform the S-matrix in Eq. (1.81) to a more useful form, then obtain the transition amplitude $\langle f|\, S\, |i\rangle$ for a particular transition $|i\rangle \to |f\rangle$ at a given order of the perturbation theory.
Having a look at Eq. (1.69), we can guess what kind of S-matrix's terms wouldn't make S_{fi} vanish. Note that each of the fields contained in the interaction Hamiltonian density

$\mathcal{H}_i(x)$ is linear in creation and annihilation operators. So for the transition from an initial state $|i\rangle$ to a final state $|f\rangle$, S must contain right annihilation operators to destroy particles in $|i\rangle$, and right creation operators to produce paricles in $|f\rangle$. Another case is when S creates a particle not in $|i\rangle$, so must then re- absorb it. These particles only exist in the intermediate states, thus are called *virtual particles*.
In addition, let us remind the important definition of the vacuum state:

$$a\,|0\rangle = 0 \text{ or } \langle 0|\,a^+ = 0. \tag{1.86}$$

Therefore, except the operators acting on virtual particles, all other ones must be ordered so that all creation operators stand to the left of all annihilation operators. Such an order is called *normal order*. ***Normal products*** $\mathcal{N}$ of functions bring them to normal order. There's a way to re- express the time- ordered product $\mathcal{T}$ in terms of the normal products $\mathcal{N}$ (and the virtual particles' contractions). That's what **Wick's theorem** does.
Let's illustrate Wick's theorem with the case of two field functions: Because the field is linear in creation and annihilation operators, without loss of generality, a field $A(x)$ can be written as: $A(x) = A^+(x) + A^-(x)$, where "+" denotes for the creation part and "-" for the annihilation part. For two boson fields A, B, we have:

$$AB = (A^+ + A^-)(B^+ + B^-) = A^+B^+ + A^+B^- + A^-B^+ + A^-B^-. \tag{1.87}$$

Then:

$$\begin{aligned}\mathcal{N}(AB) &= \mathcal{N}(A^+B^+ + A^+B^- + A^-B^+ + A^-B^-) &(1.88)\\ &= \mathcal{N}(A^+B^+) + \mathcal{N}(A^+B^-) + \mathcal{N}(A^-B^+) + \mathcal{N}(A^-B^-) &(1.89)\\ &= A^+B^+ + A^+B^- + B^+A^- + A^-B^-. &(1.90)\end{aligned}$$

Thus we can we rewrite Eq. (1.87) as:

$$AB = \mathcal{N}(AB) + A^-B^+ - B^+A^- = \mathcal{N}(AB) + \left[A^-, B^+\right]. \tag{1.91}$$

For two fermion fields, we obtain the result with the anti- commutator instead:

$$AB = \mathcal{N}(AB) + \left\{A^-, B^+\right\}. \tag{1.92}$$

We easily find that the effect of any normal product on the vacuum state vanish. And plus, the (anti-) commutators are just numbers (they do not involve operators). , thus they can be re- expressed as the vacuum expectation value of Eq. (1.91) or Eq. (1.92). Then, we obtain the result neatly as:

$$AB = \mathcal{N}(AB) + \langle 0|\,AB\,|0\rangle, \tag{1.93}$$

since

$$\mathcal{N}(AB) = \pm\mathcal{N}(BA), \tag{1.94}$$

where the minus sign is for the case of two fermion fields, the plus sign for all other cases. Applying the $\mathcal{T}$- product of two fields at different times $x_1^0 \neq x_2^0$ to Eq. (1.93) by

alternatively changing the order of fields, we get:

$$\mathcal{T}\{A(x_1)B(x_2)\} = \mathcal{N}\{A(x_1)B(x_2)\} + \overbrace{A(x_1)B}(x_2). \tag{1.95}$$

Here we have introduced a special notation for the *contraction* of $A(x_1)$ and $B(x_2)$:

$$\begin{aligned}\overbrace{A(x_1)B}(x_2) &= \langle 0|\,\mathcal{T}\{A(x_1)B(x_2)\}\,|0\rangle \\ &= \theta(t_1-t_2)\left[A^-(x_1), B^+(x_2)\right] + \theta(t_2-t_1)\left[B^-(x_2), A^+(x_1)\right],\end{aligned} \tag{1.96}$$

where the commutator are replaced by the anti-commutator for fermion fields.
Being a vacuum expectation value, the contraction will vanish unless one of the operators A and B creates particles which the other absorbs. So the contraction has the meaning of a propagator for particles from some x_1 to x_2 in space-time. And it is indeed the Feynman propagator, which we have considered before for the cases of photon and fermion fields. In much the same way, we can prove that:

$$\mathcal{T}\{A(x_1)B(x_2)C(x_3)\} = \mathcal{N}\{A(x_1)B(x_2)C(x_3)\} + A(x_1)\overbrace{B(x_2)C}(x_3) \tag{1.97}$$

$$+ B(x_2)\overbrace{A(x_1)C}(x_3) + C(x_3)\overbrace{A(x_1)B}(x_2). \tag{1.98}$$

A similar procedure can be done in the cases of three, four, ..., n fields with a definition of the generalized normal product as:

$$\mathcal{N}\left(\overbrace{ABCDEF\ldots JKLM}\ldots\right) = (-1)^P \overbrace{AK}\overbrace{BC}\overbrace{EL}\ \ \mathcal{N}(DF\ldots JM\ldots), \tag{1.99}$$

where P is the number of interchanges between neighbouring fermion fields required to change the order $(ABCDEF\ldots)$ to $(AKBCEL\ldots)$.
A general result of Eq. (1.93) for n fields is proved by Wick (Ref. [15]) by induction, which we shall not redo here. For the case of unequal times ($x_i \neq x_j$, for $i \neq j$), the Wick's theorem states:

$$\begin{aligned}T(ABCD\ldots WXYZ) = {} & N(ABCD...WXYZ) \\ & + N(\overbrace{AB}C\ldots YZ) + \ldots + N(ABC\ldots\overbrace{YZ}) \\ & + N(\overbrace{AB}\overbrace{CD}\ldots YZ) + \ldots + N(AB\ldots\overbrace{WX}\overbrace{YZ}) \\ & + \{\text{all possible contractions}\}\,.\end{aligned} \tag{1.100}$$

The right hand side of Eq. (1.100) contains all possible contractions between field operators. There the first, the second and the third lines correspond to no, one and two contractions. There, it is impossible for the case of equal-time contraction because the contraction between two fields at the same time means an instant propagation, which is prohibited in a relativistic theory.

Applying Wick's theorem to the S-matrix (Eq. (1.81)), at each order of the perturbation, we obtain a sum of generalized normal products. Each term of this sum is one- to-one with a definite process. Each of these processes is characterized by the initial and

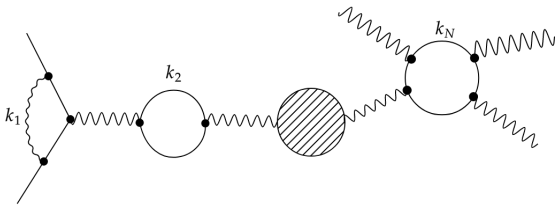
final particles contained in the $\mathcal{N}$- product, and the virtual particles which mediate the process by the non-vanishing field contractions. That's how Wick's theorem illuminates all possibilities for a process in comparison with the implicit performance of $\mathcal{T}$- product form.

1.3 Regularization

After obtaining the transition amplitude, we still have a big problem of solving divergences, which belongs to our LbyL scattering's amplitude before going to calculate physical quantities like cross section, ...

1.3.1 UV divergence

We will often encounter loop- integrals at higher order of perturbative expansion of the fields theory. That's where there will be one unrestricted momentum for each loop, i.e. this momentum can't be identified from the external momentums only by energy-momentum conservation, thus is free to vary. The problem here is that this mometum is integrated over the whole 4-momentum space (from $-\infty$ to ∞). That sometimes leads to divergent integrals. They can occur either at large loop- momentum p as an *ultra-violet (UV) divergence*, or at small $p \simeq 0$ as an *infrared (IR) divergence*. We primarily focus on UV divergences as our case of LbyL scattering.
These integrals are often expressed in this form:



$$\sim \frac{\int d^4k_1 d^4k_2 \dots d^4k_N}{\underbrace{(\not{k}_1 - m + i\varepsilon)}_{\text{fermion propagator}} \dots \underbrace{(k_j^2 + i\varepsilon)}_{\text{photon propagator}} \dots}. \tag{1.101}$$

To classify the UV divergence, we define the *superficial degree of divergence D* , it equals to the sum of power of loop momenta in the numerator minus those in the denominator of integrals like Eq. (1.101). We naively expect the integral to be *power devergent* when $D > 0$, *logarithm divergent* when $D = 0$ and UV finite when $D < 0$. From Eq. (1.101), we have:

$$D = 4L - I_f - 2I_p, \tag{1.102}$$

where L is the number of loops; I_f, I_p are the number of fermion and photon propagators. That results from each fermion propagator contribute one power of momentum in the denominator while the photon propagator's contribution is two.
Considering V as the number of vertices, we also have:

$$I = (V - 1) + L. \tag{1.103}$$

We denote the number of total internal lines, internal fermion and photon lines as I, I_f, I_p and the corresponding external ones E, E_f, E_p. Thus $I = I_f + I_p$, $E = E_f + E_p$. In case of QED, the total number of lines $N = I + E = N_f + N_p$ with N_f and N_p denote the number of fermion and photon lines. Because the QED vertices are all attached with 2 fermion lines and 1 photon line, so we always have:

$$\frac{N_f + I_f}{2} = V = N_p + I_p. \tag{1.104}$$

Substitute these relations into Eq. (1.102), we obtain:

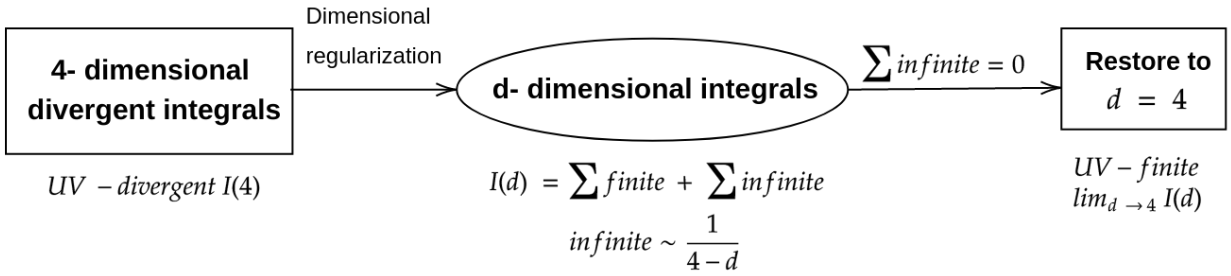
$$\boxed{D = -\frac{3}{2}E_f - E_p + 4}. \tag{1.105}$$

Hence, what is specific about QED is that the degree of divergence D in this theory is not dependent on whatever complicated internal process, but only on the number of external lines. Then it comes to us that there are just several (finite) cases where $D \geq 0$, thus we say *QED is a renormalisable theory*.

1.3.2 Dimensional regularization

Renormalization is a systematic way to formally remove the divergences. Though at the first sight, the integrals like Eq. (1.101) appear to be divergent but anyway, it must remain finite to enter to physical quantities. In the light of renormalization, by suitable modification of the integrals, we can split the infinite and finite terms hidden in those integrals seperately. After a while, we may find that the sum of all infinite parts equals to zero, i.e. the divergence is removed.
To accomplish that, regularization is used as the *first step*. There're different methods of regularization, e.g. cut-off method, Pauli- Villars method, or dimensional regularization method, ... Because all those above methods must lead to the same physical results, we choose to regularize the divergences by the latest one, ***dimensional regularization*** (DR). That is because not only it is easier to apply but much more important, DR preserve *gauge invariance* and the validity of Ward identities automatically. In this method, the number of space dimensions is set to d. Then the divergent terms are expressed as poles in the form $\sim 1/(4-d)$. *In the next step*, the Lagrangian is modified so that such poles vanish. It is achieved by adding terms, called counter terms to the original Lagrangian. It will go along with the re- definition of fields and some parameters (masses, coupling constants) of the theory, so they're no longer fixed but depend on a scale. *At last*, when all calculations are done, after renormalization, we will return to $d = 4$, these integrals are now completely removed from divergences, i.e. become finite [4] .

[4]For a full discussion of the technique of dimensional regularization, see Ref. [1]



In this thesis, we will limit ourselves to just concentrate on DR without further step. We will see later how just DR is enough to put an end to the divergences of LbyL amplitude. How could it be, or just by accident? Applying Eq. (1.105), our process $\lambda + \lambda \to \lambda + \lambda$ has the degree of divergence $D = 0$, or it is logarithm divergent. But the true degree of divergence of our four- photon amplitude is indeed smaller than 0. That happens when we consider more on the physical constraints on the integrals as Lorentz invariance, gauge invariance, Bose symmetry,... (see Ref. [8] for the case of 3-photon amplitude. We can follow the same procedure to obtain the covariant expression for the finite 4-photon amplitude, but it contains a large number of covariant tensors, thus will take much time). Consequently, the amplitude of LbyL scattering is indeed finite.

Dimensional regularization

The UV-divergent integrals live in four- dimensional space. As the factor d^4p contribute to the degree of freedom, we should avoid the divergences by lowering the space-time dimension to $d < 4$ (see Ref. [4]) as a (continous) positive variable, thus:

$$\int \frac{d^4p}{(2\pi)^4} \longrightarrow \int \frac{d^dp}{(2\pi)^d}. \tag{1.106}$$

This change comes together with the following "*axioms*":

- In d dimensions, the action is still given by:

$$S = \int dx^d \mathcal{L} \tag{1.107}$$

  as to be dimensionless. In natural units, $[x] = -1$, thus $[\mathcal{L}] = d$. Due to the QED Lagrangian Eq. (1.61), we get the dimension of the fields $[\Psi] = \frac{d-1}{2}$, $[A_\mu] = \frac{d-2}{2}$ and the electric charge $[e] = \frac{4-d}{2}$. For the dimension of the electric charge to be dimensionless at any d, we introduce the ***mass parameter*** μ ($[\mu] = 1$). Thus the coupling strength for QED theory is replaced by:

$$e_{DR} = \mu^{\frac{d-4}{2}} e. \tag{1.108}$$

  That is a characteristic feature of DR.

- The **metric tensor** in this space is defined:

$$\begin{cases} g_{00} = -g_{ii} = 1 & i = 1, 2, 3, \ldots, d-1, \\ g_{\mu\nu} = 0 & \mu \neq \nu. \end{cases} \tag{1.109}$$

So

$$g_{\mu\nu} g^{\mu\nu} = d. \tag{1.110}$$

Accordingly, the d-vector k_μ is represented as:

$$k_\mu = (k_0, k_1, \ldots, k_{d-1}), \tag{1.111}$$

and

$$k^2 = k_\mu k^\mu = k_0^2 - \sum_{i=1}^{d-1} k_i^2. \tag{1.112}$$

Note that the definiton of metric tensor in a non-integer d- dimensional space is somehow unacceptable. But the poles don't originate from the metric tensor, so in the last step, when we return to $d = 4$, the metric will be well- defined.

- We must also know how to handle the expressions involved **gamma matrices** in d dimensions. We introduce a set of d matrices $\gamma_0, \gamma_1, , \ldots, \gamma_{d-1}$, which obey the anti-commutation relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}. \tag{1.113}$$

These matrices are considered as $f(d)$ x $f(d)$ dimension(s), and I is the $f(d)$ x $f(d)$ unit matrix. With those new definitions, it is sufficent to obtain the contraction identities:

$$\gamma_\mu \gamma^\mu = dI, \tag{1.114}$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = (2 - d)\gamma^\nu, \tag{1.115}$$

$$\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu = 4 g^{\alpha\beta} - (4 - d)\gamma^\alpha \gamma^\beta. \tag{1.116}$$

We can choose $f(d) = 4$, then $\mathrm{Tr}(I) = 4$, i.e. as in 4 dimensions. So the trace relations in this d- dimensional space read:

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4 g_{\mu\nu}, \tag{1.117}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\delta) = 4(g_{\mu\nu} g_{\rho\delta} - g_{\mu\rho} g_{\nu\delta} + g_{\mu\delta} g_{\nu\rho}), \tag{1.118}$$

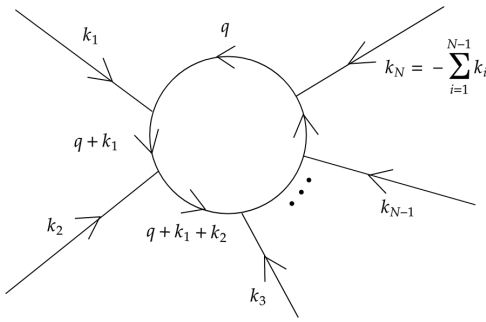
$$\mathrm{Tr}(\underbrace{\gamma_\alpha \gamma_\beta \ldots \gamma_\mu \gamma_\nu}_{\text{odd number of } \gamma-\text{matrices}}) = 0. \tag{1.119}$$

Again, the existence of γ- matrices in a non-integer number of dimensions is far from clear. And despite that, since the gamma- matrices relations are not poles in the limit $d \to 4$, thus only their familiar expressions enter to the final results by replacing $d = 4$ in the above relations.

Tensor integrals

One-loop integrals can always be expandeded in terms of the following N-point tensor integrals (we follow the conventions all from Ref. [7], also for purpose of using COLLIER's later)

$$T^{N,\mu_1\ldots\mu_M}(p_1,\ldots,p_{N-1},m_0,\ldots,m_{N-1}) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{q^{\mu_1}\ldots q^{\mu_M}}{N_0N_1\ldots N_{N-1}}, \tag{1.120}$$



where

$$N_i = (q+p_i)^2 - m_i^2 + i\varepsilon, \qquad p_i = k_1 + k_2 + \ldots + k_i, \qquad i = 1,2,\ldots,N-1. \tag{1.121}$$

with $k_1, k_2, \ldots, k_{N-1}$ are the external momenta that appear in turn along the direction of internal momentum (the N^{th} external momentum is obtained from $(N-1)$ others' by the energy-momentum conservation); μ is the mass parameter Eq. (1.108); $i\varepsilon$ is the infinitesimally small imaginary part from the propagators. The rank of tensor is M, thus the scalar integral corresponds to $M = 0$, i.e. the numerator inside the integral is 1.
Following the notations from Ref. [7], we have these following conventions:

- The N-point integrals is re-symbolized as:

$$T^1 = A, \quad T^2 = B, \quad T^3 = C, \quad T^4 = D, \quad T^5 = E, \ldots \tag{1.122}$$

  Their corresponding scalars are $A_0, B_0, C_0, \ldots$

- Due to Lorentz invariance, the tensor integral Eq. (1.120) must equal to tensors formed from the external momenta $\{k_1, k_2, \ldots, k_{N-1}\}$ and the metric tensor with

corresponding indices. So we have the most general decomposition of a tensor integral into Lorentz-covariant structures:

$$T^{N,\mu_1\cdots\mu_M} = \sum_{n=0}^{[\frac{M}{2}]} \sum_{i_{2n+1},\ldots,i_M=1}^{N-1} \{\underbrace{g\cdots g}_{n} p\cdots p\}^{\mu_1\cdots\mu_M}_{i_{2n+1}\ldots i_M} T^N_{\underbrace{0\ldots0}_{2n} i_{2n+1}\ldots i_M} \tag{1.123}$$

$$\begin{aligned} &= \sum_{i_1,\ldots,i_M=1}^{N-1} p^{\mu_1}_{i_1}\cdots p^{\mu_M}_{i_M} T^N_{i_1\ldots i_p} + \sum_{i_3,\ldots,i_M=1}^{N-1} \{g p_{i_3}\cdots p_{i_M}\}^{\mu_1\cdots\mu_M}_{i_3\ldots i_M} T^N_{00 i_3\ldots i_M} \\ &+ \sum_{i_5,\ldots,i_M=1}^{N-1} \{gg p_{i_5}\cdots p_{i_M}\}^{\mu_1\cdots\mu_M}_{i_5\ldots i_M} T^N_{0000 i_5\ldots i_M} + \cdots \\ &+ \begin{cases} \sum_{i_M=1}^{N-1} \{g\cdots gp\}^{\mu_1\cdots\mu_M}_{i_M} T^N_{\underbrace{0\ldots0}_{M-1} i_M}, & \text{for M odd,} \\ \{g\cdots g\}^{\mu_1\cdots\mu_M} T^N_{\underbrace{0\ldots0}_{M}}, & \text{for M even.} \end{cases} \end{aligned} \tag{1.124}$$

where $T^N_{0\ldots0 i_{2n+1}\ldots i_M}$ are the tensor coefficients. One thing to remark is the tensor integral Eq. (1.120) is *symmetric* under any exchange of the tensor indices $\{\mu_1,\ldots,\mu_M\}$. So the tensor coefficients are unique up to any exchange of the indices i_j. Thus we conveniently choose one representative of them by sorting i_j to the ascending order. So are the coefficients of the curly brackets introduced here, they are totally symmetric with all the lower indices of i_j. Taking some examples of the curly brackets:

$$\{pp\}^{\mu\nu}_{i_1 i_2} = p^\mu_{i_1} p^\nu_{i_2} = \{pp\}^{\nu\mu}_{i_2 i_1}, \tag{1.125}$$

$$\{gp\}^{\mu\nu\rho}_{i} = g^{\mu\nu} p^\rho_i + g^{\mu\rho} p^\nu_i + g^{\nu\rho} p^\mu_i, \tag{1.126}$$

$$\{gg\}^{\mu\nu\rho\delta} = g^{\mu\nu} g^{\rho\delta} + g^{\mu\rho} g^{\nu\delta} + g^{\mu\delta} g^{\nu\rho}, \tag{1.127}$$

$$\begin{aligned} \{gpp\}^{\mu\nu\rho\delta}_{i_1 i_2} = {} & g^{\mu\nu} p^\rho_{i_1} p^\delta_{i_2} + g^{\mu\rho} p^\nu_{i_1} p^\delta_{i_2} + g^{\mu\delta} p^\nu_{i_1} p^\rho_{i_2} + \\ & g^{\nu\rho} p^\mu_{i_1} p^\delta_{i_2} + g^{\nu\delta} p^\mu_{i_1} p^\rho_{i_2} + g^{\rho\delta} p^\mu_{i_1} p^\nu_{i_2} = \{gpp\}^{\mu\nu\rho\delta}_{i_2 i_1}, \end{aligned} \tag{1.128}$$

Follow those conventions, here are some examples of the tensor integral and their possible coefficents:

– 1-point function: $\{A_0, A_{00}, A_{0000}, \ldots\}$

$$A^{\mu\nu}(m) = A_{00} g^{\mu\nu}, \qquad \forall\, n = 1, 2, \ldots \tag{1.129}$$

$$A^{\mu_1\cdots\mu_{2n+1}} = 0, \tag{1.130}$$

for the 1-point function doesn't relate to any external momentum.

– 2-point function: $\{B_0, B_1, B_{00}, B_{11}, B_{001}, B_{111}, \ldots\}$

$$B^\mu(p, m_0, m_1) = B_1 p^\mu, \tag{1.131}$$
$$B^{\mu\nu} = B_{00} g^{\mu\nu} + B_{11} p^\mu p^\nu, \tag{1.132}$$
$$B^{\mu\nu\rho} = B_{111} p^\mu p^\nu p^\rho + B_{001}\{gp\}_1^{\mu\nu\rho}. \tag{1.133}$$

– 3-point function: $\{C_0, C_1, C_2, C_{00}, C_{11}, C_{12}, C_{22}, C_{001}, C_{002}, C_{111}, C_{112}, C_{122}, C_{222} \ldots\}$

$$C^{\mu\nu}(p_1, p_2, m_0, m_1, m_2) = C_{00} g^{\mu\nu} + C_{11} p_1^\mu p_1^\nu + C_{12}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) + C_{22} p_2^\mu p_2^\nu, \tag{1.134}$$

$$\begin{aligned} C^{\mu\nu\rho} = {} & C_{001}\{gp\}_1^{\mu\nu\rho} + C_{002}\{gp\}_2^{\mu\nu\rho} + C_{111} p_1^\mu p_1^\nu p_1^\rho + \\ & C_{112}(p_1^\mu p_1^\nu p_2^\rho + p_1^\mu p_2^\nu p_1^\rho + p_2^\mu p_2^\nu p_1^\rho) + C_{122}(p_1^\mu p_2^\nu p_2^\rho + \\ & p_2^\mu p_1^\nu p_2^\rho + p_2^\mu p_2^\nu p_1^\rho) + C_{222} p_2^\mu p_2^\nu p_2^\rho. \end{aligned} \tag{1.135}$$

– 4- point function: $\{D_0, D_i, D_{00}, D_{ij}, D_{00i}, D_{ijk}, D_{0000}, D_{00ij}, D_{ijkl}, \ldots\}$ where $i, j, k, l = 1, 2, 3$

$$\begin{aligned} D^{\mu\nu\rho\delta}(p_1, p_2, p_3, m_0, m_1, m_2, m_3) = {} & D_{0000}\{gg\}^{\mu\nu\rho\delta} + D_{00ij}\{gpp\}_{ij}^{\mu\nu\rho\delta} \\ & + D_{ijkl}\{pppp\}_{ijkl}^{\mu\nu\rho\delta}. \end{aligned} \tag{1.136}$$

For the method to calculate the tensor integral and deriving simple tensor coefficents, see Appendix D. For a complicated process as LbyL scattering, its highest tensor integrals are the rank-4, which correspond to thousands of terms. Thus, for later purpose of numerical analysis, we use COLLIER's library to calculate scalar and tensor coefficients.

1.4 Cross section. Two-particle scattering problem

The next step is to relate the (finite) transition amplitude that we have obtained from the previous sections to our most concerned physical quantity, *cross section*.

1.4.1 Fermi's golden rule

The assumptions about the asymptotically free states and the weak perturbation happening in a very short time are well suitable to apply Fermi's golden rules (Ref. [11]). Due to this well- known rule, the transition rate (the probability per unit of time) for such a process to take place is:

$$W = W_{fi} = 2\pi \left|\mathcal{M}_{fi}\right|^2 \rho(E_f), \tag{1.137}$$

where $\mathcal{M}_{fi}$ is the Feynman amplitude (Eq. (1.85)) for the transition from the given initial state $|i\rangle$ to some final state $|f\rangle$. The multiplication factor $\rho(E_f)$ is what we call the phase space or density of state factor. ρ is a function of the total energy E_f and masses of the

individual particles in the final state. From this rule, we will derive the cross section formula based on its Lorentz invariance.

Firstly, the density of state $\rho(E)dE$ is the number of states with the energy lies in (E, $E + dE$):

$$\rho(E) = \frac{dN(E)}{dE}|_{E=E_f} = \int dE\delta(E - E_i)\frac{dN(E)}{dE} = \int \delta(E - E_i)dN(E), \tag{1.138}$$

because of the conservation of energy $E_f = E_i$ and the property of δ- function $\int dE\ \delta(E - E_f) = 1$, where E_i is the initial total energy. Next we consider the number of states $dN(E)$, i.e. the number of possible states in the phase space. Each state in the phase space is identified by a point with six coordinates $(x_1, x_2, x_3, p_1, p_2, p_3)$. But the Heisenberg's principle $\Delta x_i \Delta p_i \geq 2\pi$ restricts the definiteness of those variables so that each state is confined into an element cell with the volume $(2\pi)^3$. So the number of states is:

$$dN(E) = \prod_{i=1}^{N} \frac{dx_i^3 d\vec{p}_i^3}{(2\pi)^3} = V^N \prod_{i=1}^{N} \frac{d\vec{p}_i^3}{(2\pi)^3}, \tag{1.139}$$

where N is the number of particles in the final state, and we suppose each particle is confined within a same volume V in the space configuration. We conveniently just consider in each unit of volume, so we dismiss V then. The spin of paticles also need to be considered and was all brought to the transition amplitude $\mathcal{M}$.

However, there's also one important thing to state is the conservation of the momentums, so that $\vec{P}_f = \vec{P}_i = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_N$. Then we just need to integrate over $(N - 1)$ particles. But again, we can make use of the δ- function to express the integral over all final particles:

$$dN(E) = \prod_{i=1}^{N-1} \frac{dp_i^3}{(2\pi)^3} = (2\pi)^3 \prod_{i=1}^{N} \frac{d\vec{p}_i^3}{(2\pi)^3} \delta^3 \left[\vec{p}_N - (\vec{p}_i - \vec{P}_1 - \vec{p}_2 - \ldots - \vec{p}_{N-1})\right]. \tag{1.140}$$

Now the transition rate (Eq. (1.137)) becomes an integral over all possibilities for the final state $|f\rangle$:

$$W = (2\pi)^4 \int |\mathcal{M}_{fi}|^2 \delta^4(p - p_i) \prod_{i=1}^{N} \frac{d\vec{p}_i^3}{(2\pi)^3}. \tag{1.141}$$

One thing to note, the state defined in Fermi's rule (Eq. (1.137)) is normalized to one per unit of volume. Meanwhile, we conventionally normalize the states to their energies (Eq. (1.26), Eq. (1.53), Eq. (1.54)). then we can replace:

$$W \to \frac{W}{\langle i|i\rangle \langle f|f\rangle} = \frac{W}{2E_{i1}2E_{i2}\ldots 2E_{iN_i}2E_{f1}2E_{f2}\ldots 2E_{fN_f}}. \tag{1.142}$$

Thus, we rewrite the transition rate as:

$$W = \frac{(2\pi)^4}{2E_{i1}\ldots 2E_{iN_i}} \int |\mathcal{M}_{fi}|^2 \delta^4(p - p_i) \prod_{i=1}^{N} \frac{d\vec{p}_i^3}{(2\pi)^3 2E_{fi}}. \tag{1.143}$$

1.4.2 Cross section definition

The cross section σ is defined as the probability for a particle from an incoming beam to interact with a target particle in one unit of time (Ref. [13]):

$$\sigma = \frac{\text{number of interactions /unit time/ target particle}}{\text{incident flux}}, \tag{1.144}$$

where:

$$\text{flux} = \text{number of incident particles/ unit area/ unit time}, \tag{1.145}$$

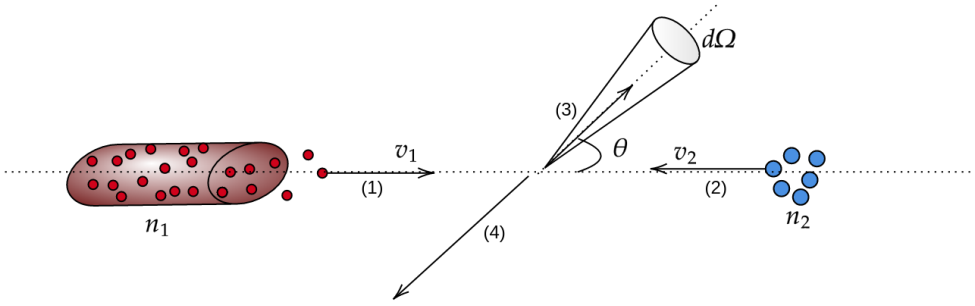
where the area here is the effective cross sectional area of the incident beam. From the definition above, we have the relation between the transition rate W (number of interactions/unit time) and the cross section σ (all in one unit of volume) as:

$$W = \sigma \text{ x incident flux x density of target particles.} \tag{1.146}$$

1.4.3 Cross section of two- particle scattering

Now we particularly consider the problem of two- particle scattering into two other particles , which is the case of LbyL scattering:

$$\underset{(E_1,\vec{p}_1)}{(1)} + \underset{(E_2,\vec{p}_2)}{(2)} \longrightarrow \underset{(E_3,\vec{p}_3)}{(3)} + \underset{(E_4,\vec{p}_4)}{(4)}.$$



Assume that in an arbitrary inerital reference of system, a particle from beam (1) moves with the velocity $\vec{v}_1$ and hits the target (2) moving with velocity $\vec{v}_2$ towards (1), results in the production of particles (3), (4). Their energy-momentums are denoted as above. Eq. (1.146) in this case becomes:

$$W = \sigma \text{ x } n_1(v_1 + v_2) \text{ x } n_2 = \sigma \ (v_1 + v_2). \tag{1.147}$$

where $v_2 = |\vec{v}_1|$, $v_2 = |\vec{v}_2|$, ; and n_1, n_2 are the densities of the incident beam and the target. But the states are normalized to one particle per unit of volume, then $n_1 = n_2 = 1$. And so, in combination with Eq. (1.143), we obtain the expression for the cross section of

the 2-to-2 scattering as:

$$\sigma = \frac{(2\pi)^{-2}}{4E_1E_2(v_1+v_2)} \int \left|\mathcal{M}_{fi}\right|^{2*} \delta^4(p_3+p_4-p_1-p_2) \frac{d\vec{p}_3^3}{2E_3} \frac{d\vec{p}_4^3}{2E_4}, \tag{1.148}$$

where the squared amplitude $\left|\mathcal{M}_{fi}\right|^{2*}$ is defined from $\left|\mathcal{M}_{fi}\right|^2$ in different ways depending on what particles participating in the interaction and what polarization case is considered. For the case of LbyL scattering, both the initial and final particles are photons, which are bosons and have two polarizations, its squared amplitude taking part in the cross section is defined in Subsection 3.3.2 and Subsection 3.4.1. Since now, we denote $\left|\mathcal{M}_{fi}\right|^2$ for $\left|\mathcal{M}_{fi}\right|^{2*}$ with that definition being implied .

After a while, it's worthwhile to make some comments on the Lorentz invariance of the cross section (Eq. (1.148)):

- The Feynman amplitude $\mathcal{M}_{fi}$ is Lorentz invariant.
- The final particles obey the on-shell condition then

$$\int d^4p\delta(p^2-m^2)\Big|_{p_0>0} = \int d^3\vec{p} \int dp_0 \delta\left[p_0^2 - (\vec{p}^2+m^2)\right]\theta(p_0) = \int \frac{d^3\vec{p}}{2p_0} \tag{1.149}$$

  is invariant.

- The factor at the denominator is also Lorentz invariant because:

$$E_1E_2(v_1+v_2) = \sqrt{\left(p_{1\mu}p_2^{\mu}\right)^2 - m_1^2m_2^2}. \tag{1.150}$$

  The right hand side of Eq. (1.150) is variant so is the left hand side.

Note that, the cross section (Eq. (1.148)) includes all possibilities for the momentums of outgoing particles, so we can define the *differential cross section* to measure the probability for the scatterd particle to satisfy some conditions on a physical quantity A as $\frac{d\sigma}{dA}$. The differential cross section according to the scattering angles is most often concerned. By which, we measure the probability for the scatterd particle to go to the differential solid angle $d\Omega$:

$$\frac{d\sigma}{d\Omega} \qquad \longrightarrow \qquad \sigma = \int \frac{d\sigma}{d\Omega} d\Omega, \tag{1.151}$$

where $d\Omega = d\cos\theta d\phi$; and θ, ϕ correspond to the polar and azimuth angles when the direction of Oz axis coincides with the direction of the inital particle moving toward the target.

The cross section in the Centre of Mass (C.o.M.) system
Now we apply the Lorentz invariant formula Eq. (1.148) for the interaction cos section in the C.o.M. system, and thus the result is unchanged in any other (inertial) system. In the C.o.M. system, together with the conservation law for energy- momentums , our isolated

system has some "good- looking" characteristics as follow:

$$\begin{cases} E_1 + E_2 = E_3 + E_4 = \sqrt{s}, \\ \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = 0. \end{cases} \tag{1.152}$$

Set $\vec{p}_1 = -\vec{p}_2 = \vec{p}_i$ and $\vec{p}_3 = -\vec{p}_4 = \vec{p}_f$, the factor in the denominator is simplified as:

$$E_1 E_2 (v_1 + v_2) = E_1 E_2 \left(\frac{|\vec{p}_i|}{E_1} + \frac{|\vec{p}_i|}{E_2} \right) = E_1 |\vec{p}_i| + E_2 |\vec{p}_i| = |\vec{p}_i| \sqrt{s}. \tag{1.153}$$

Thus, using Eq. (1.152) and transfrom the three- dimentional integral of $\vec{p}_4$ to its corresponding 4- dimensional integral as Eq. (1.149), Eq. (1.148) now becomes:

$$\sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i|\sqrt{s}} \int d\Gamma, \tag{1.154}$$

where

$$\int d\Gamma = \int \frac{d\vec{p}_3^3}{2E_3} \frac{d\vec{p}_4^3}{2E_4} \left| \mathcal{M}_{fi} \right|^2 \delta^4 (p_3 + p_4 - p_1 - p_2) \tag{1.155}$$

$$= \int \frac{d\vec{p}_3^3}{2E_3} \int d^4 p_4 \delta^4 \left[p_4 - (p_1 + p_2 - p_3) \right] \left| \mathcal{M}_{fi} \right|^2 \delta (p_4^2 - m_4^2) \theta (p_{40}) \tag{1.156}$$

$$= \int \frac{d\vec{p}_3^3}{2E_3} \left| \mathcal{M}_{fi} \right|^2 \delta \left[(p_1 + p_2 - p_3)^2 - m_4^2 \right] \theta (\sqrt{s} - E_3) \tag{1.157}$$

$$= \int \frac{d\vec{p}_3^3}{2E_3} \left| \mathcal{M}_{fi} \right|^2 \delta \left[(\sqrt{s} - E_3)^2 - |\vec{p}_3|^2 - m_4^2 \right] \qquad \text{(hence, } E_3 < \sqrt{s}) \tag{1.158}$$

$$= \int d|\vec{p}_3| |\vec{p}_3|^2 \frac{1}{2E_3} \delta \left[(\sqrt{s} - E_3)^2 - |\vec{p}_3|^2 - m_4^2 \right] \int d\Omega \left| \mathcal{M}_{fi} \right|^2 \tag{1.159}$$

$$= \int d|\vec{p}_3| \frac{|\vec{p}_3|^2}{2E_3} \delta \left[\left(\sqrt{s} - \sqrt{|\vec{p}_3|^2 + m_3^2} \right)^2 - |\vec{p}_3|^2 - m_4^2 \right] \int d\Omega \left| \mathcal{M}_{fi} \right|^2 \tag{1.160}$$

$$= \frac{|\vec{p}_3|^2}{2E_3} \left[2|\vec{p}_3| + 2(\sqrt{s} - E_3) \frac{|\vec{p}_3|}{E_3} \right]^{-1} \int d\Omega \left| \mathcal{M}_{fi} \right|^2 \tag{1.161}$$

$$= \frac{|\vec{p}_f|}{4\sqrt{s}} \int d\Omega \left| \mathcal{M}_{fi} \right|^2. \tag{1.162}$$

$$\tag{1.163}$$

Finally, we obtain the cross section for 2-to-2 body scattering as:

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} \int d\Omega \left| \mathcal{M}_{fi} \right|^2, \tag{1.164}$$

or in the differential form:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} \left| \mathcal{M}_{fi} \right|^2. \tag{1.165}$$

Chapter 2

Light-by-light scattering in QED

We are now equipped with essential theories to do the most important work of calculating the transition amplitude for the LbyL scattering. We will see that our process $\lambda + \lambda \rightarrow \lambda + \lambda$ exhibits some interesting features due to its symmetry between the incoming and outgoing particles (photons). We choose to derive the amplitude by Wick's theorem instead of a much more easy- to - obtain way of applying Feynman's rule. But that option makes it more natural to get all possibilities for our process at LO.

2.1 Amplitudes at Leading-order

2.1.1 The S-matrix

We rewrite the Lagrangian of QED:

$$\mathcal{L}_{QED}(x) = \sum_f \left[\bar{\Psi}_f(x)(i\gamma^\alpha \partial_\alpha - m_f)\Psi_f(x) - Q_f \bar{\Psi}_f(x)\gamma^\alpha A_\alpha \Psi_f(x)\right] - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x), \tag{2.1}$$

where the field functions $A_\alpha(x)$, $\Psi(x)$ are already explained in Section 1.1. The theory is applied with all charged elementary particles (i.e.leptons and quarks), which the index f indicates. Then m_f, Q_f are the mass and the electric charge corresponding to each fermion[1]. We will see that one of these LO processes attached with only one type of particle. So for simplicity, we just first consider e.g. the (anti-) electron as the only particle participating in the interaction.
Because the interaction Lagrangian does not contain the derivatives of fields, then the interaction Hamiltonian is simply as (only electron considered, so the index f is dismissed):

$$\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x) = e\bar{\Psi}(x)\gamma^\alpha A_\alpha \Psi(x). \tag{2.2}$$

The S-matrix Eq. (1.82) is now:

$$S = \mathcal{T}\exp\left\{-ie\int d^4x \bar{\Psi}(x)\gamma^\alpha A_\alpha \Psi(x)\right\}. \tag{2.3}$$

[1]In case of quarks, we must also include in the Lagrangian the running indices on three colors for each.

2.1.2 The S-matrix element at leading order

Our LbyL scattering is presented as follow:

$$\begin{array}{ccccccc} \lambda_1 & + & \lambda_2 & \longrightarrow & \lambda_3 & + & \lambda_4, \quad (2.4) \\ (E_1,\vec{k}_1),\eta_1 & & (E_2,\vec{k}_2),\eta_2 & & (E_3,\vec{k}_3),\eta_3 & & (E_4,\vec{k}_4),\eta_4 \quad (2.5) \end{array}$$

where $(E_i,\vec{k}_i),\eta_i$ is the four energy-momentum and spin (polarization) of each corresponding particle. With the S-matrix as above, to find its element S_{fi}, we need to identify the intial and final states. They are no more than these asymtotically free states as:

$$\text{Initial state:} \quad |i\rangle = |\lambda_1,\lambda_2\rangle = \sqrt{2E_1}\sqrt{2E_2}a^+_{\eta_1}(\vec{k}_1)a^+_{\eta_2}(\vec{k}_2)\,|0\rangle\,, \tag{2.6}$$

$$\text{Fianl state:} \quad |f\rangle = |\lambda_3,\lambda_4\rangle = \sqrt{2E_3}\sqrt{2E_4}a^+_{\eta_3}(\vec{k}_3)a^+_{\eta_4}(\vec{k}_4)\,|0\rangle \tag{2.7}$$

$$\Rightarrow \langle\lambda_3,\lambda_4| = \sqrt{2E_3}\sqrt{2E_4}\,\langle 0|\,a_{\eta_3}(\vec{k}_3)a_{\eta_4}(\vec{k}_4). \tag{2.8}$$

Whereas the S-matrix Eq. (2.3) appears in groups including { $\bar{\Psi}$, Ψ, A }, there are mainly three processes belong to the propagation of electrons, positrons and photons. Therefore, a non-trivial S-matrix element for our transition must in some way contain the right number of $A(x)$ fields to combine with the external particle states $|\lambda_1,\lambda_2\rangle$ and $|\lambda_3,\lambda_4\rangle$.

1. It's easy to verify that the S-matrix element at odd orders vanishes:

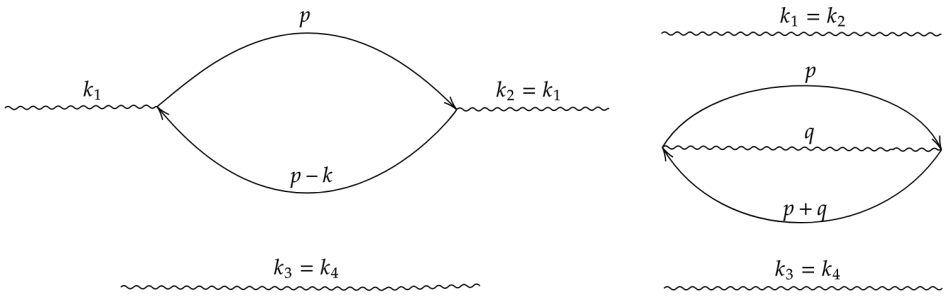
$$S^{(\text{odd order})}_{fi} = \langle\lambda_3,\lambda_4|\ldots A(x_1)\ldots A(x_2)\ldots A(x_{2n+1})\ldots|\lambda_1,\lambda_2\rangle = 0, \tag{2.9}$$

   because there's always one photon field is not contracted in such these orders. So we exclude all odd orders of perturbation.

2. So let's continue with the second order:

$$\begin{aligned} S^{(2)}_{fi} &= -e^2\int dx_1^4dx_2^4\,\langle\lambda_3,\lambda_4|\,\mathcal{T}\left\{[\bar{\Psi}(x_1)\not{A}(x_1)\Psi(x_1)]\,[\bar{\Psi}(x_2)\not{A}(x_2)\Psi(x_2)]\right\}|\lambda_1,\lambda_2\rangle \\ &= -e^2\int dx_1^4dx_2^4\,\langle\lambda_3,\lambda_4|\,\mathcal{N}\Big\{\ldots\overbrace{A(x_1)\ldots A}(x_2)\ldots A(x_3)\ldots A(x_4)\ldots\Big\}+ \\ &\mathcal{N}\Big\{\ldots\overbrace{A(x_1)\ldots A}(x_2)\ldots\overbrace{A(x_3)\ldots A}(x_4)\ldots\Big\}+\ldots|\lambda_1,\lambda_2\rangle\,. \end{aligned} \tag{2.10}$$

   Once again, the element at this order doesn't contribute to the transition. That is due to there are four external photon fields needed to be contracted, while we just have two internal photon fields. Thus, this term just lead to disconnected diagrams. The figure below corresponds to the second and the third lines of Eq. (2.10), all other term at $n = 2$ give the same diconnected diagrams:



3. There, at the order $n = 4$:

$$S_{fi}^{(4)} = \frac{e^4}{24} \int dx_1^4 dx_2^4 dx_3^4 dx_4^4 \, \langle \lambda_3, \lambda_4 | \, \mathcal{T} \left\{ [\bar{\Psi} \not{A} \Psi]_1 \, [\bar{\Psi} \not{A} \Psi]_2 \, [\bar{\Psi} \not{A} \Psi]_3 \, [\bar{\Psi} \not{A} \Psi]_4 \right\} | \lambda_1, \lambda_2 \rangle \, , \tag{2.11}$$

where the notation $[\bar{\Psi} \not{A} \Psi]_i$ is for the fields at x_i: $\bar{\Psi}(x_i) \not{A}(x_i) \Psi(x_i)$. .
Putting the disconnected parts aside, there is just one way for the legal terms. That is when all the four internal photon fields are sufficiently contracted with the four external ones (where all the internal fermion fields are certainly full contracted with each other). Thus, this is the leading order for our transtion.
Because the integral variables x_1, x_2, x_3, x_4 are completely equivalent to each other and the external particles are all photons, we can combine any internal $A(x)$ field to any one of the external particles. That leads to $4! = 24$ ways to combine the photon fields.
As one of those 24 cases, we can conveniently fix so as λ_1, λ_2 to be annihilated at x_1, x_2 and λ_3, λ_4 to be emitted at x_3, x_4 (the rest of them give the same results). Thus, we can eliminate the symmetric factor 24 at the denominator. Note that, there're also 6 ways (cyclic permutations) to contract the fermion fields in one specific configuration of the photon fields:

(a) Cyclic $\{1-2-3-4\}$: $[\bar{\Psi} \not{A} \Psi]_1 [\bar{\Psi} \not{A} \Psi]_2 [\bar{\Psi} \not{A} \Psi]_3 [\bar{\Psi} \not{A} \Psi]_4$,

(b) Cyclic $\{1-2-4-3\}$: $[\bar{\Psi} \not{A} \Psi]_1 [\bar{\Psi} \not{A} \Psi]_2 [\bar{\Psi} \not{A} \Psi]_3 [\bar{\Psi} \not{A} \Psi]_4$,

(c) Cyclic $\{1-3-2-4\}$: $[\bar{\Psi} \not{A} \Psi]_1 [\bar{\Psi} \not{A} \Psi]_2 [\bar{\Psi} \not{A} \Psi]_3 [\bar{\Psi} \not{A} \Psi]_4$,

(d) Cyclic $\{1-3-4-2\}$: $[\bar{\Psi} \not{A} \Psi]_1 [\bar{\Psi} \not{A} \Psi]_2 [\bar{\Psi} \not{A} \Psi]_3 [\bar{\Psi} \not{A} \Psi]_4$,

(e) Cyclic $\{1-4-2-3\}$: $[\bar{\Psi} \not{A} \Psi]_1 [\bar{\Psi} \not{A} \Psi]_2 [\bar{\Psi} \not{A} \Psi]_3 [\bar{\Psi} \not{A} \Psi]_4$,

(f) Cyclic $\{1-4-3-2\}$: $[\bar{\Psi} \not{A} \Psi]_1 [\bar{\Psi} \not{A} \Psi]_2 [\bar{\Psi} \not{A} \Psi]_3 [\bar{\Psi} \not{A} \Psi]_4$.

- Firstly, we couple the fermion fields along the cyclic $\{1-2-3-4\}$:

$$S^{(4)}_{fi,1} = \frac{e^4}{24}\int dx_1^4\dots dx_4^4\langle\lambda_3,\lambda_4|\mathcal{N}\Big\{\overbrace{[\bar\Psi \not{A}\Psi]_1[\bar\Psi \not{A}\Psi]_2[\bar\Psi \not{A}\Psi]_3[\bar\Psi \not{A}\Psi]_4}\Big\}|\lambda_1,\lambda_2\rangle \tag{2.12}$$

$$= e^4\int dx_1^4\dots dx_4^4\langle\lambda_3,\lambda_4|\Big\{\overbrace{[\bar\Psi \not{A}^+\Psi]_3[\bar\Psi \not{A}^+\Psi]_4[\bar\Psi \not{A}^-\Psi]_1[\bar\Psi \not{A}^-\Psi]_2}\Big\}|\lambda_1,\lambda_2\rangle \tag{2.13}$$

$$\begin{aligned}= -e^4\int dx_1^4\dots dx_4^4\,&\langle\lambda_3,\lambda_4|\, iS_F(x_2-x_3)\not{A}^+(x_3)iS_F(x_3-x_4)\\ &\not{A}^+(x_4)iS_F(x_4-x_1)\not{A}^-(x_1)iS_F(x_1-x_2)\not{A}^-(x_2)\,|\lambda_1,\lambda_2\rangle\end{aligned} \tag{2.14}$$

$$\begin{aligned}= -e^4\int dx_1^4\dots dx_4^4&\sqrt{2E_3}\sqrt{2E_4}\,\langle 0|\, a_{\eta_3}(\vec{k}_3)a_{\eta_4}(\vec{k}_4)\\ &iS_F(x_2-x_3)\Big[\sum_{\theta_3}\int\frac{d^3\vec{q}_3}{(2\pi)^3\sqrt{2\omega_3}}a^+_{\theta_3}(\vec{q}_3)\not{\epsilon}^*(\vec{q}_3,\theta_3)e^{iq_3x_3}\\ &iS_F(x_3-x_4)\sum_{\theta_4}\int\frac{d^3\vec{q}_4}{(2\pi)^3\sqrt{2\omega_4}}a^+_{\theta_4}(\vec{q}_4)\not{\epsilon}^*(\vec{q}_4,\theta_4)e^{iq_4x_4}\\ &iS_F(x_4-x_1)\sum_{\theta_1}\int\frac{d^3\vec{q}_1}{(2\pi)^3\sqrt{2\omega_1}}a_{\theta_1}(\vec{q}_1)\not{\epsilon}(\vec{q}_1,\theta_1)e^{-iq_1x_1}\\ &iS_F(x_1-x_2)\sum_{\theta_4}\int\frac{d^3\vec{q}_2}{(2\pi)^3\sqrt{2\omega_2}}a_{\theta_2}(\vec{q}_2)\not{\epsilon}(\vec{q}_2,\theta_2)e^{-iq_2x_2}\Big]\\ &\sqrt{2E_1}\sqrt{2E_2}a^+_{\eta_1}(\vec{k}_1)a^+_{\eta_2}(\vec{k}_2)\,|0\rangle\end{aligned} \tag{2.15}$$

$$\begin{aligned}= -e^4\sum_{\theta_1,\theta_2,\theta_3,\theta_4}&\int\frac{d^3\vec{q}_1}{(2\pi)^3}\int\frac{d^3\vec{q}_2}{(2\pi)^3}\int\frac{d^3\vec{q}_3}{(2\pi)^3}\int\frac{d^3\vec{q}_4}{(2\pi)^3}\sqrt{\frac{E_1E_2E_3E_4}{\omega_1\omega_2\omega_3\omega_4}}\\ &\langle 0|\, a_{\eta_3}(\vec{k}_3)a_{\eta_4}(\vec{k}_4)a^+_{\theta_3}(\vec{q}_3)a^+_{\theta_4}(\vec{q}_4)a_{\theta_1}(\vec{q}_1)a_{\theta_2}(\vec{q}_2)a^+_{\eta_1}(\vec{k}_1)a^+_{\eta_2}(\vec{k}_2)\,|0\rangle\\ &\int dx_1^4\dots dx_4^4 iS_F(x_2-x_3)\not{\epsilon}^*(\vec{q}_3,\theta_3)e^{iq_3x_3}iS_F(x_3-x_4)\not{\epsilon}^*(\vec{q}_4,\theta_4)e^{iq_4x_4}\\ &iS_F(x_4-x_1)\not{\epsilon}(\vec{q}_1,\theta_1)e^{-iq_1x_1}iS_F(x_1-x_2)\not{\epsilon}(\vec{q}_2,\theta_2)e^{-iq_2x_2}.\end{aligned} \tag{2.16}$$

Before going on, note that for the Eq. (2.13) to happen, we have eliminated all other terms because they don't suit with the external states, i.e. give zero contributions and we have fixed our space configuration. For the same reason, the "bra-ket" factor in Eq. (2.16) indeed corresponds to several terms because of the symmetry of λ_1 and λ_2, λ_3 and λ_4, i.e. by exchange the operators followed the commutation relation Eq. (1.24), but then we just consider one term that relates to our fixed configuration, thus:

$$\begin{aligned}&\langle 0|\, a_{\eta_3}(\vec{k}_3)a_{\eta_4}(\vec{k}_4)a^+_{\theta_3}(\vec{q}_3)a^+_{\theta_4}(\vec{q}_4)a_{\theta_1}(\vec{q}_1)a_{\theta_2}(\vec{q}_2)a^+_{\eta_1}(\vec{k}_1)a^+_{\eta_2}(\vec{k}_2)\,|0\rangle\\ &\supset (2\pi)^3\delta^3(\vec{q}_3-\vec{k}_3)\delta^{\eta_3,\theta_3}(2\pi)^3\delta^3(\vec{q}_4-\vec{k}_4)\delta^{\eta_4,\theta_4}\\ &\quad(2\pi)^3\delta^3(\vec{q}_1-\vec{k}_1)\delta^{\eta_1,\theta_1}(2\pi)^3\delta^3(\vec{q}_2-\vec{k}_2)\delta^{\eta_2,\theta_2}.\end{aligned} \tag{2.17}$$

Then Eq. (2.16) becomes neatly as:

$$S^{(4)}_{fi,1} = -e^4 \int dx_1^4 \dots dx_4^4 iS_F(x_2 - x_3) \not{\epsilon}^*(\vec{k}_3, \eta_3) e^{ik_3 x_3} iS_F(x_3 - x_4) \not{\epsilon}^*(\vec{k}_4, \eta_4) e^{ik_4 x_4} iS_F(x_4 - x_1) \not{\epsilon}(\vec{k}_1, \eta_1) e^{-ik_1 x_1} iS_F(x_1 - x_2) \not{\epsilon}(\vec{k}_2, \eta_2) e^{-ik_2 x_2}. \tag{2.18}$$

Now we transform the propagators into the momentum space, then:

$$\begin{aligned} S^{(4)}_{fi,1} = -e^4 \int dx_1^4 \dots dx_4^4 & \left(\int \frac{dp_3^4}{(2\pi)^4} iS_F(p_3) e^{-ip_3(x_2 - x_3)} \right) \not{\epsilon}^*(\vec{k}_3, \eta_3) e^{ik_3 x_3} \\ & \left(\int \frac{dp_4^4}{(2\pi)^4} iS_F(p_4) e^{-ip_4(x_3 - x_4)} \right) \not{\epsilon}^*(\vec{k}_4, \eta_4) e^{ik_4 x_4} \\ & \left(\int \frac{dp_1^4}{(2\pi)^4} iS_F(p_1) e^{-ip_1(x_4 - x_1)} \right) \not{\epsilon}(\vec{k}_1, \eta_1) e^{-ik_1 x_1} \\ & \left(\int \frac{dp_2^4}{(2\pi)^4} iS_F(p_2) e^{-ip_2(x_1 - x_2)} \right) \not{\epsilon}(\vec{k}_2, \eta_2) e^{-ik_2 x_2} \end{aligned} \tag{2.19}$$

$$\begin{aligned} = -e^4 & \int \frac{dp_1^4}{(2\pi)^4} \int \frac{dp_2^4}{(2\pi)^4} \int \frac{dp_3^4}{(2\pi)^4} \int \frac{dp_4^4}{(2\pi)^4} iS_F(p_3) \not{\epsilon}^*(\vec{k}_3, \eta_3) \\ & iS_F(p_4) \not{\epsilon}^*(\vec{k}_4, \eta_4) iS_F(p_1) \not{\epsilon}(\vec{k}_1, \eta_1) iS_F(p_2) \not{\epsilon}(\vec{k}_2, \eta_2) \int dx_1^4 e^{ix_1(p_1 - p_2 - k_1)} \\ & \int dx_2^4 e^{ix_2(-p_3 + p_2 - k_2)} \int dx_3^4 e^{ix_3(p_3 + k_3 - p_4)} \int dx_4^4 e^{ix_4(p_4 + k_4 - p1)} \end{aligned} \tag{2.20}$$

$$\begin{aligned} = -e^4 & \int \frac{dp_1^4}{(2\pi)^4} \int \frac{dp_2^4}{(2\pi)^4} \int \frac{dp_3^4}{(2\pi)^4} \int \frac{dp_4^4}{(2\pi)^4} iS_F(p_3) \not{\epsilon}^*(\vec{k}_3, \eta_3) iS_F(p_4) \\ & \not{\epsilon}^*(\vec{k}_4, \eta_4) iS_F(p_1) \not{\epsilon}(\vec{k}_1, \eta_1) iS_F(p_2) \not{\epsilon}(\vec{k}_2, \eta_2) (2\pi)^4 \delta^4(p_1 - p_2 - k_1) \\ & (2\pi)^4 \delta^4(-p_3 + p_2 - k_2) (2\pi)^4 \delta^4(p_3 + k_3 - p_4) (2\pi)^4 \delta^4(p_4 + k_4 - p_1) \end{aligned} \tag{2.21}$$

$$\begin{aligned} = -e^4 & \int \frac{dp_1^4}{(2\pi)^4} \int \frac{dp_2^4}{(2\pi)^4} \int \frac{dp_3^4}{(2\pi)^4} \int \frac{dp_4^4}{(2\pi)^4} iS_F(p_3) \not{\epsilon}^*(\vec{k}_3, \eta_3) \\ & iS_F(p_4) \not{\epsilon}^*(\vec{k}_4, \eta_4) iS_F(p_1) \not{\epsilon}(\vec{k}_1, \eta_1) iS_F(p_2) \not{\epsilon}(\vec{k}_2, \eta_2) \\ & (2\pi)^4 \delta^4 \left[p_2 - (p_1 - k_1) \right] (2\pi)^4 \delta^4 \left[p_3 - (p_1 - k_1 - k_2) \right] \\ & (2\pi)^4 \delta^4 \left[p_4 - (p_1 - k_4) \right] (2\pi)^4 \delta^4(k_1 + k_2 - k_3 - k_4) \end{aligned} \tag{2.22}$$

$$\begin{aligned} \overset{p_1 = p}{=} & -e^4 \delta^4(k_1 + k_2 - k_3 - k_4) \int dp^4 iS_F(p_3 = p - k_1 - k_2) \not{\epsilon}^*(\vec{k}_3, \eta_3) \\ & iS_F(p_4 = p - k_4) \not{\epsilon}^*(\vec{k}_4, \eta_4) iS_F(p) \not{\epsilon}(\vec{k}_1, \eta_1) iS_F(p_2 = p - k_1) \not{\epsilon}(\vec{k}_2, \eta_2). \end{aligned} \tag{2.23}$$

From the relation between the S-matrix element and the Feynman amplitude Eq. (1.85), we have the first amplitude:

$$\mathcal{M}^{(4)}_{fi,1} = -\frac{e^4}{(2\pi)^4}\int dp^4 iS_F(p-k_1-k_2)\not{\epsilon}^*(\vec{k}_3,\eta_3)iS_F(p-k_1-k_2+k_3)$$
$$\not{\epsilon}^*(\vec{k}_1+\vec{k}_2-\vec{k}_3,\eta_4)iS_F(p)\not{\epsilon}(\vec{k}_1,\eta_1)iS_F(p-k_1)\not{\epsilon}(\vec{k}_2,\eta_2). \quad (2.24)$$

It corresponds to the Fig. 2.2. We can also check the amplitude Eq. (2.12) with the one derived from Feynman's rules (see Appendix C).

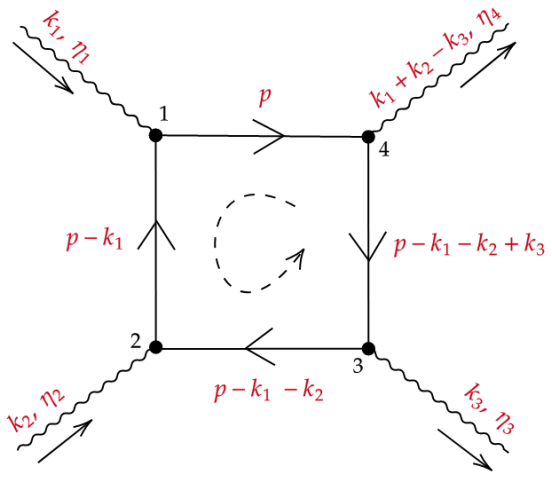


Figure 2.2: The first diagram

- It's obvious that we can proceed in the same (long) way to obtain the ampitude for other cylic permutations. However, we can make use the correspondence between the amplitude and its diagram to symplify things. Note that the order in the cylic is in an opposite direction to the one of internal momentum, and the momentum is preserved at each vertex. Then similarly, the amplitudes for the second and the third cylics $\{1-2-4-3\}$, $\{1-3-2-4\}$ can be easily obtained from the first cylic's. We can make a little deform on each of them to have a clear vision (the order of x_1, x_2, x_3, x_4 doesn't matter): Thus, their

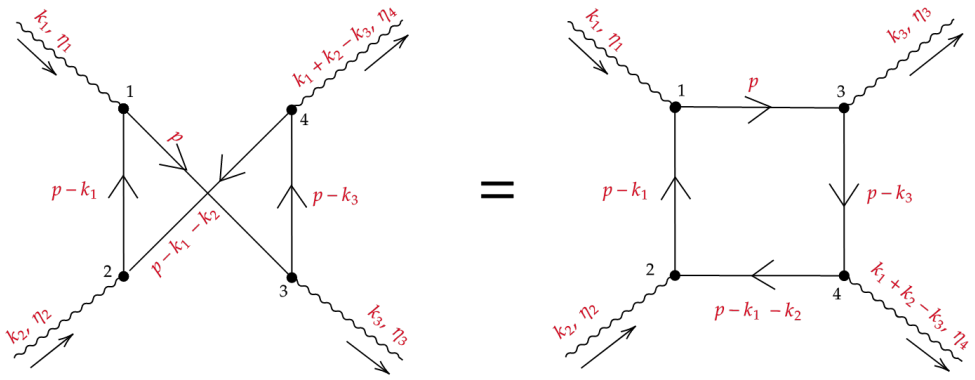


Figure 2.3: The second diagram

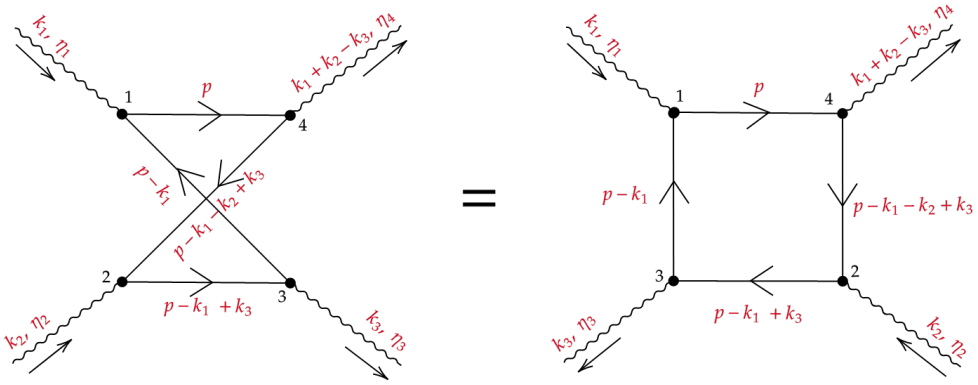


Figure 2.4: The third diagram

amplitudes are easily written down from the Fig. 2.3 and the Fig. 2.4 as:

$$\mathcal{M}^{(4)}_{fi,2} = -\frac{e^4}{(2\pi)^4}\int dp^4 iS_F(p-k_1-k_2)\not{\epsilon}^*(\vec{k}_1+\vec{k}_2-\vec{k}_3,\eta_4)iS_F(p-k_3)$$
$$\not{\epsilon}^*(\vec{k}_3,\eta_3)iS_F(p)\not{\epsilon}(\vec{k}_1,\eta_1)iS_F(p-k_1)\not{\epsilon}(\vec{k}_2,\eta_2), \tag{2.25}$$

and

$$\mathcal{M}^{(4)}_{fi,3} = -\frac{e^4}{(2\pi)^4}\int dp^4 iS_F(p-k_1+k_3)\not{\epsilon}(\vec{k}_2,\eta_2)iS_F(p-k_1-k_2+k_3)$$
$$\not{\epsilon}^*(\vec{k}_1+\vec{k}_2-\vec{k}_3,\eta_4)iS_F(p)\not{\epsilon}(\vec{k}_1,\eta_1)iS_F(p-k_1)\not{\epsilon}^*(\vec{k}_3,\eta_3). \tag{2.26}$$

- The fourth, fifth and sixth cylics $\{1-4-3-2\}$, $\{1-3-4-2\}$, $\{1-4-2-3\}$ are correspondingly similar to the first, second and third ones while the direction of interal momentum is reversed. For example in the fourth case (Fig. 2.5):

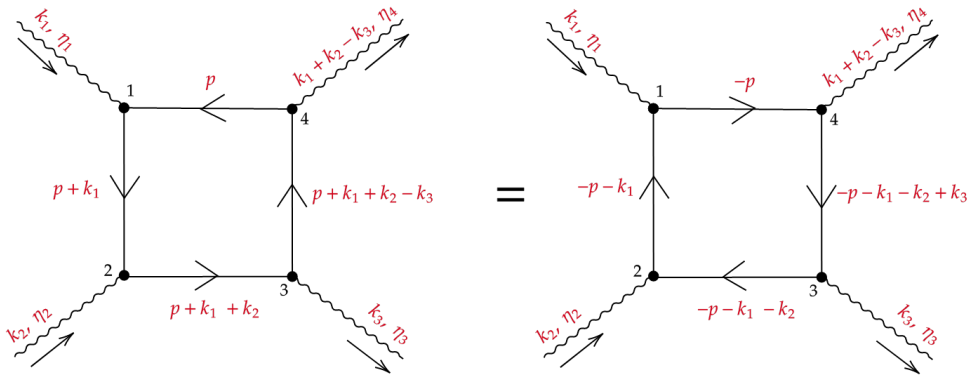


Figure 2.5: The fourth diagram is indeed the same with the first

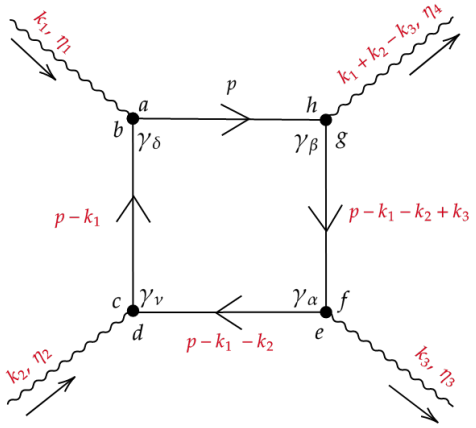
Then we just need to replace $p_\mu \to -p_\mu$ in Eq. (2.24), Eq. (2.25), Eq. (2.26). And that leads to the same amplitudes in pairs :

$$\mathcal{M}^{(4)}_{fi,4} = \mathcal{M}^{(4)}_{fi,1},$$
$$\mathcal{M}^{(4)}_{fi,5} = \mathcal{M}^{(4)}_{fi,2},$$
$$\mathcal{M}^{(4)}_{fi,6} = \mathcal{M}^{(4)}_{fi,3}.$$

Therefore, we can say that there're totally 3 processes in our LbyL scattering (so the first three amplitudes are multiplied by an additional factor 2). Because of the diagrams corresponding to the amplitudes, we call them *1-loop integrals*.

Trace integrals

We should write the amplitudes in the "trace" form for later purpose. To do so, we rewrite each component of the amplitude in terms of its element with indices. By this way, we can move those elements freely. For instance, with the first amplitude obtained from Eq. (2.24), we have :



$$\mathcal{M}^{(4)}_{fi,1} = -\frac{2e^4}{(2\pi)^4}\int dp^4 \Big[\frac{\not p - \not k_1 - \not k_2}{(p-k_1-k_2)^2 - m^2}\Big]_{ab}[\gamma_\alpha]_{bc}(\varepsilon_3^*)^\alpha \Big[\frac{\not p - \not k_1 - \not k_2 + \not k_3}{(p-k_1-k_2+k_3)^2 - m^2}\Big]_{cd}$$
$$[\gamma_\beta]_{de}(\varepsilon_4^*)^\beta \Big[\frac{\not p}{p^2 - m^2}\Big]_{ef}[\gamma_\delta]_{fg}(\varepsilon_1)^\delta \Big[\frac{\not p - \not k_1}{(p-k_1)^2 - m^2}\Big]_{gh}[\gamma_\nu]_{ha}(\varepsilon_2)^\nu \tag{2.27}$$

$$= -\frac{2e^4}{(2\pi)^4}\int dp^4 \mathrm{Tr}\Big[\frac{\not p - \not k_1 - \not k_2}{(p-k_1-k_2)^2 - m^2}\gamma_\alpha \frac{\not p - \not k_1 - \not k_2 + \not k_3}{(p-k_1-k_2+k_3)^2 - m^2}\gamma_\beta$$
$$\frac{\not p}{p^2 - m^2}\gamma_\delta \frac{\not p - \not k_1}{(p-k_1)^2 - m^2}\gamma_\nu\Big](\varepsilon_1)^\delta(\varepsilon_2)^\nu(\varepsilon_3^*)^\alpha(\varepsilon_4^*)^\beta, \tag{2.28}$$

where we conveniently denote ε_i instead of $\varepsilon_i(\vec{k}_i, \eta_i)$. The reason why the last index of γ_ν coincidents with the first index of $\frac{\not p - \not k_1 - \not k_2}{(p-k_1-k_2)^2 - m^2}$ in Eq. (2.27) originates from the essence of QED Lagrangian, where each photon field is sandwiched between two fermion fields

and from the fact that the LO term associates with a 1-loop propagation of fermions (see Eq. (2.13)). That's why the trace appears in the next equation Eq. (2.28).

Similarly, the trace integrals for the other two:

$$\mathcal{M}^{(4)}_{fi,2} = -\frac{2e^4}{(2\pi)^4}\int dp^4 \mathrm{Tr}\Big[\frac{\not p - \not k_1 - \not k_2}{(p-k_1-k_2)^2-m^2}\gamma_\beta \frac{\not p - \not k_3}{(p-k_3)^2-m^2}\gamma_\alpha \frac{\not p}{p^2-m^2}\gamma_\delta \frac{\not p - \not k_1}{(p-k_1)^2-m^2}\gamma_\nu\Big](\varepsilon_1)^\delta(\varepsilon_2)^\nu(\varepsilon_3^*)^\alpha(\varepsilon_4^*)^\beta, \tag{2.29}$$

and

$$\mathcal{M}^{(4)}_{fi,3} = -\frac{2e^4}{(2\pi)^4}\int dp^4 \mathrm{Tr}\Big[\frac{\not p - \not k_1 + \not k_3}{(p-k_1+k_3)^2-m^2}\gamma_\nu \frac{\not p - \not k_1 - \not k_2 + \not k_3}{(p-k_1-k_2+k_3)^2-m^2}\gamma_\beta \frac{\not p}{p^2-m^2}\gamma_\delta \frac{\not p - \not k_1}{(p-k_1)^2-m^2}\gamma_\alpha\Big](\varepsilon_1)^\delta(\varepsilon_2)^\nu(\varepsilon_3^*)^\alpha(\varepsilon_4^*)^\beta. \tag{2.30}$$

Thus, the total amplitude at LO is the sum of all partial amplitudes from Eq. (2.28), Eq. (2.29),Eq. (2.30):

$$\mathcal{M}_{LO} = \mathcal{M}^{(4)}_{fi,1} + \mathcal{M}^{(4)}_{fi,2} + \mathcal{M}^{(4)}_{fi,3} = (\varepsilon_1)^\delta(\varepsilon_2)^\nu(\varepsilon_3^*)^\alpha(\varepsilon_4^*)^\beta \Pi_{\alpha\beta\delta\nu}(\vec{k_1},\vec{k_2},\vec{k_3}), \tag{2.31}$$

where Π is the (total) tensor amplitude. It is completely seperated from the polarization vectors and will be the main consideration in the next section. The product of polarization vectors will be brought in at the very last step.

Now, let us have some evaluations about how the amplitudes derived by Wick's theorem consistent with the attributes of a one- fermion loop. As reflected from the one-loop amplitudes above, they are always:

- The integrals with the internal momentum p of a fermion inside the fermion loop as the integral variable. It is integrated over all possible values of p in momentum space $(-\infty, +\infty)$. That sometimes leads to bad behaviors of an infinite integral, that will be mentioned later in our case.
- Attached with a negative sign "-". It is a consequence of QED Langrangian where the fermion fields appear in pairs and are orders as $(\bar{\Psi}\dots\Psi)\dots(\bar{\Psi}\dots\Psi)$.
- Expressed as a trace (mentioned above).
- Never standing alone. Because the fermion in the loop can choose an arbitrary path to connect the vertices, so there will be different processes possible for the same transition at loop order.

In other words, they're all the attributes of QED in perturbation approximation.

2.2 UV divergence cancellation of the LbyL scattering

2.2.1 Ultraviolet- divergent parts

Before the next stage to give numerical result for our total amplitude (Eq. (2.31)), there are some serious points to make clear about the 1- loop integrals in Eq. (2.28),

Eq. (2.29),Eq. (2.30). First, because we just need to consider the difference between those three amplitudes, for simplicity, we define three corresponding tensor quantities I_i ($i = 1, 2, 3$) such that:

$$\mathcal{M}^4_{fi,i} = -\frac{2e^4}{(2\pi)^4} I_{i,\alpha\beta\delta\nu} (\varepsilon_1)^\delta (\varepsilon_2)^\nu (\varepsilon_3^*)^\alpha (\varepsilon_4^*)^\beta, \tag{2.32}$$

i.e. they are indeed the integrals only. Let's have a look at one of them:

$$I_1 = I_{1,\alpha\beta\delta\nu} = \int dp^4 \mathrm{Tr}\Big[\frac{\not{p} - \not{k}_1 - \not{k}_2}{(p - k_1 - k_2)^2 - m^2}\gamma_\alpha \frac{\not{p} - \not{k}_1 - \not{k}_2 + \not{k}_3}{(p - k_1 - k_2 + k_3)^2 - m^2}\gamma_\beta \frac{\not{p}}{p^2 - m^2}\gamma_\delta \frac{\not{p} - \not{k}_1}{(p - k_1)^2 - m^2}\gamma_\nu\Big], \tag{2.33}$$

and consider only the term with highest order of p:

$$I_1 \supset \int d^4p \frac{\mathrm{Tr}\left(\not{p}\gamma_\alpha \not{p}\gamma_\beta \not{p}\gamma_\delta \not{p}\gamma_\nu\right)}{[p^2 - m^2][(p - k_1)^2 - m^2][(p - k_1 - k_2)^2 - m^2][(p - k_1 - k_2 + k_3)^2 - m^2]}. \tag{2.34}$$

Taking the dimensionality of I_1:

$$[I_1] = \frac{[p]^4[p]^4}{[p]^8} = \text{constant}. \tag{2.35}$$

Thus, the integral tends to infinity when $p_\mu \to \pm\infty$, also for I_2, I_3. So they are called to be ultraviolet- divergent ("UV" is denoted for this divergent part). That is our problem now to handle these divergences.
One way to cure this problem is using the ***dimensional regularization method*** (see Section 1.3). Since then, we will work in d- dimensional space (in this case, we choose $d < 4 \to [I_1] < 1$, so the integral becomes finite):

$$d^4p \to d^dp, \tag{2.36}$$

together with the new gamma matrices and trace identities in d- dimensional space (Eq. (1.113)-Eq. (1.119)). Then, we obtain the UV parts as functions of $\frac{1}{4-d}$. After that, we will find that the sum of all UV terms are zero, i.e. *the UV cancellation*. What is left is the finite parts which are well- behaved when $d = 4$. Thus, we would restore to our original 4-dimensional space without worry about the divergences.

2.2.2 Decomposition of the UV part in d- dimensions

Using the gamma matrices and trace identities in Eq. (1.113), Eq. (1.119), we can rewrite the numerator inside the integral of Eq. (2.44) as [2]:

$$\begin{aligned}\mathrm{Tr}\left(\not{p}\gamma_\alpha\not{p}\gamma_\beta\not{p}\gamma_\delta\not{p}\gamma_\nu\right) &= 4p^4 g_{\alpha\nu}g_{\beta\delta} - 4p^4 g_{\alpha\delta}g_{\beta\nu} + 4p^4 g_{\alpha\beta}g_{\delta\nu} \\ &\quad - 8p^2 p_\alpha p_\beta g_{\delta\nu} - 8p^2 p_\beta p_\delta g_{\alpha\nu} - 8p^2 p_\delta p_\nu g_{\alpha\beta} - 8p^2 p_\alpha p_\nu g_{\beta\delta} \\ &\quad + 32 p^\alpha p^\beta p^\delta p^\nu. \end{aligned} \tag{2.37}$$

Based on the terms which have the same structure in the RHS of Eq. (2.37), we can divide them into three groups as:

- The first three terms are all proportional to p^4, so we first evaluate the integral:

$$I_a = \int d^d p \frac{p^4}{[p^2-m^2][(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]}, \tag{2.38}$$

  where r_1, r_2, r_3 are combinations of external momenta for general consideration. We now may want to express this integral in terms of one- loop tensor N-point integrals (Eq. (1.120)). To do that, we define an operator UV such that:

$$UV(\text{UV terms} + \text{finite terms}) = \text{UV terms}, \tag{2.39}$$

  i.e. UV only keeps the UV part of the function it acts on.
  We have:

$$\begin{aligned} UV(I_a) &= UV\Big\{ \int d^d p \frac{(p^4-m^4)+m^4}{[p^2-m^2][(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]} \Big\} \\ &= UV\Big\{ \int d^d p \frac{(p^2-m^2)(p^2+m^2)}{[p^2-m^2][(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]} \Big\} \\ &= UV\Big\{ \int d^d p \frac{p^2}{[(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]} \Big\} \\ &= UV\Big\{ \int d^d p \frac{[(p+r_1)^2-m^2]-2pr_1-r_1^2+m^2}{[(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]} \Big\} \\ &= UV\Big\{ \int d^d p \frac{1}{[(p+r_2)^2-m^2][(p+r_3)^2-m^2]} \Big\} \\ &\overset{p+r_2=p}{=} \int d^d p \frac{1}{[p^2-m^2][(p+r_3-r_2)^2-m^2]} \Big\} \\ &= B_0^{UV}(m, r_3-r_2) \\ &= \frac{2}{4-D}, \end{aligned} \tag{2.40}$$

  note that the Eq. (2.40) should multiply by an overall factor (see Appendix D), but we can now ignore it. In the same way, *we can always express a general integral in which the numerator is a polynomial of p in terms of the tensor integrals.*

[2] Because it is the trace of eight gamma matrices, we may use FORM or other numerical tools to manipulate the expression instead

- Analogously, we consider the integral for the next four terms $\sim p^2 p_\mu p_\nu$:

$$I_b = \int d^d p \frac{p^2 p_\mu p_\nu}{[p^2-m^2][(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]}. \tag{2.41}$$

Then:

$$\begin{aligned}
UV(I_b) &= UV\Big\{\int d^d p \frac{(p^2-m^2)p_\mu p_\nu + m^2 p_\mu p_\nu}{[p^2-m^2][(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]}\Big\} \\
&= UV\Big\{\int d^d p \frac{p_\mu p_\nu}{[(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]}\Big\} \\
&\overset{p+r_1=p}{=} UV\Big\{\int d^d p \frac{(p_\mu - r_{1\mu})(p_\nu - r_{1\nu})}{[p^2-m^2][(p+r_2-r_1)^2-m^2][(p+r_3-r_1)^2-m^2]}\Big\} \\
&= UV\Big\{\int d^d p \frac{p_\mu p_\nu}{[p^2-m^2][(p+r_2-r_1)^2-m^2][(p+r_3-r_1)^2-m^2]}\Big\} \\
&= C^{UV}_{\mu\nu}(m, r_2-r_1, r_3-r_1) \\
&= \frac{g_{\mu\nu}}{2(4-D)}.
\end{aligned} \tag{2.42}$$

To obtain those results for the tensor integrals, we would follow the canonical methods in Appendix D (here we give the results for 1 and 2-point functions, for the higher- point functions, we use the results from Ref. [5]).

- The integral for the last term $\sim p_\alpha p_\beta p_\delta p_\nu$ is already a primary tensor integral:

$$\begin{aligned}
UV(I_c) &= UV\Big\{\int d^d p \frac{p_\alpha p_\beta p_\delta p_\nu}{[p^2-m^2][(p+r_1)^2-m^2][(p+r_2)^2-m^2][(p+r_3)^2-m^2]}\Big\} \\
&= UV\Big\{D_{\alpha\beta\delta\nu}(r_1, r_2, r_3)\Big\} \\
&= \{gg\}_{\alpha\beta\delta\nu} D^{UV}_{0000}(m, r_1, r_2, r_3) \\
&= \frac{1}{12(4-D)}(g_{\alpha\beta}g_{\delta\nu} + g_{\alpha\delta}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\delta}).
\end{aligned} \tag{2.43}$$

2.2.3 Cancellation of the total UV tensor integral

After all, now we are able to rewrite the UV part of the first integral (Eq. (2.33)):

$$\begin{aligned}
UV(I_1) &= 4\frac{2}{4-d}g_{\alpha\nu}g_{\beta\delta} - 4\frac{2}{4-d}g_{\alpha\delta}g_{\beta\nu} + 4\frac{2}{4-d}g_{\alpha\beta}g_{\delta\nu} \\
&\quad - 8\frac{g_{\alpha\beta}}{2(4-d)}g_{\delta\nu} - 8\frac{g_{\beta\delta}}{2(4-d)}g_{\alpha\nu} - 8\frac{g_{\delta\nu}}{2(4-d)}g_{\alpha\beta} - 8\frac{g_{\alpha\nu}}{2(4-d)}g_{\beta\delta} \\
&\quad + 32\frac{1}{12(4-d)}(g_{\alpha\beta}g_{\delta\nu} + g_{\alpha\delta}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\delta}) \\
&= \frac{8}{3(4-d)}(g_{\alpha\beta}g_{\delta\nu} + g_{\alpha\nu}g_{\beta\delta} - 2g_{\alpha\delta}g_{\beta\nu}).
\end{aligned} \tag{2.44}$$

One thing to remark is that the UV part is independent of external momenta. Thus, from Eq. (2.28), Eq. (2.29), Eq. (2.30), the UV parts of the second and the third can be obtained from the first by exchanging the tensor indices, in particular:

- $UV(I_1) \xrightarrow{\alpha \leftrightarrow \beta} UV(I_2)$, thus:

$$UV(I_2) = \frac{8}{3(4-d)}(g_{\beta\alpha}g_{\delta\nu} + g_{\beta\nu}g_{\alpha\delta} - 2g_{\beta\delta}g_{\alpha\nu}). \tag{2.45}$$

- $UV(I_1) \xrightarrow{\alpha \leftrightarrow \nu} UV(I_3)$, thus:

$$UV(I_3) = \frac{8}{3(4-d)}(g_{\nu\beta}g_{\delta\alpha} + g_{\nu\alpha}g_{\beta\delta} - 2g_{\nu\delta}g_{\beta\alpha}). \tag{2.46}$$

Taking the sum of all three UV- divergent parts (Eq. (2.44), Eq. (2.45), Eq. (2.46)), we obtain it to be vanishing:

$$UV(\mathcal{M}_{total}) = UV(I) = UV(I_1) + UV(I_2) + UV(I_3) = 0. \tag{2.47}$$

In conclusion, though each of the transition amplitudes in our process is divergent but by using dimensional regularization, we found that *these divergences have vanished as considering totally*. Therefore, when we come back to $d = 4$, the divergence is no more a problem.

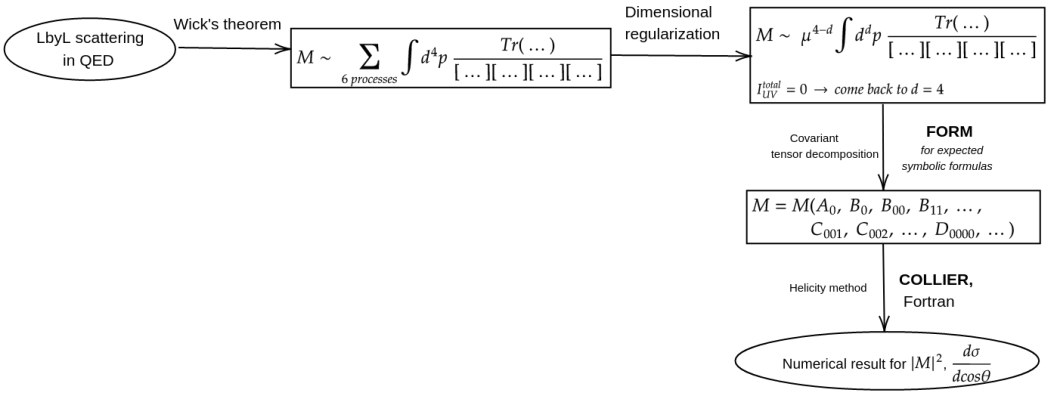
Now then, what should be considered is the remaining UV-finite terms. Based on the expressions of 3 amplitudes (Eq. (2.28), Eq. (2.29),Eq. (2.30)) , they potentially contain up to thousands of terms (i.e. N-point tensor integrals). So that seems impossible to write down all those terms by hand, except with the help of a specific tool as FORM, which will be introduced in the next chapter.

Chapter 3

Numerical methods

3.1 Introduction to FORM

FORM is a symbolic manipulation system (Ref. [14]). From an original (mathematical) expression, if we want to transform it into the desired form, we need to have sufficent knowledge about the rules of transformation based on some basic algebraic principles (+,-,*,/,...). FORM is not only equipped with some of those basic rules but also with the *user- defined rules* (i.e. it is implemented with a limited range of built-in rules and functions. But it is able to work on more complicated problems by the user, who defines the rules and "teaches" FORM how to execute the expressions) . That is what FORM would do things for us, by *symbolic manipulation* more than numeric computings (like *MATLAB*, Mathematica, ...).
FORM is widely used in various fields but mostly in theoretical physics. As one of its strengths, FORM supports useful ways to handle with gamma matrices and tensors, even in a general d-dimensional space. Due to its charaterictics to just do what we ask for, it is simple and fast. Thus, it suits well with our purpose to give numerical results of LbyL scattering's amplitude with large symbolic manipulations. More particular, FORM rewrites the amplitudes (Eq. (2.28), Eq. (2.29),Eq. (2.30)), which are removed from the UV-divergent part already, in terms of the tensor integrals as the standard form of COLLIER[1]. The figure below shows the whole process to obtain the numerical amplitude of LbyL scattering:



[1]Our full FORM code to decompose a general 4-point loop integral into tensor integrals with their covariant coefficients following the convention is presented here

3.2 Introduction to COLLIER

COLLIER library provides numerical results for arbitrary one- loop tensor integrals for a scattering process in a general quantum fields theory (Ref. [6]). The results are evaluated in dimensional regularization and include both the UV (IR) -divergent and -finite parts. As the next step, COLLIER is used to provide numerical values for our one- loop tensor integrals in each tensor component. The basis of code of COLLIER is the reduction methods (see the illustration of this methods in Section D.3). The conventions used in COLLIER on the notations of covariant tensor components is introduced throughout the thesis. For more information about COLLIER and its usage, see Ref. [7].
The input of COLLIER obviously contain all the variables of a tensor integral (Eq. (1.120)), i.e. masses and external momenta sorted in a definite order [2]. One feature of COLLIER is that beside those necessary inputs, it allows the user to freely set the values of several parameters. Two of these parameters are the UV pole $\Delta_{UV} \sim 1/(d-4)$; and the mass scale μ (Eq. (1.108)) (otherwise, they are set to default values by COLLIER). Thus it will be more convenient to check the UV- divergences cancellation of a scattering amplitude, i.e the independence of μ and the poles.

3.3 Polarization

Now, the numerical result of 4-point integrals in our process has been obtained by FORM + COLLIER embedded in a Fortran program. For the result of the transition amplitudes in Eq. (2.28), Eq. (2.29) and Eq. (2.30) , we still need more clarifications on the polarization vectors.

3.3.1 Polarization basis

We rewrite here the wave solution of the photon field:

$$A^{\mu}(x) = \sum_{\eta} \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}} \left[a(\vec{k})\varepsilon_{\eta}^{\mu}(\vec{k})e^{-ikx} + a^{+}(\vec{k})\varepsilon_{\eta}^{\mu *}(\vec{k})e^{ikx} \right], \tag{3.1}$$

where η covers all possible polarization states of the photon. We apply Lorentz gauge condition on the photon field:

$$\partial_{\mu} A^{\mu}(x) = 0. \tag{3.2}$$

One convenient way to obtain Eq. (3.2) is to set the polarization vectors such that :

$$\varepsilon_{\mu} k^{\mu} = \varepsilon.k = 0. \tag{3.3}$$

In a reference frame, where the 4-momentum vector of photon is:

$$k_{\mu} = (E, 0, 0, E), \tag{3.4}$$

[2] The masses inputted to COLLIER are the invariant masses instead of $k_1, k_2, \ldots$ only (see Ref. [7]), thus the expressions are independent of our chosen system of reference.

i.e. E is the energy of photon and its 3-momentum vector directs along the Oz axis. In this frame, we choose the polarization vectors, and also the basis vectors ε_η as:

$$\varepsilon_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \qquad \varepsilon_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}; \qquad \varepsilon_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \qquad \varepsilon_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{3.5}$$

We can check that ε_1, ε_2 satisfy the Eq. (3.3). They are identified as the two physical polarization states, while ε_0 and ε_3 are the unphysical states. It has been proved (Ref. [12]) that the contributions of ε_0 and ε_3 cancel for physical observables. We will therefore ignore ε_0 and ε_3 in our calculation.
One thing to note is that the choice of basis vectors is in one-to-one coresspondence to the k-vector of photon. Thus, if we want to find different polarization vector sets for several photons in the same system of reference, we must boost the chosen basis vectors of one of the photons as the same way we boost its k-vectors to the others . Applying those results to our problem of LbyL process Eq. (2.4) in a preferential C.o.M. system, we obtain [3] :

$$\begin{cases} k_1 &= (E,0,0,E); \\ k_2 &= (E,0,0,-E) = \tilde{L}(\pi,0)k_1; \\ k_3 &= (E, E\sin(\theta)\cos(\phi), E\sin(\theta)\sin(\phi), E\cos\theta) = \tilde{L}(\theta,\phi)k_1; \\ k_4 &= k_1+k_2-k_3 = (E, -E\sin(\theta)\cos(\phi), -E\sin(\theta)\sin(\phi), -E\cos\theta) = \tilde{L}(\pi-\theta,\pi+\phi)k_1, \end{cases} \tag{3.6}$$

where $\tilde{L}$ is the Lorentz transformation:

$$\tilde{L}(\alpha,\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha)\cos(\beta) & -\sin(\beta) & \sin(\alpha)\cos(\beta) \\ 0 & \cos(\alpha)\sin(\beta) & \cos(\beta) & \sin(\alpha)\sin(\beta) \\ 0 & -\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}, \tag{3.7}$$

i.e. they are just the space rotations. If we choose two basis vectors corresponding to k_1 as ε_1 and ε_2 in Eq. (3.5), they will be denoted again as $\varepsilon_{1,1}$ and $\varepsilon_{1,2}$ (the first index refers to the photon considered, and the latter index is for the polarization of that photon). Then, by applying the same transformations on $\varepsilon_{1,1}$ and $\varepsilon_{1,2}$, we obtain the basis vectors in the C.o.M:

- For k_1:

$$\varepsilon_{1,1} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}; \quad \varepsilon_{1,2} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \Rightarrow \quad \varepsilon_1^+ = -\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}; \quad \varepsilon_1^- = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}. \tag{3.8}$$

[3] Throughout this section, to describe LbyL scattering, we will work in the C.o.M. system. These notations will be always be used with the same physical meaning: E is the energy of one incoming photon; θ, ϕ correspond to the polar and azimuth angles of the scattering

- For k_2:

$$\varepsilon_{2,1} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}; \quad \varepsilon_{2,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \varepsilon_2^+ = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ i \\ 0 \end{pmatrix}; \quad \varepsilon_2^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix}. \tag{3.9}$$

- For k_3:

$$\varepsilon_{3,1} = \begin{pmatrix} 0 \\ \cos\theta\cos(\phi) \\ \cos\theta\sin(\phi) \\ -\sin(\theta) \end{pmatrix}; \qquad \varepsilon_{3,2} = \begin{pmatrix} 0 \\ -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon_3^+ = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos(\phi) - i\sin(\phi) \\ \cos\theta\sin(\phi) + i\cos(\phi) \\ -\sin(\theta) \end{pmatrix}; \qquad \varepsilon_3^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos(\phi) + i\sin(\phi) \\ \cos\theta\sin(\phi) - i\cos(\phi) \\ -\sin(\theta) \end{pmatrix}. \tag{3.10}$$

- And for k_4:

$$\varepsilon_{4,1} = \begin{pmatrix} 0 \\ \cos\theta\cos(\phi) \\ \cos\theta\sin(\phi) \\ -\sin(\theta) \end{pmatrix}; \qquad \varepsilon_{4,2} = \begin{pmatrix} 0 \\ \sin(\phi) \\ -\cos(\phi) \\ 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon_4^+ = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos(\phi) + i\sin(\phi) \\ \cos\theta\sin(\phi) - i\cos(\phi) \\ -\sin(\theta) \end{pmatrix}; \qquad \varepsilon_4^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos(\phi) - i\sin(\phi) \\ \cos\theta\sin(\phi) + i\cos(\phi) \\ -\sin(\theta) \end{pmatrix}. \tag{3.11}$$

where we have instead used the new set of basis vectors from the old one by the linear combination:

$$\varepsilon_i^{\pm} = \mp\frac{1}{\sqrt{2}}(\varepsilon_{i,1} \pm i\varepsilon_{i,2}), \tag{3.12}$$

with $i = 1, 2, 3, 4$. Since then, we will denote two physical states (or spin states, or polarization states) of a photon as the state "+" and the state "-".
The above results of k-vectors Eq. (3.6) and their basis vectors Eq. (3.8)-Eq. (3.11) will all be used as the inputs of the main Fortran-based program.

3.3.2 Average squared amplitude over polarization states

It is realized that each case of the process corresponds to one set of the polarization vectors. For each photon is possible with two polarization states, so our 4-photon process includes up to $2^4 = 16$ cases of polarizations, which are:

$$\mathcal{M}_1 = \mathcal{M}_{++++}; \qquad \mathcal{M}_9 = \mathcal{M}_{----}; \tag{3.13}$$
$$\mathcal{M}_2 = \mathcal{M}_{-+++}; \qquad \mathcal{M}_{10} = \mathcal{M}_{+---};$$
$$\mathcal{M}_3 = \mathcal{M}_{+-++}; \qquad \mathcal{M}_{11} = \mathcal{M}_{-+--};$$
$$\mathcal{M}_4 = \mathcal{M}_{++-+}; \qquad \mathcal{M}_{12} = \mathcal{M}_{--+-};$$
$$\mathcal{M}_5 = \mathcal{M}_{+++-}; \qquad \mathcal{M}_{13} = \mathcal{M}_{---+};$$
$$\mathcal{M}_6 = \mathcal{M}_{--++}; \qquad \mathcal{M}_{14} = \mathcal{M}_{++--};$$
$$\mathcal{M}_7 = \mathcal{M}_{-+-+}; \qquad \mathcal{M}_{15} = \mathcal{M}_{+-+-};$$
$$\mathcal{M}_8 = \mathcal{M}_{-++-}; \qquad \mathcal{M}_{16} = \mathcal{M}_{+--+}; \tag{3.14}$$

where the lower notations on the RHSs tell us the polarizaion states of λ_1, λ_2, λ_3 and λ_4 correspondingly. Since those states are orthogonal to each other and equally contribute to the process, then the ***average squared amplitude*** can be built from their combination as an *unpolarized* quantity:

$$\left|\overline{\mathcal{M}}\right|^2 = \text{average of} \left(\sum_{i=1}^{16} |\mathcal{M}_i|^2 \right), \tag{3.15}$$

which will be brought to the calculation of unpolarized (differential) cross section later (Subsection 3.4.2). There are some important features of photons define the way how we take the average of Eq. (3.15):

- Firstly, it is often that we don't know the polarizations of the initial particles. Because *each photon corresponds to two polarizations* , thus we multiply the sum in Eq. (3.15) by $(1/2)^2 = 1/4$ as considering its average:

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{1}{4} \left(\sum_{i=1}^{16} |\mathcal{M}_i|^2 \right). \tag{3.16}$$

- Besides that, in case there's no way to distinguish between two final photons or we really don't care about their polarizations, we then multiply Eq. (3.16) by an additional factor $1/2$, thus:

$$\boxed{\left|\overline{\mathcal{M}}\right|^2 = \frac{1}{2}\frac{1}{4} \left(\sum_{i=1}^{16} |\mathcal{M}_i|^2 \right) = \frac{1}{8} \left(\sum_{i=1}^{16} |\mathcal{M}_i|^2 \right)}. \tag{3.17}$$

  Indeed, that is due to the fact that *photons without considering their polarizations are "pure" bosons, which are interchangeable while making no difference*. Moreover, the (differential) cross section is calculated from $|\mathcal{M}|^2$ over the phase space for all possibilities of final states. That means the result will be counted twice, one for (λ_3, λ_4) and

one for (λ_4, λ_3) at some θ, ϕ. So the factor $1/2$ is necessary as the consequence of boson symmetry.

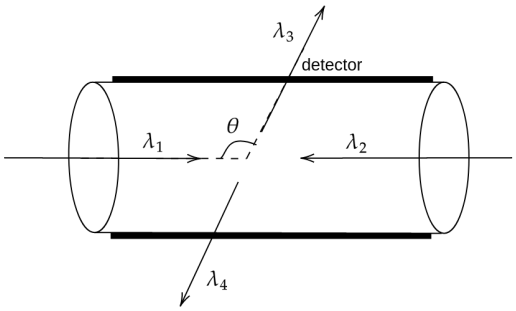
The average squared amplitude delivered by this way refers to the *helicity method* for photons. This method not only makes it easier for numerical calculation later but also differentiates the results in the polarized and the unpolarized cases.

3.4 Numerical results

Except the case if we consider the change of some quantities versus the initial energy of photon and with no more mention, we would set the *invariant diphoton mass* $m_{\lambda\lambda} = 2E = 2(GeV)$ [4] . The main inputs delivered to our main Fortran program to calculate the numerical results of LbyL scattering are presented in Appendix E.

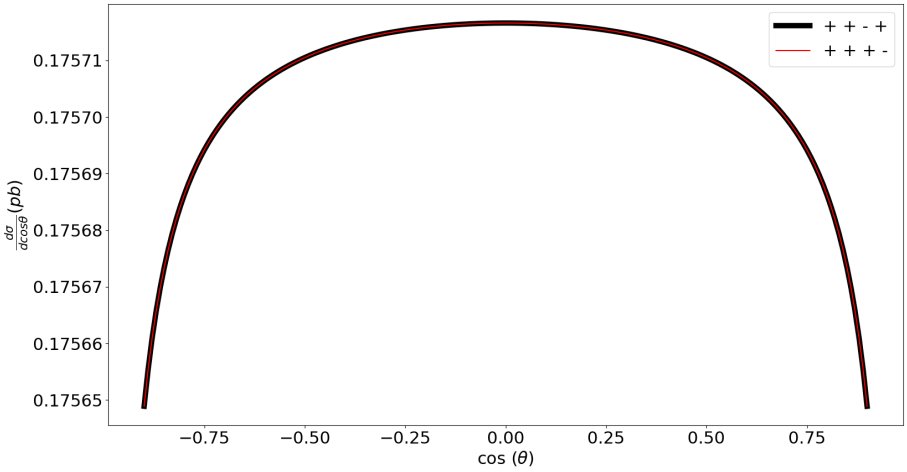
3.4.1 Differential cross section of polarized states

When considering the square of amplitude for individual polarization case, we should make clear about the physical meaning of it.

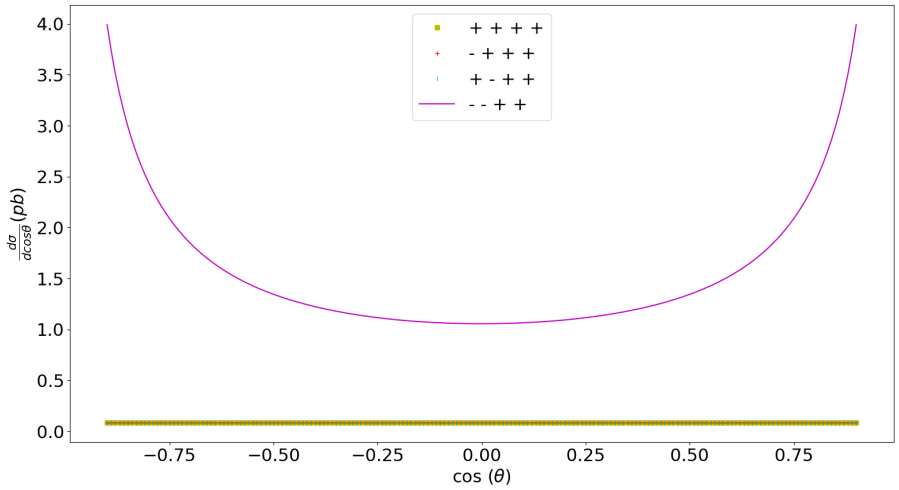


The above figure is an illustration of how the detector is installed so that they can recognize all possibilities of the outgoing photons. Before going on, we should remark on the symmetry between different states in Eq. (3.13), Eq. (3.14). We can obtain the last eight cases from the first eight ones by simply changing "+" to "-" and vice versa. These differences make no affect on the measured results, e.g. $(++--)$ and $(--++)$ give the same results. So it makes sense to just consider the first eight cases of Eq. (3.13). One more thing to note, *if two outgoing photons have the same polarizations*, there's no way for the detector to distinguish between the event that λ_3 scattering at some θ and the one that λ_4 scattering at the same position. Therefore, the integrated cross section in those polarization cases make it double. In the meantime, no same thing happens to the other cases. Then, to suit our numerical results with the experimental results, we have to multiply the squared amplitude by a factor $1/2$ in the 1^{st}, 2^{nd}, 3^{rd} and 6^{th} polarizations to obtain the polarized cross sections.
Below (Fig. 3.2) are the results of the differential cross section in $cos\theta$ of eight polarization cases in that manner:

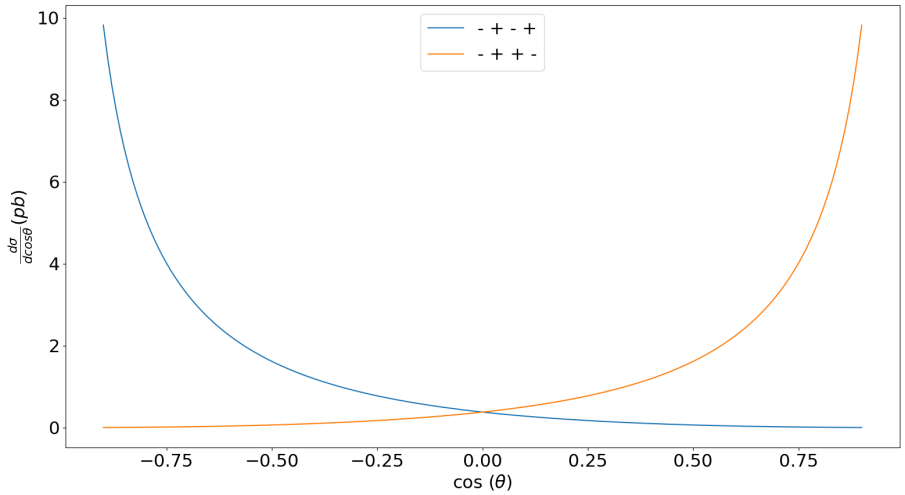
[4] $m^2_{\lambda\lambda} = (k_1 + k_2)^2 = (2E)^2$ is Lorentz invariant.



(a)



(b)



(c)

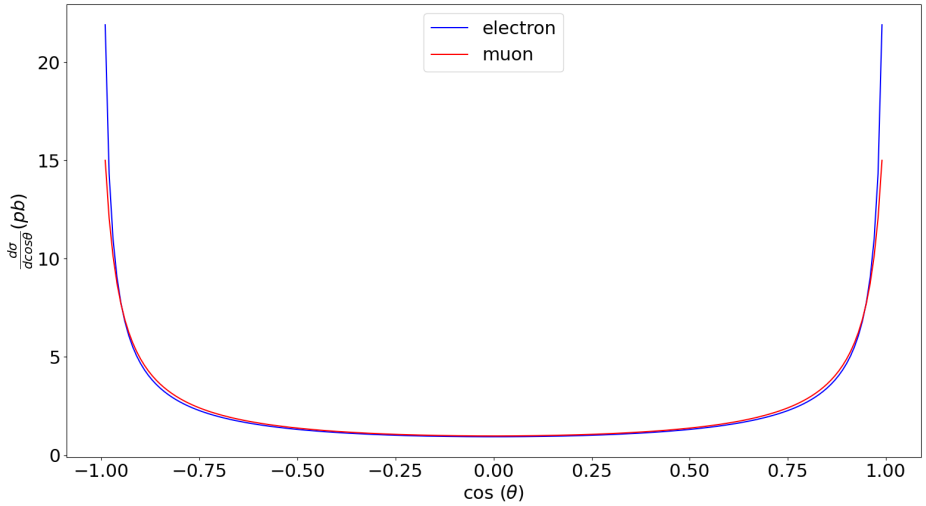
Figure 3.1: The differential cross section in $cos\theta$ for each of 8 polarization states at $m_{\lambda\lambda} = 2E = 2(GeV)$, in case electron (positron) is the loop's fermion. The first Fig. 3.1a and Fig. 3.1b show the symmetry of $|\mathcal{M}|^2$ in $\cos\theta = 0$ axis. That is due to either the 2 initial photons or the 2 final ones have the same spin. That also explains why the third figure Fig. 3.1c doesn't exhibit that feature. We see that the cross sections for some of these eight cases even give the same results because the symmetry when we exchange the two final (or initial) states. For example, the polarization cases $++-+$ and $+++-$ give us identical plots (Fig. 3.1a) .

3.4.2 Differential cross section in $\cos\theta$

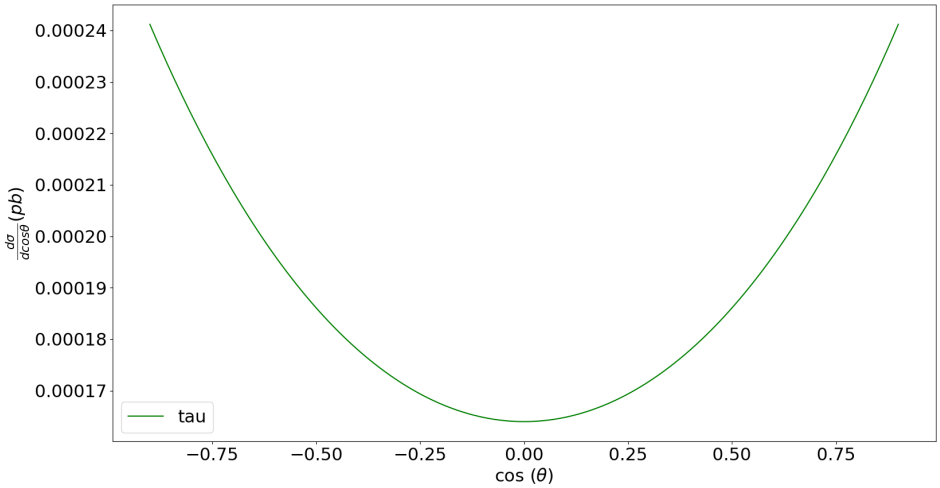
Obtaining the differential cross section with respect to the scattering solid angle from Eq. (1.165) for LbyL scattering, we can also derive that with repect to $\cos\theta$ as:

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{64\pi^2 s}\left|\overline{\mathcal{M}_{fi}}\right|^2 \int_0^{2\pi} d\phi = \frac{1}{128\pi E^2}\left|\overline{\mathcal{M}_{fi}}\right|^2, \tag{3.18}$$

where we obtain $\left|\overline{\mathcal{M}_{fi}}\right|^2$ from Eq. (3.17) as considering the unpolarized cross section. Note that we have acquired Eq. (3.18) because our process is cylindrical symmetry.



(a)



(b)

Figure 3.2: The differential cross section for leptons at $m_{\lambda\lambda} = 2E = 2(GeV)$. The tau's mass is thousand times more than the electron's while its differential cross section is much more smaller. Muon, which is lighter than tau but at leat is hundred times heavier than electron, exhibit its differential cross section slightly above the electron's for the region of small $cos\theta$ but for the large $cos\theta$, electron obtain a much bigger cross section.

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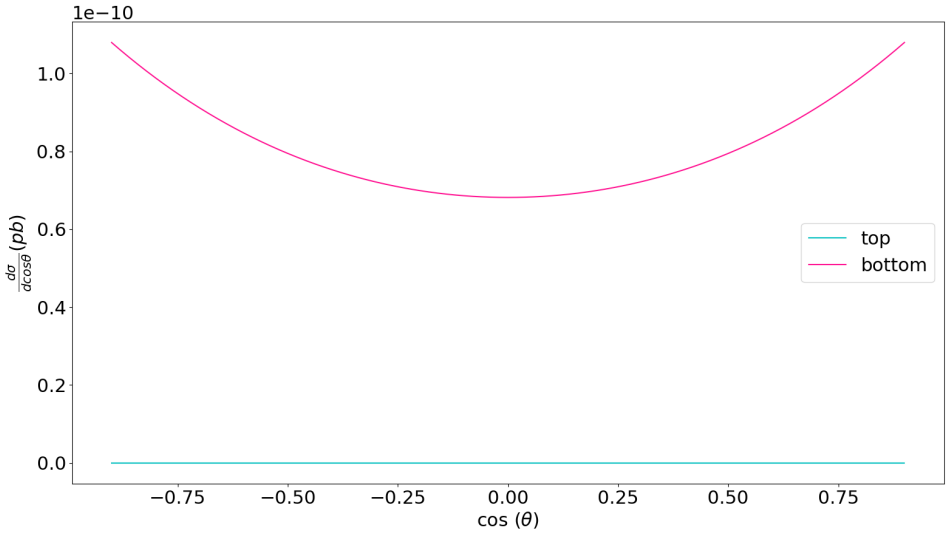


Figure 3.3: The differential cross section of top and bottom quarks. As expected, the top quark and the bottom quarks- the two heaviest fermions, make negligible contribution ($\sim 10^{-11} pb$) to the LbyL scattering at LO

We present above the results of the differential cross section $\frac{d\sigma}{d\cos\theta}$ for different fermions[5]. Consequently, we conclude that the backscattering ($\theta = \pm\pi$) has the largest cross section; and there's a symmetry of the differential cross section about $\cos\theta = 0$. We also observe that there's a remarkable dicrepancy on this quantity between the fermions. Actually, the impact of the fermion on the amplitude is only from its mass and the corresponding coupling strength (see Appendix E). At the first sight, the cross section decreases as the fermion mass increases. So it would make sense when we consider electron (positron), which is much lighter than the others, to be the main contribution to LbyL scattering at LO.

3.4.3 The total cross section

To obtain the total cross section from the differential cross section $f(\cos\theta) = \frac{d\sigma}{d\cos\theta}$, we have used a simple numerical integration method, *Simpson's 1/3 rule* (Ref. [3]) . By which, we divide the range of $\cos\theta$ into 10000 subintervals, that is equivalent to 10000 steps of equal length $h = 2\text{x}10^{-4}$ in $\cos\theta$. The bounded value for the error committed by Simpson's rule is

$$\varepsilon = \frac{h^4}{180}(b-a)f''''(\cos\theta),$$

for the approximation of the integral $\int_a^b f(\cos\theta)d(\cos\theta)$. Because the explicit derivative of $f(\cos\theta)$ is not available, so we have not evaluated the error of the integration here. Besides that, we observe the rapidly large values of the differential cross sections when $\cos\theta \to \pm 1$ (Fig. 3.2a). Thus, to avoid this not well behaved region, we choose the

[5]In calculations of the cross section for quarks, note that each quark has three colors with the same electric charge.

range of the integral to calculate the total cross section is on $[a, b] = [-0.999, 0.999]$. Following those simplification just for comparisions, we obtain the contribution of different fermions to LbyL scattering in total cross section at $m_{\lambda\lambda} = 2(GeV)$ as:

Fermion	Total cross section (pb)
electron	5.0283834739989475
muon	4.7518384393064705
tau	$3.8925224355615970 * 10^{-4}$
u quark	0.58954951200570982
c quark	$9.1128361689925329 * 10^{-4}$
t quark	$6.1704902160624335 * 10^{-21}$
d quark	$2.3116621375507394 * 10^{-3}$
s quark	$2.2247877622484486 * 10^{-3}$
b quark	$1.6790300633556275 * 10^{-10}$

Table 3.1: Total cross section for different fermion's loop, with $m_{\lambda\lambda} = 2(GeV)$. Though the muon's differential cross section is slightly above the electron's on a wide range (Fig. 3.2a), but when considering on a sufficiently larger range, the total cross section of electron indeed contributes most.

3.4.4 Tests

For simplicity, we do the tests considering the cross section of electron (positron) only.

Comparisons

Below is the comparision table about the squared amplitude at E= 0.5 GeV between our program (FORM + COLLIER) and FormCalc + LoopInts programs [6] [7]. The results is obtained at supposedly different mass of electron $\tilde{m}_e$ at a specific phase space point where:

$$\begin{aligned} k_1 &= (0.5, 0, 0, -0.5), \\ k_2 &= (0.5, 0, 0, 0.5), \\ k_3 &= (0.5, -0.136535251025801, 0, 0.480997011661530), \\ k_4 &= (0.5, 0.136535251025801, 0, -0.480997011661530), \end{aligned} \tag{3.19}$$

with all momenta in GeV.

[6] These results are kindly provided by Dr. LE Duc Ninh.

[7] FC + LI-2, FC + LI-4 are denoted for *FormCalc + LoopInts-double-precision* and *FormCalc + LoopInts-quadruple-precision* programs.

$\tilde{m}_e(GeV)$	$0.510998928 * 10^{-3}$	0.510998928
FC+ LI-2	$9.684598879667990 * 10^{-6}$	$7.287224477651457 * 10^{-8}$
FC + LI-4	$9.684598879667734 * 10^{-6}$	$7.287224477651214 * 10^{-8}$
FORM + COLLIER	$\mathbf{9.6845988798}078927 * 10^{-6}$	$\mathbf{7.2872244776}753966 * 10^{-8}$

Table 3.2: Comparison between the squared Feynman amplitude obtained from our program and the ones from FormCalc + LoopInts programs, where we have included the factor $1/8$ as an unpolarized case.

UV- divergence cancellation test

To test the UV- divergence independence, we compare the differential cross section at $\cos\theta = 0.5$ for different values of the pole $\Delta_{UV} \sim \frac{1}{4-d}$ (while other parameters are fixed).

	$\frac{d\sigma}{d\cos\theta}$(pbarn) at $\cos\theta = 0.5$
$\Delta_{UV} = 0$	1.3121686627033162
$\Delta_{UV} = 10$	1.3121686627033402
$\Delta_{UV} = 1000$	1.3121686627032534

Table 3.3: Differential cross sections at different Δ_{UV} values shows LbyL process is independent of UV- divergences, with $m_{\lambda\lambda} = 2(GeV)$.

Gauge invariance test

For any process involving external photons, its amplitude can be always written in the form:

$$\mathcal{M} = \varepsilon^{\alpha}_{\eta_1}(\vec{k}_1)\varepsilon^{\beta}_{\eta_2}(\vec{k}_2)\ldots\Pi_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots), \tag{3.20}$$

where $\varepsilon(\vec{k})$ is the polarization vector for each external photon (we just consider the real vectors to make it simple); Π is the tensor amplitude which doesn't depend on the polarization vectors. For the photon field is expressed as:

$$A^{\mu}(x) \sim \int d^3\vec{k}\left(a(\vec{k})\varepsilon^{\mu}(\vec{k})e^{-ikx} + \text{h.c}\right). \tag{3.21}$$

Under the gauge transformation, the field changes as:

$$A^{\mu}(x) \longrightarrow A^{\mu}(x) + \partial^{\mu}f(x), \tag{3.22}$$

or

$$\varepsilon^{\mu}(\vec{k})e^{\pm ikx} \longrightarrow \left[\varepsilon^{\mu}(\vec{k}) \pm ik^{\mu}\tilde{f}(k)\right]e^{\pm ikx}, \tag{3.23}$$

if we choose $f(x)$ such that:

$$f(x) = \int d^3\vec{k}\left[\tilde{f}(\vec{k})e^{-ikx} + \text{h.c}\right]. \tag{3.24}$$

Thus, for the amplitude Eq. (3.20) to be gauge invariant, we should obtain:

$$k_1^{\alpha}\Pi_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots) = k_2^{\beta}\Pi_{\alpha\beta\ldots}(\vec{k}_1,\vec{k}_2,\ldots) = \ldots = 0. \tag{3.25}$$

All the above conclusions must be applied to our LbyL scattering- a 4-photon process. Accordingly, from the total amplitude of LbyL scattering in Eq. (2.31), we obtain:

$$k_1^{\delta}\Pi_{\alpha\beta\delta\nu} = k_2^{\nu}\Pi_{\alpha\beta\delta\nu} = k_3^{\alpha}\Pi_{\alpha\beta\delta\nu} = k_4^{\beta}\Pi_{\alpha\beta\delta\nu} = 0, \tag{3.26}$$

where $k_4 = k_1 + k_2 - k_3$.
Because gauge invariance is the essence of QED theory, the test on this for our numerical result of LbyL's amplitude would strongly support its correctness. The test can be proceeded by replace $\varepsilon^{\mu}(\vec{k}_i)$ by k_i^{μ} in the program. We should expect the amplitude after the replacement tends to zero. Indeed,

	$\frac{d\sigma}{d\cos\theta}$(pbarn) at $\cos\theta = 0.5$
$\varepsilon_1 = k_1$	$4.9352286252834168 * 10^{-26}$
$\varepsilon_2 = k_2$	$3.5196658122656454 * 10^{-25}$
$\varepsilon_3 = k_3$	$2.6153938811898192 * 10^{-25}$
$\varepsilon_4 = k_4$	$6.0975909315549352 * 10^{-25}$
$\varepsilon_i = \varepsilon_i$	1.3121686627033162

Table 3.4: The cross sections, for $m_{\lambda\lambda} = 2(GeV)$, are almost zero by setting $\varepsilon_i = k_i$ (i=1,2,3,4) as the consequence of gauge invariance.

3.5 Conclusion and Outlook

So far, we have obtained the transition amplitude of LbyL scattering at LO in QED. Considering the interaction as a perturbation, Wick's theorem makes clear about this scattering process as the one mediated by pairs of fermions and anti-fermions in four-leg loops. The UV divergences appeared at first in the individual loop diagrams but are then cancelled by using dimensional regularization. To achieve it, we have decomposed the one-loop integrals into tensor integrals, which can be calculated by the reduction method. After that, FORM program is used to transform the amplitudes into a new form written in terms of scalar and tensor integral coefficients. These coefficients are then calculated by using COLLIER library. Accordingly, the results of differential cross section are derived and show us that LbyL scattering brings a symmetry about $cos\theta = 0$. Comparisions are also made on different fermions, so we can see electron (positron), the lightest charged fermion, contributes most to the process at LO.

Till now, the total cross sections has not really been obtained yet. One problem here is the drastic large values of cross section at $\cos\theta$ close to the bound values ± 1. As the next step, we expect to derive the total cross section by a more efficiently treatable method. Additionally, due to the characteristic of this scattering, which begins in photons and ends in photons, there may be many possible electrically charged particles and anti-particles to mediate the process. For example, we would extend the problem to $W^{\pm}$ bosons in Standard Model or furthermore, the axion- like particles in processes beyond the Standard Model. Up to this point, what we can say about LbyL scattering is that it is not only interesting as a manifestation of a very rare QED phenomenon but also potentially opens the door to further studies on particles in new physics.

Appendix A

Natural units

In relativistic quantum field theory, it's very often that the expressions contain the universal constants as Planck's constant $\hbar$ and the velocity of light c. So we can choose to work in *natural units* (n.u.), which comes along with the imposition:

$$\boxed{\hbar = c = 1.} \tag{A.1}$$

The dimensions of most physical quantities can always represented through the dimensions of *mass* [M], *length* [L], *time* [T] and *electric charge* [Q]. We will see that imposing Eq. (A.1) allows us to describe those above primary dimensions by the dimension of mass [M], so do all other units. Let us clarify that fact:

1. $$c = 1 \Rightarrow [L][T]^{-1} = 1. \tag{A.2}$$

2. From the principle of uncertainty : $\triangle x.\triangle p \sim \hbar$ and $[p] = [M][L][T]^{-1}$, thus:

$$[L]^2.[T]^{-1} = [M]. \tag{A.3}$$

3. The coupling constant in c.g.s units:

$$\alpha = \frac{e^2}{4\pi\hbar c}. \tag{A.4}$$

   For this constant to be dimensionless, i.e. indepedent of the unit system, the dimension of charge in n.u. must be also dimensionless, $[Q] = 0$.

Those result in:

$$[L] = [T] = [M]^{-1}, \text{ and } [Q] = 0. \tag{A.5}$$

Hence, any quantity in n.u. has the dimension $[M]^n$. Here is the table for the value of n of commonly used quantities in n.u. :

Quantity (dimension = $[M]^n$)	n
Velocity	0
Length	-1
Time	-1
Mass	1
Energy- momentum	1
Action	0
Lagrangian or Hamiltonian densities	4
Fine structure constant	0
Electric charge	0
Electromagnetic field $A_\mu(x)$	1
Dirac fields $\Psi(x)$ and $\overline{\Psi}(x)$	$\frac{3}{2}$

All those features make natural unit to be the best unit system to work on theoretical calculations. To transform these to another system, we need the conversion factors between them. For instance, if we want to express the dimensions in units of MeV, centimeter (cm) and second (s) (to make a comparision with experimental results, for instance), here are the useful conversion factors:

$$\hbar = 6.582 \text{ x } 10^{-22} MeV.s,$$
$$\hbar c = 1.973 \text{ x } 10^{-11} MeV.cm\ .$$

For examples:

- If the decay rate of ϕ- meson is $\Gamma = 4.0 MeV$ then its value in second unit is

$$4.0 \text{ x } \frac{\hbar}{6.582 \text{ x } 10^{-22} s} = 6.08 \text{ x } 10^{21} (s^{-1}),$$

- For the length scale $r = 0.001 MeV^{-1}$, it equals to

$$0.001 \text{ x } \frac{1.973 \text{ x } 10^{-11} cm}{\hbar c} = 0.01973 \text{ x } 10^{-12} (cm) = 0.01973 (pm),$$

where $\hbar = c = \hbar c = 1$ in natural units.

Appendix B

Conventions and Notations

We re-introduce here the notations and conventions that are implicitly used throughout the thesis, where we follow Peskin's textbook in most cases (Ref. [12]).

1. Tensors in relativity:

   - The metric tensor of Minkowski space-time is chosen to be:

   $$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{B.1}$$

   where we use the Greek indices $\mu, \nu, \alpha, \beta, \ldots = 0, 1, 2, 3$ to denote four components in space- time, while the Latin indices $i, j, \ldots = 1, 2, 3$ denote for three spatial components.

   - An event in the space-time with that metric is described by a 4-vector as:

   $$x_\mu = (x_0, \vec{x}) = (t, x_1, x_2, x_3), \tag{B.2}$$

   and the corresponding contravariant 4-vector:

   $$x^\mu = g^{\mu\nu} x_\nu = (t, -x_1, -x_2, -x_3). \tag{B.3}$$

   - For other 4-vectors or 4-tensors, we use the same presentation, e.g the energy-momentum vector:

   $$p_\mu = (E, \vec{p}) = (E, p_1, p_2, p_3), \tag{B.4}$$

   and its corresponding invariance for a particle of mass m:

   $$p^2 = p_\mu p_\nu g^{\mu\nu} = p_\mu p^\mu = E^2 - |\vec{p}|^2 = m^2. \tag{B.5}$$

   - The derivative operator is treated as 4-vector in the following way:

   $$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, \nabla \right). \tag{B.6}$$

   That means we have chosen the displacement vector x_μ to be in lower indices naturally, while the derivative vectors are naturally in upper indices.

- The totally antisymmetric Levi-Civita tensor is defined so that: $\epsilon^{0123} = +1$

2. Special functions:

   - Heaviside step function:
   $$\theta(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0, \end{cases} \tag{B.7}$$

   - Dirac delta function:
   $$\delta(x) = \frac{d}{dx}\theta(x) \tag{B.8}$$

   The delta function in n dimensions, denoted as $\delta^{(n)}(x)$ is zero everywhere except at $x = 0$, and the area under its curve always equals to 1:

   $$\int d^n x \delta^{(n)}(x) = 1 \tag{B.9}$$

3. Fourier transformation: The Fourier transformation in momentum integral always attaches with the factors of 2π. The transformations in four dimensional space:

   $$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{f}(k), \tag{B.10}$$

   $$\tilde{f}(k) = \int d^4x e^{ikx} f(x), \tag{B.11}$$

   An important Fourier tranformation is of the delta Dirac function:

   $$\int d^4x e^{ikx} = (2\pi)^4 \delta^{(4)}(k). \tag{B.12}$$

4. Gamma matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are often used when working with Dirac spinors. They can be chosen to be expressed in different representation, as long as satisfy Eq. (1.39). Two of the common representations are

   - Dirac representation:
   $$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{B.13}$$

   - Chiral representation:
   $$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{B.14}$$

   where σ^i are Pauli matrices such that:

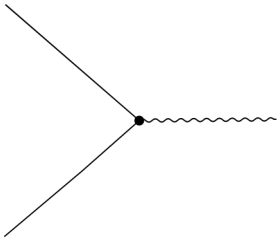
   $$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.15}$$

Appendix C

Feynman Rules in QED

We present here the procedure to obtain the Feynman amplitude for a QED process by Feynman rules. They are indeed a system of rules to derive the amplitude from its corresponding Feynman graph. It is shown that these rules always give us the right amplitude, but in a much more time- saving way in comparision with the one using Wick's theorem (see Subsection 1.2.4). So it is especially advantegeous for a complicated process at high order of perturbation.
To apply Feynman rules for a process at a given order n, follow these steps:

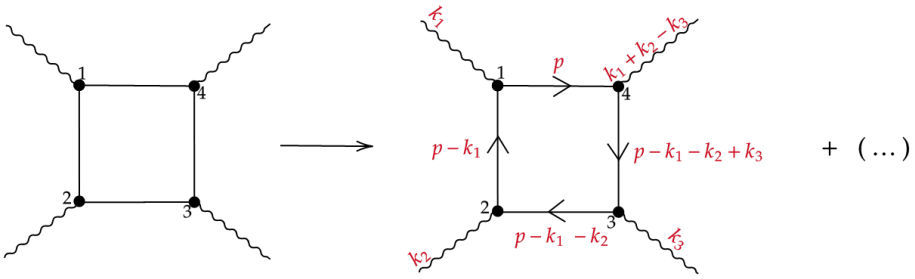
1. The interaction part of QED Lagrangian ($\sim \overline{\Psi} \not{A} \Psi$) forces each vertex (point of interaction) to be the meeting- place of two fermion fields with one photon field. So the vertex in QED should be:



   where the straight line denotes the fermion propagation and the wavy line is for the photon propagation.

2. Depending on the particular transition and the order of perturbation considered, we try to draw possible connected graphs for it. They should include all differently topological diagrams. That can be done by exchanging the momenta between the fields of the same kind and the propagation direction of fermion fields towards the vertices (the amplitude doesn't change when we reverse the propagation direction of photon lines)

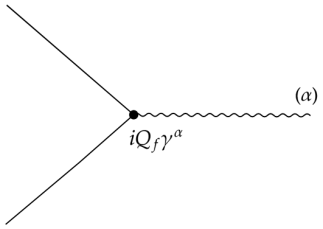
   Take the example of LbyL scattering at order $n = 4$, there can be just one configuration for it (because $n = 4$, all four photon fields should combine with 4 external photons; each two of the 8 remaining fermion fields shouls couple to each other for a fermion propagation). From this configuration, we would obtain totally 6 differently topological diagrams, as expected results from Wick's theorem (see Subsection 2.1.2). Here we just illustrate one diagram of them



3. We label the momentum vector for each line as long as the momentum conservation is obtained at each vertex.

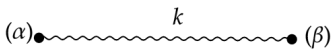
4. As the next step, we admit these rules to "translate" the diagram into its amplitude, i.e. each line corresponds with a factor in the amplitude.

   - At each vertex, write a factor $iQ_f\gamma^\alpha$, where Q_f is the electric charge corresponding to the fermion at the vertex: :



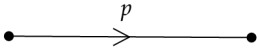
   - For each internal photon line labelled with the momentum k, write a factor:

$$iD_{F\alpha\beta}(k) = i\frac{-g_{\alpha\beta}}{k^2+i\varepsilon}$$

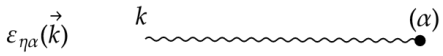


   - For each internal fermion line labelled with the momentum p, write a factor:

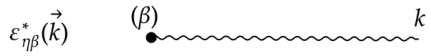
$$iS_{Fab}(p) = i\left(\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\right)_{ab}$$



   - For an external line, write one of the following factors:
     - For an initial photon line:

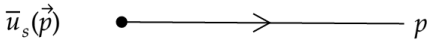


– For a final photon line:

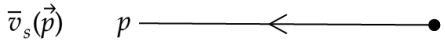


– For an initial fermion line:

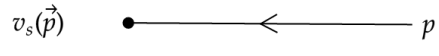
– For a final fermion line:



– For an initial anti–fermion line:



– For a final anti-fermion line:



Note that the indices η or s denote for the spin of photon and (anti-) fermion corresponding to the lines.
After that, we write down these factors one by one while we read the diagram along the opposite direction to the propagation direction of fermions.

5. We divide the amplitude that we have obtained by $n!$. That is due to there are $n!$ ways to permute n vertices and the fact that each vertex should be treated equally.

6. If there is one mometum of an internal line not fixed by the mometum conservation at some vertex, there will be a loop for each one. That is because when a momentum is free to vary, it must cover all values of momenta in the 4-momentum space. Thus, each loop corresponds to the unconstrained momentum p carries out an integral $(2\pi)^{-4}d^4p$. If the loop is a closed fermion loop, take the trace of the amplitude and multiply by a factor (-1) (see the conclusion at the end of Section 2.1)

7. At the final step, multiply the amplitude by a factor $(-1)^r$, where r is the number of required permutations between fermion fields so that they are in normal order.

Appendix D

Dimensionally Regularized Integrals

D.1 Math preliminaries

D.1.1 Feynman's trick

Pay attention on the denominator of the tensor integral Eq. (1.120):

$$T^N \sim \int d^4q \frac{(\dots)}{(q^2 - m_0^2 + i\varepsilon)[(q+p_1)^2 - m_1^2 + i\varepsilon]\dots[(q+p_{N-1})^2 - m_1^2 + i\varepsilon]}. \tag{D.1}$$

The N factors in the denominator have different q- denpendences. Now we want the denominator appear as factors which have the same q- dependences. That's where ***Feynman's trick*** helps.
Firstly, we need to prove the identity:

$$\frac{1}{q^2 - m^2 + i\varepsilon} = (-i)\int_0^\infty d\alpha e^{i\alpha(q^2 - m^2 + i\varepsilon)}. \tag{D.2}$$

Indeed,

$$\begin{aligned}(-i)\int_0^\infty d\alpha e^{i\alpha(q^2-m^2+i\varepsilon)} &= (-i)\frac{e^{i\alpha(q^2-m^2+i\varepsilon)}}{i(q^2-m^2+i\varepsilon)}\Big|_0^\infty \\ &= \frac{1}{q^2-m^2+i\varepsilon} - \lim_{\alpha\to\infty}\frac{\overbrace{e^{-\alpha\varepsilon}}^{=0,\text{with } \varepsilon>0}\, e^{i\alpha(q^2-m^2)}}{i(q^2-m^2+i\varepsilon)} \\ &= \frac{1}{q^2-m^2+i\varepsilon}.\end{aligned} \tag{D.3}$$

Note the requirement that the infinitesimal $\varepsilon > 0$.

For the functions $A_1, A_2, \dots, A_n$ which have the same form as the LHS of Eq. (D.2), we obtain the general result:

$$\frac{1}{A_1 A_2 \dots A_n} = (-i)^n \int_0^\infty d\alpha_1 \dots d\alpha_n e^{i(\alpha_1 A_1 + \dots + \alpha_n A_n)}. \tag{D.4}$$

Using the identity:

$$\int_0^\infty \frac{dq}{q}\delta\Big(1 - \frac{\sum_i \alpha_i}{q}\Big) = 1. \tag{D.5}$$

Applying Eq. (D.5) to Eq. (D.4), we have:

$$\begin{aligned}
\frac{1}{A_1 A_2 \dots A_n} &= (-i)^n \int_0^\infty \frac{dq}{q} \int_0^\infty d\alpha_1 \dots d\alpha_n \delta\Big(1 - \frac{\sum_i \alpha_i}{q}\Big) e^{i(\alpha_1 A_1 + \dots + \alpha_n A_n)} \\
&= (-i)^n \int_0^\infty \frac{dq}{q} \int_0^\infty q^n d\left(\frac{\alpha_1}{q}\right) \dots d\left(\frac{\alpha_n}{q}\right) \delta\Big(1 - \frac{\sum_i \alpha_i}{q}\Big) e^{i(\alpha_1 A_1 + \dots + \alpha_n A_n)} \\
&\overset{\alpha_i = \frac{\alpha_i}{q}}{=} (-i)^n \int_0^\infty dq q^{n-1} \int_0^\infty d\alpha_1 \dots d\alpha_n \delta(1 - \sum_i \alpha_i) e^{iq(\alpha_1 A_1 + \dots + \alpha_n A_n)} \\
&= (-i)^n \int_0^1 d\alpha_1 \dots d\alpha_n \delta(1 - \sum_i \alpha_i) \underbrace{\int_0^\infty dq q^{n-1} e^{iq(\alpha_1 A_1 + \dots + \alpha_n A_n)}}_{= \frac{\Gamma(n)}{(-i)^n (\alpha_1 A_1 + \dots + \alpha_n A_n)^n}},
\end{aligned} \tag{D.6}$$

where we denote:

$$\int_0^1 d\alpha_1 \dots d\alpha_n \delta(1 - \sum_i \alpha_i) = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \dots \int_0^{1-\alpha_1-\alpha_2-\dots-\alpha_{n-1}} d\alpha_n \delta(1 - \sum_i \alpha_i).$$

Thus, we get the ***Feynman's lemma***:

$$\boxed{\frac{1}{A_1 A_2 \dots A_n} = \Gamma(n) \int_0^1 d\alpha_1 \dots d\alpha_n \frac{\delta(1 - \sum_i \alpha_i)}{(\alpha_1 A_1 + \dots + \alpha_n A_n)^n}}, \tag{D.7}$$

with the Gamma function $\Gamma(n) = (n-1)!$, $n = 1, 2, 3, \dots$.
Applying Eq. (D.7) to the case of $n = 2$ and for simplicity, we set $m_0 = m_1 = \dots = m_n = m$:

$$\begin{aligned}
\frac{1}{(q^2 - m^2 + i\varepsilon)[(q+p)^2 - m^2 + i\varepsilon]} &= \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \frac{\delta(1 - \alpha_1 - \alpha_2)}{\Big\{\alpha_1 (q^2 - m^2 + i\varepsilon) + \alpha_2 [(q+p)^2 - m^2 + i\varepsilon]\Big\}^2} \\
&= \int_0^1 d\alpha \frac{1}{\Big\{(1-\alpha)(q^2 - m^2 + i\varepsilon) + \alpha [(q+p)^2 - m^2 + i\varepsilon]\Big\}^2} \\
&= \int_0^1 d\alpha \frac{1}{\Big[q^2 - m^2 + 2\alpha qp + \alpha p^2 + i\varepsilon\Big]^2} \\
&= \int_0^1 d\alpha \frac{1}{\Big[(q + \alpha p)^2 + \alpha p^2 - \alpha^2 p^2 - m^2 + i\varepsilon\Big]^2} \\
&\overset{q + \alpha p = q}{=} \int_0^1 d\alpha \frac{1}{\Big[q^2 + \alpha p^2 (1 - \alpha) - m^2 + i\varepsilon\Big]^2},
\end{aligned} \tag{D.8}$$

and similarly for $n = 3$:

$$\frac{1}{(q^2 - m^2 + i\varepsilon)[(q+p_1)^2 - m^2 + i\varepsilon][(q+p_2)^2 - m^2 + i\varepsilon]} = \int_0^1 d\alpha_1 d\alpha_2 \frac{2}{\left[q^2 + \alpha_1 p_1^2(1-\alpha_1) + \alpha_2 p_2^2(1-\alpha_2) - 2\alpha_1\alpha_2 p_1 p_2 - m^2 + i\varepsilon\right]^3}. \tag{D.9}$$

We can also obtain similar results for $n = 4, 5, \ldots$. We can see the common pattern of Eq. (D.8), Eq. (D.9) so there's a need to calculate the integral:

$$I_n(A) = \int d^d q \frac{1}{(q^2 - A + i\varepsilon)^n} \tag{D.10}$$

before further steps to calculate the N-point tensor integrals.

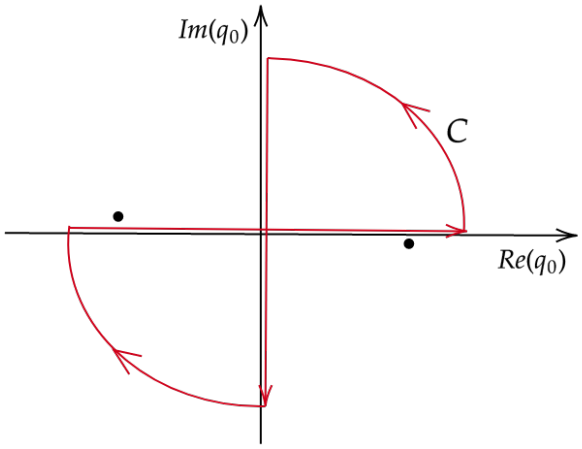
D.1.2 Wick rotation. d-dimensional Euclidean space

Wick roration

The purpose of this section is to calcualte the integral Eq. (D.10). First note that, the poles of Eq. (D.10) in q_0 are:

$$(q_0^P)^2 - |\vec{q}|^2 - A + i\varepsilon = 0 \Rightarrow q_0^P = \pm\sqrt{|\vec{q}|^2 + A - i\varepsilon} = \pm\left(\sqrt{|\vec{q}|^2 + A} - i\varepsilon^*\right), \tag{D.11}$$

with positive infinitesimal $\varepsilon, \varepsilon^*$. Thus the poles are always stay in the second and the fourth quarter of the complex plane. It allows us to change the integral along the real axis to that along the imaginary axis, i.e. ***Wick rotation***:



$$\oint_C (q^2 - A + i\varepsilon)^{-n} = 0$$
$$\Rightarrow \int_{-\infty}^{\infty} dq_0(\ldots) = \int_{-i\infty}^{i\infty} dq_0(\ldots). \tag{D.12}$$

d-dimensional Euclidean space integral

Taking the integral Eq. (D.10) in d-dimesional space with the Lorentz signature $(1,-1,-1,\ldots,$ is somehow hard to work with. By the substitution:

$$q_0 \to iq_0^E. \tag{D.13}$$

Thus leads to the changes:

1. By the above substitution, our d-momentum vetor $q_E = (q_0^E, \vec{q})$ has the signature $(1,1,1,\ldots,1)$ of an Euclidean space:

$$q^2 = q_0^2 - |\vec{q}|^2 = -(q_0^E)^2 - |\vec{q}|^2 = -q_E^2. \tag{D.14}$$

2. We also have:

$$\int_{-\infty}^{\infty} dq_0 = \int_{-i\infty}^{i\infty} dq_0 = i\int_{-\infty}^{\infty} dq_0^E. \tag{D.15}$$

Thus the d-dimensional integral in new space:

$$\int d^d q = \int dq_0 \int d^{d-1}\vec{q} = i\int dq_0^E \int d^{d-1}\vec{q} = i\int d^d q_E. \tag{D.16}$$

Here we introduce the familiar symmetrically spherical coordinates:

$$d^d q_E = q_E^{d-1} dq_E (\sin\theta_{d-1})^{d-2} d\theta_{d-1} \ldots \sin\theta_2 d\theta_2 d\theta_1, \tag{D.17}$$

where $0 \le \theta_1 \le 2\pi$, $0 \le \theta_2, \ldots, \theta_{d-1} \le \pi$.
Using:

$$\int_0^{2\pi} d\theta_1 = 2\pi; \qquad \int_0^{\pi} d\theta \sin^m\theta = \sqrt{\pi}\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}. \tag{D.18}$$

Thus, the d- dimensional solid angle:

$$\int d^{d-1}\Omega = \int_0^{\pi} (\sin\theta_{d-1})^{d-2} d\theta_{d-1} \ldots \int_0^{\pi} \sin\theta_2 d\theta_2 \int_0^{2\pi} d\theta_1 = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{D.19}$$

At last, we obtain the transition of d- dimensional integral in Minkowski space to the one in Euclidean space:

$$\boxed{\int d^d q \to i\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} dq_E q_E^{d-1} = \frac{i\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} dq_E^2 (q_E^2)^{\frac{d}{2}-1}}. \tag{D.20}$$

D.2 Scalar integrals

Now, the integral Eq. (D.10) is rewritten as:

$$\begin{aligned} I_n(A) &= \frac{i\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty dq_E^2 (q_E^2)^{\frac{d}{2}-1} \frac{(-1)^n}{(q_E^2 + A - i\varepsilon)^n} \\ &\overset{q_E^2 = x}{=} (-1)^n \frac{i\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty x^{\frac{d}{2}-1} (x + A - i\varepsilon)^{-n} dx \\ &\overset{\frac{x}{A-i\varepsilon} = y}{=} (-1)^n \frac{i\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty (A - i\varepsilon)^{\frac{d}{2}-n} \underbrace{y^{\frac{d}{2}-1}(1+y)^{-n} dy}_{= B(\frac{d}{2}, n-\frac{d}{2})} \\ &= i(-1)^n \pi^{d/2} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} (A - i\varepsilon)^{\frac{d}{2}-n} \\ &= i(-1)^n \pi^{2-\eta} \frac{\Gamma(n - 2 + \eta)}{\Gamma(n)} (A - i\varepsilon)^{2-n-\eta}, \end{aligned} \tag{D.21}$$

where we have set $d = 4 - 2\eta$. The last equal sign of Eq. (D.21) is due to the Beta- function: $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. We still need a little discussion about the Gamma function before going on:

1. $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ for complex number z which has positive real part. Gamma-function has poles at $z = 0, -1, -2, -3, \ldots$.
2. $\Gamma(z + 1) = z\Gamma(z)$.
3. $\Gamma(n) = (n - 1)!$ for $n = 0, 1, 2, 3, \ldots$.
4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
5. It is also well- known that: $\lim_{z\to 0} \Gamma(z) \sim \frac{1}{z} - \gamma_E$
   where $\gamma_E = \lim_{n\to\infty} \left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - ln(n)\right] \approx 0.577$ is the Euler constant.

D.2.1 1-point function

Following the definition of N-point tensor integrals from Eq. (1.120). We first derive the scalar 1-point function:

$$\begin{aligned} A_0(m) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d q \frac{1}{q^2 - m^2 + i\varepsilon} \\ &= \frac{(2\pi\mu)^{2\eta}}{i\pi^2} I_1(m^2) \\ &= -m^2 \left(\frac{m^2 - i\varepsilon}{4\pi\mu^2}\right)^{-\eta} \Gamma(\eta - 1) + \mathcal{O}\eta. \end{aligned} \tag{D.22}$$

As expected, this function has a pole at $d = 4$ and anylitic for $d < 2$. For we need to evaluate all those integrals at the limit $d \to 4$ ($\lim \eta \to 0^+$) in the final stage, using these

approximations:

$$\left(\frac{m^2-i\varepsilon}{4\pi\mu^2}\right)^{-\eta} = 1 - \underbrace{\eta \ln\left(\frac{m^2-i\varepsilon}{4\pi\mu^2}\right)}_{\ln(z)=\ln(|z|)+i\arg z,\ \text{choose}\ \arg z=0} + \mathcal{O}\left((d-4)^2\right) \approx 1-\eta\ln\left(\frac{m^2}{4\pi\mu^2}\right),$$

by the Taylor expansion: $a^x \approx 1 + xlnx$ when $x \to 0$, and

$$\begin{aligned}\Gamma(\eta-1) &= \frac{1}{\eta-1}\Gamma(\eta) = \frac{1}{\eta-1}\left(\frac{1}{\eta}-\gamma_E+\mathcal{O}(\eta)\right)\\ &\overset{\eta\to 0}{=} -\left(1+\eta+\eta^2+\mathcal{O}(\eta)\right)\left(\frac{1}{\eta}-\gamma_E+\mathcal{O}(\eta)\right) \approx -\left(\frac{1}{\eta}-\gamma_E+1\right).\end{aligned}$$

Thus, we have:

$$A_0(m) = m^2\left[\underbrace{\frac{2}{4-d}-\gamma_E+\ln(4\pi)}_{\triangle_{UV}} - \ln\left(\frac{m^2}{\mu^2}\right)+1\right]+\mathcal{O}(4-d). \tag{D.23}$$

Here we denote $\triangle_{UV}$ as the UV part of the integral.
Hence:

$$(4-d)A_0(m) = 2m^2+\mathcal{O}(4-d). \tag{D.24}$$

D.2.2 2-point function

In a much similar way, the 2-point scalar integral reads (see Eq. (D.8)):

$$\begin{aligned}B_0(m,p) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{1}{(q^2-m^2+i\varepsilon)[(q+p)^2-m^2+i\varepsilon]}\\ &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int_0^1 dx\int d^dq\frac{1}{\left[q^2+xp^2(1-x)-m^2+i\varepsilon\right]^2}\\ &= \frac{(2\pi\mu)^{2\eta}}{i\pi^2}\int_0^1 dx I_2\left(m^2-xp^2(1-x)\right)\\ &= (4\pi\mu^2)^\eta\Gamma(\eta)\int_0^1 dx\left(m^2-xp^2(1-x)-i\varepsilon\right)^{-\eta}\\ &= \frac{1}{\eta}-\gamma_E+\ln(4\pi)-\int_0^1 dx\ln\left(\frac{m^2-xp^2(1-x)-i\varepsilon}{\mu^2}\right)+\mathcal{O}(4-d).\end{aligned} \tag{D.25}$$

(the calculation of the integral in x is not mentioned here, for details about the mathematical methods to calculate more higher N-point integral , see Ref. [9]). Thus,

$$(4-d)B_0(m,p) = 2+\mathcal{O}(4-d) \tag{D.26}$$

D.2.3 3-point and 4-point functions

Though the bigger N of a N-point function, the more complicated is its expression, we can also obtain it as the two previous ones by the same procedure, e.g. the 3- and 4- point scalar integrals:

$$C_0(m,p_1,p_2) = -\int_0^1 dx \int_0^{1-x} dy \left[x(x-1)p_1^2 + y(y-1)p_2^2 + 2xyp_1p_2 + m^2 - i\varepsilon\right]^{-1}, \tag{D.27}$$

and

$$D_0(m,p_1,p_2,p_3) = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \Big[x(x-1)p_1^2 + y(y-1)p_2^2 + z(z-1)p_3^2 + 2xyp_1p_2 + 2xzp_1p_3 + 2yzp_2p_3 + m^2 - i\varepsilon\Big]^{-2}. \tag{D.28}$$

As well as it could be, the 3- and higher- point scalar integrals are all convergent, i.e. their degree of divergence $D < 0$, so there's no pole $\sim 1/(4-d)$ in their expressions.

D.3 Tensor integrals

In this section, we will see that we can derive the tensor integrals from the scalar ones, which we have known the solutions. That method, called the *reduction method*, is simple to apply, especially for low number of $N(= 1, 2)$, but become tougher for the numerical analysis of most physical processes $(N > 3)$.

1. For the first step, note that whenever we encouter such kind of integrals:

$$\int d^dq \frac{p_iq}{(q^2-m^2)\left[(q+p_1)^2-m^2\right]\ldots\left[(q+p_i)^2-m^2\right]\ldots}, \tag{D.29}$$

(where i is the position number of the external momenta p_i in the loop, not Lorentz indices) or

$$\int d^dq \frac{q^2}{(q^2-m^2)\left[(q+p_1)^2-m^2\right]\ldots}. \tag{D.30}$$

We can always bring them back to (linear combination of) other scalar integrals. by transforming the numerator in to terms that conincident with any factor in the numerator. Making some derivations:

$$p_iq = \frac{1}{2}\Big\{\Big[(q+p_i)^2-m^2\Big] - (q^2-m^2) - p_i^2\Big\}, \tag{D.31}$$

$$q^2 = \Big[(q^2-m^2)+m^2\Big]. \tag{D.32}$$

Other similar terms as $(p_ip_j\ldots qq\ldots)$ can also be derived in the same way by implementing Eq. (D.31) and Eq. (D.32) consecutively.

2. As the next step, multiplying the RHS of Eq. (1.120) by the corresponding covariant structures from the external momenta ($p^\mu, p^\mu p^\nu, \ldots$) and metric tensor ($g^{\mu\nu}, \ldots$), we will obtain such terms similar to Eq. (D.29) or Eq. (D.30). Meanwhile, the covariant structures in the tensor decomposition Eq. (1.124) must be multiplied by the same factor. Thus may lead to a homogeneous system of equations, which helps us to find the solutions of the tensor coefficent from the scalar integrals.

3. Also note that, for the integral is taken over all d-momentum space, if the denominator of the tensor integral shown its symmetry to the integral variable q, thus the odd rank of the integral must be zero (odd function), e.g.

$$\int d^d q \frac{\overbrace{q^\mu q^\nu \ldots q^\delta}^{\text{odd number of tensor rank}}}{(q^2 - m^2)} = 0. \tag{D.33}$$

Now, let us illustrate the reduction method for some simple tensor integrals:

- $\underline{B^\mu(m,p)}$:

$$B^\mu(m,p) = \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d q \frac{q^\mu}{(q^2 - m^2)[(q+p)^2 - m^2]} = B_1 p^\mu, \tag{D.34}$$

there we have neglect the infinitesimally imaginary part $i\varepsilon$. Multiply both sides of Eq. (D.34) by p_μ, we obtain:

$$\begin{aligned}
B_1 p^2 &= \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d q \frac{q.p}{(q^2 - m^2)[(q+p)^2 - m^2]} \\
&= \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d q \frac{1}{2} \frac{[(q+p)^2 - m^2] - (q^2 - m^2) - p^2}{(q^2 - m^2)[(q+p)^2 - m^2]} \\
&= \frac{1}{2} \frac{(2\pi\mu)^{4-d}}{i\pi^2} \Big\{ \int d^d q \frac{1}{q^2 - m^2} - \int d^d q \frac{1}{(q+p)^2 - m^2} \\
&\qquad - p^2 \int d^d q \frac{1}{(q^2 - m^2)[(q+p)^2 - m^2]} \Big\} \\
&= \frac{1}{2}(A_0(m) - A_0(m) - p^2 B_0(p,m)) \\
&= -\frac{B_0(m,p)}{2} p^2.
\end{aligned} \tag{D.35}$$

Thus,

$$B_1 = -\frac{B_0(m,p)}{2}. \tag{D.36}$$

And its UV part from the one of B_0:

$$(4-d)B_1 = -1 + \mathcal{O}(4-d). \tag{D.37}$$

- $\underline{B^{\mu\nu}(m, p_1, p_2)}$:

$$B\mu\nu(m,p) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{q^\mu q^\nu}{(q^2-m^2)[(q+p)^2-m^2]} = B_{00}g^{\mu\nu} + B_{11}p^\mu p^\nu. \tag{D.38}$$

At first, multiplying both sides of Eq. (D.38) by $g_{\mu\nu}$ gives us:

$$\begin{aligned} B_{00}d + B_{11}p^2 &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{q^2}{(q^2-m^2)[(q+p)^2-m^2]} \\ &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{(q^2-m^2)+m^2}{(q^2-m^2)[(q+p)^2-m^2]} \\ &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\Big\{\int d^dq \frac{1}{(q+p)^2-m^2} + m^2\int d^dq \frac{1}{(q^2-m^2)[(q+p)^2-m^2]}\Big\} \\ &= A_0(m) + m^2B_0(m,p). \end{aligned} \tag{D.39}$$

To obtain the second equation, make a contraction of Eq. (D.38) with p_μ:

$$\begin{aligned} B_{00}p^\nu + B_{11}(p^2)p^\nu &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{(q.p)q^\nu}{(q^2-m^2)[(q+p)^2-m^2]} \\ &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dq \frac{1}{2}\frac{[(q+p)^2-m^2]-(q^2-m^2)-p^2}{(q^2-m^2)[(q+p)^2-m^2]}q^\nu \\ &= \frac{1}{2}\frac{(2\pi\mu)^{4-d}}{i\pi^2}\Big\{\underbrace{\int d^dq \frac{q^\nu}{q^2-m^2}}_{=0} - \underbrace{\int d^dq \frac{q^\nu}{(q+p)^2-m^2}}_{q^\mu+p^\mu=q'^\mu} \\ &\qquad - p^2\int d^dq \frac{q^\nu}{(q^2-m^2)[(q+p)^2-m^2]}\Big\} \\ &= \frac{1}{2}\frac{(2\pi\mu)^{4-d}}{i\pi^2}\Big\{-\underbrace{\int d^dq' \frac{q'^\nu}{q'^2-m^2}}_{=0} + p^\nu\int d^dq \frac{1}{(q+p)^2-m^2} \\ &\qquad - p^2\underbrace{\int d^dq \frac{q^\nu}{(q^2-m^2)[(q+p)^2-m^2]}}_{=B^\nu(m,p)}\Big\} \\ &= \frac{1}{2}(p^\nu A_0(m) - p^2B^\nu(m,p)) \\ &= \frac{p^\nu}{2}(A_0(m) + \frac{p^2}{2}B_0(m,p)). \end{aligned} \tag{D.40}$$

Thus,

$$B_{00} + B_{11}(p^2) = \frac{1}{2}A_0(m) + \frac{p^2}{4}B_0(m,p). \tag{D.41}$$

Using both Eq. (D.39) and Eq. (D.41), we obtain the tensor coefficents for rank-2 2-point loop integral as:

$$B_{00} = \frac{1}{4(d-1)}\left[2A_0 + (4m^2 - p^2)B_0\right], \tag{D.42}$$

$$B_{11} = \frac{1}{4(d-1)p^2}\left[2(d-2)A_0 + (p^2 d - 4m^2)B_0\right]. \tag{D.43}$$

Consider the limit $d \to 4$, from the UV parts of A_0 and B_0 (Eq. (D.24),Eq. (D.26)), we obtain:

$$(4-d)B_{00} = m^2 - \frac{p^2}{6} + \mathcal{O}(4-d), \tag{D.44}$$

$$(4-d)B_{11} = \frac{2}{3} + \mathcal{O}(4-d). \tag{D.45}$$

Appendix E

Inputs to Fortran program

- Mass of electron : $m_e = 0.510998950 * 10^{-3}(GeV)$ [a]
- Mass of muon: $m_\mu = 105.6583715 * 10^{-3}(GeV)$
- Mass of tau: $m_{tau} = 1776.86 * 10^{-3}(GeV)$
- Mass of up quark $m_u = 2.16 * 10^{-3}(GeV)$
- Mass of down quark $m_d = 4.67 * 10^{-3}(GeV)$
- Mass of strange quark $m_s = 93 * 10^{-3}(GeV)$
- Mass of charm quark $m_c = 1.27(GeV)$
- Mass of bottom quark $m_b = 4.18(GeV)$
- Mass of top quark $m_t = 172.76(GeV)$
- Pi constant $\pi = 3.141592653589793238$
- QED coupling constant for leptons: $\alpha = 1/137.035999139$
- QED coupling constant for up, charm and top quarks: $= \frac{4}{9}\alpha$
- QED coupling constant for down, strange and bottom quarks: $= \frac{1}{9}\alpha$
- Cross section unit conversion formula: $1GeV^{-2} = 0.3893793721(mbarn)$

[a] Those values is taken from Particle Data Group's data in the website: *https://pdg.lbl.gov/*

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