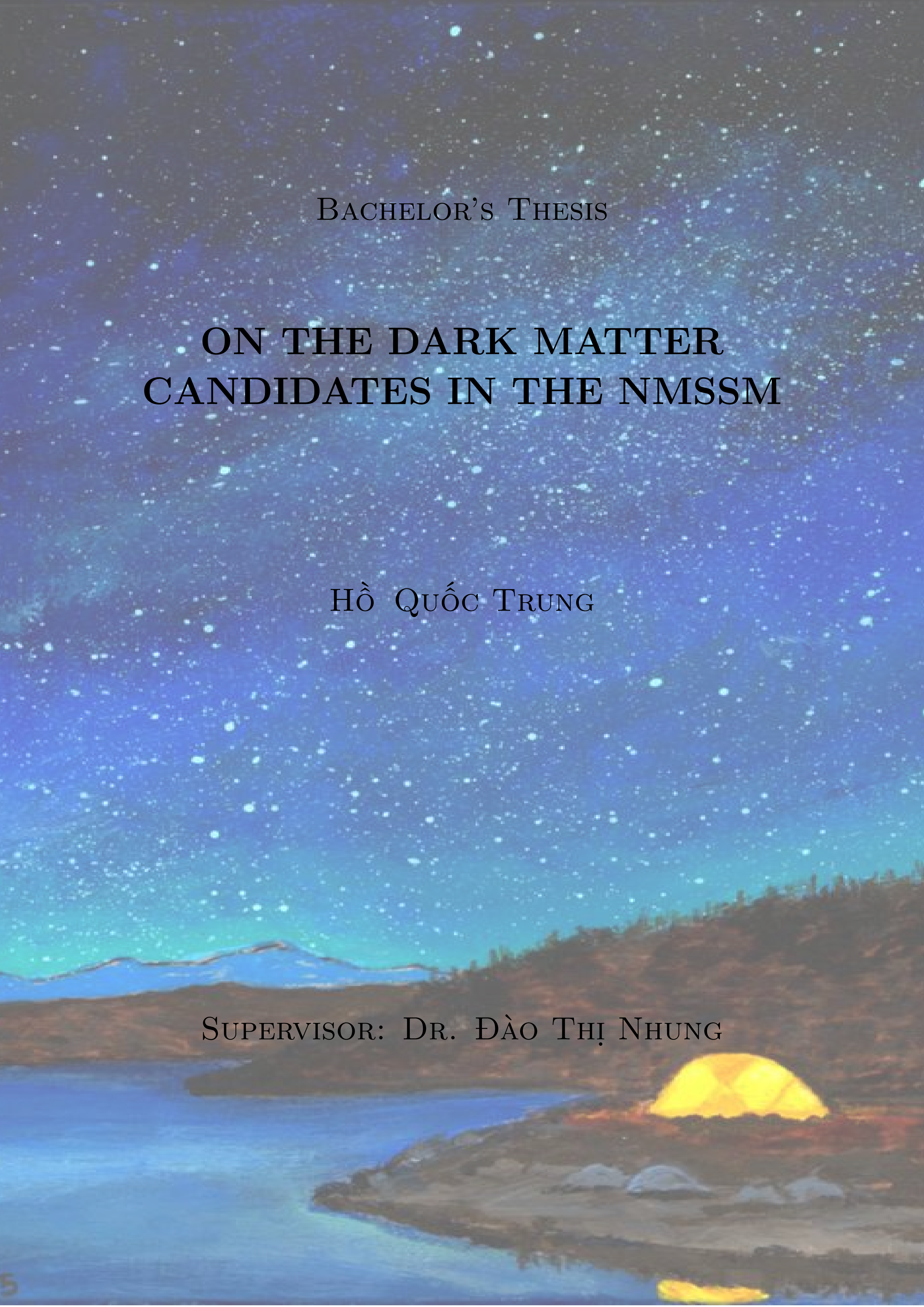

Bachelor's Thesis

ON THE DARK MATTER CANDIDATES IN THE NMSSM

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Introduction

The main purpose of this thesis is to calculate the relic density of the lightest neutralino, assumed to be a DM candidate, in the NMSSM. All of the calculations of the relic density will take into account all possible neutralino pair-annihilation into SM-like final states, with the lightest neutralino is chosen as the most propable DM particle. Various approximations are made in order to achieve reasonable results, which can be compared with cosmological data.

In the first chapter, we reexamined the astrophysical evidences which support the idea that a large amount of dark matter should exists to compensate the missing mass distribution, as well as the basic properties deduced from our observations. A well-known dark matter candidate so-called WIMP is mentioned with the reason why WIMPs together with freeze-out mechanism are considered in many contexts as the very likely mechanism for describing and explaining the nature of the DM.

Follows up the idea of DM is the calculation framework for the relic density of DM species in Chapter II where we review the detailed derivation of the reformulation of the Boltzmann equation and examine carefully various approximation schemes.

The next three chapters provide the basic descriptions of the Standard Model and its supersymmetric extensions. A brief review of the Standard Model and its unsolved questions are presented in Chapter III, which leads us to an extend the SM by incorporating a supersymmetry between fermions and bosons. Basic constructions of a supersymmetric model using superspace and superfields language are represent in Chapter IV, which will be used to build the simplest supersymmetric extension of SM namely the MSSM, and a modification called NMSSM to solve the μ problem arising internally from the MSSM. The full Lagrangian, mass spectrum and related quantities are considered in both models.

With the dark matter properties listed in the first chapter, a type of massive and electrically neutral particle within the context of MSSM and NMSSM so-called neutralino is very well-suited as being a dark matter candidate. Up to tree-level, we consider all possible 2 to 2 processes of neutralino pair annihilation into SM-like particles in Chapter VI. The analytic expressions of scattering amplitude of all mentioned processes are represented together with the analysis on their cross sections. Some tricks on the helicity amplitude method are described at the beginning of this chapter and are implemented into the code.

Chapter VII discuss our numerical results for a particular parameter point. The public package **FeynCALC** is used to generate and simplify helicity amplitudes which are then implemented in our private code. The numerical integrals are taken care of by VEGAS algorithm from package **CUBA** due to its convergence rate, especially when we deal with multidimensional integration range and the integrand contains multiple peaks due to resonances.

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I

Dark Matter in the Universe

Outline

Let us start by a brief discussion about the abundance of particles called *dark matter* (DM), whose existence has not been experimentally confirmed but plays an essential role in the answers of various questions related to the formation of the Universe, as well as providing a reasonable explanation of other astronomical observations.

The main purpose of this chapter is to deliver a review of DM, evidences and motivations. We also represent the common genesis mechanism of DM so-called freeze-out mechanism and a class of DM candidates namely WIMPs. These are relevant to later discussion on the calculation framework of DM abundance as well as the DM candidate we choose to proceed those calculations.

I.1 Observatory Evidences for Dark Matter

Here we want to have a bird's eye view of the most well-known astronomical evidences of the DM. We would not discuss them in detail, as there are many other works done that we mention in the references.

Peculiar motion of galaxy clusters

The very first evidence of dark matter came from the work of Fritz Zwicky [1] when he examined the Coma galaxy cluster in 1933. Theoretically the velocity dispersion of galaxies can be related to the average mass of the system using the virial theorem. He noticed that the outer members of the Coma cluster were moving far too quickly to be merely tracing the gravitational potential of the visible cluster mass. His calculations showed that if the Newtonian gravity governs the cluster kinematic then the measured velocities must corresponds to a cluster mass of 400 times larger than the one inferred from the observed luminous matter. He proposed that there exists a large amount of unseen matter which is called dark matter and popularized that terminology.

Rotation curves of spiral galaxies

The study of rotation curvers of the spiral galaxies confirmed further the existence of a large amount of non-luminous matter. A rotation curve represents the evolution of the velocity of stars with respect to their distance from the center of the galaxy. Assuming that the mass is spherically distributed in the considered galaxy (meaning we ignore the contribution of the spiral arms), the Gauss Law indicates that the relation between the radial velocity of a star which is moving under the gravitational effect of the given galaxy is

$$v_{\text{rot}} = \sqrt{\frac{GM(r)}{r}}, \tag{I.1.1}$$

where r is the distance between the rotating object and the center of mass of the galaxy, G is the gravitational constant and $M(r)$ is the total mass distribution of the star. At large distance where majority of the galaxy mass is enclosed, we expect the mass term M is no longer depend on r, yields the simple relation $v_{\text{rot}} \propto r^{-1/2}$.

However, the work of V. C. Rubin and J. Ford, W. Kent on the rotation curve of Andromeda galaxies ([2]) showed a very slow decrease of rotational velocity with respect to the distance to the galaxy. In Ref. [3], Bosma and his colleagues showed that the observed flatness of rotational curve cannot be solved by modifying the relative weight of the diverse galactic components. It requires a new type of matter component to compensate the missing mass in calculations. The mass distribution $M(r) \propto r$ can be introduced to solve the above flatness, which is the expected relations from a self-gravitational gas of non-interacting particles.

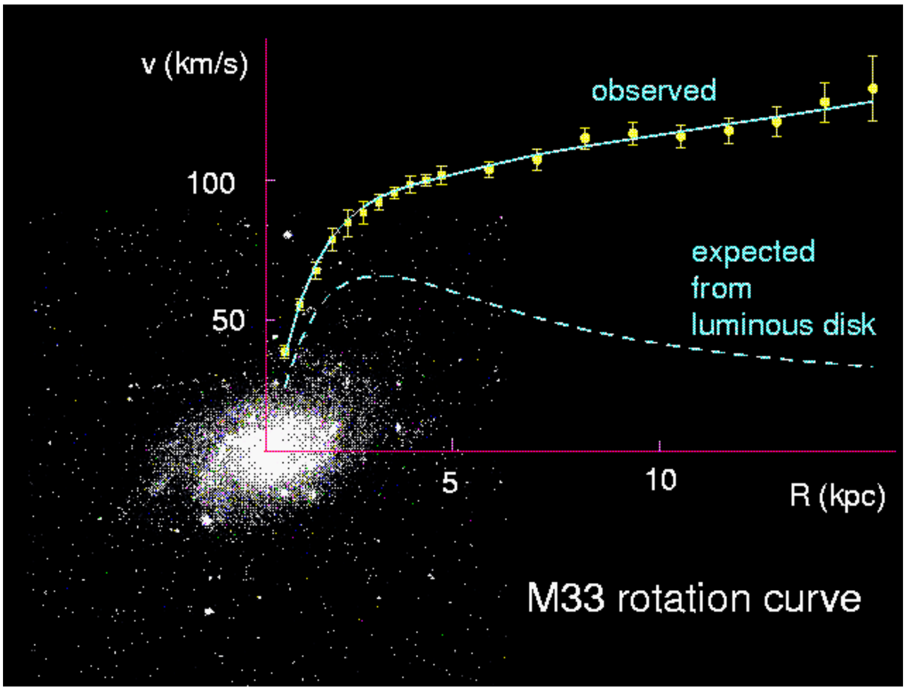


Figure I.1: Comparison between rotation curve from the observation and the prediction based on the luminous disk of the dwarf spiral galaxy M33, with the observed flat curve indicates that the presence of the DM within the cluster. Figure taken from [4].

It should be mentioned that the discrepancies in velocity rotation curves could also arise by modifying the Newtonian gravity. Such MOdified Newtonian Dynamics (MOND) theory, first suggested by M. Milgrom (Ref. [5]), is proposed as an alternative to DM.

Gravitational lensing and the Bullet cluster

The *gravitational lensing* occurs when the light from a more-distant galaxy is bent around an massive object before being observed. This phenomenon is predicted theoretically by the Einstein's General Relativity, in which the gravity of massive object distorts the space-time. This phenomenon is similar to the optical lensing where light goes through inhomogeneous medium, creating a translated and distorted image of the distant light source. The image of the more-distant galaxy looks like a ring, so-called the Einstein's ring. Depending on the ring radius the effect is classified as micro-, weak- or strong-lensing. All of them are exploited to find the distribution of mass (both visible and invisible) in the Universe. Many past and current surveys such as Cosmic Lens All-Skey Survey, Sloan Digital Sky Survey, Sloan Lens ACS have provided a lot of data which support the idea that individual galaxy is built of baryonic material encased inside a much larger halo of dark matter. Interestingly, the observed mas distribution shows that the visible matter concentrates at the core of the galaxy and dominates over the dark matter, however the falls at large radius. This is consistent with requirements to explain the flatness of rotation curves at the outer part of galaxies. For a comprehensive review of this effect we refer the reader to, for instance, [6].

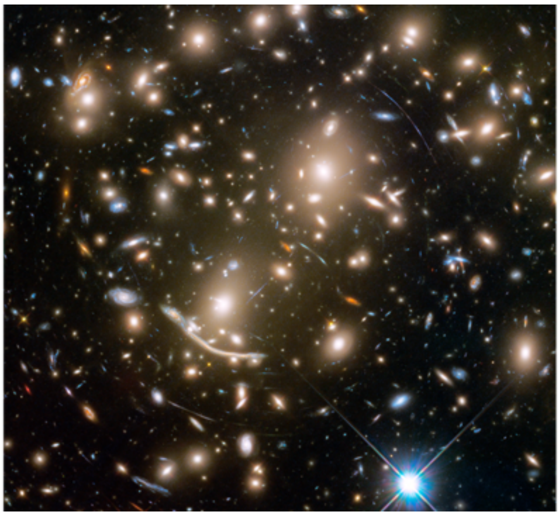


Figure I.2: An example of gravitational lensing effect of a galaxy cluster called Abell 370. Astronomers chose Abell 370 as a target for Hubble because its gravitational lensing effects can be used for probing remote galaxies that inhabited the early Universe. Source: https://hubblesite.org/image/4024/gallery/18-gravitational-lensing.

The observation of the Bullet Cluster provides the best current evidence for the nature of DM and disfavor the MOND solutions. In observing the collision between two clusters of galaxies, the major ingredients of these clusters behave differently during the collision. The stars of the galaxies (objects that emit visible light) is not significantly affected by the collision, most of these are slowed down by the impact of gravity but its distribution is not charged during and after the collision. In contrast, the intergalatic gases made up the majority of the visible matter largely diffuse, heat up and emit X-ray. They seem to move slower compared to the stars and concentrate at the center of the Bullet Cluster. Observations of mass distribution of the Bullet Cluster through gravitational lensing shows that the dominant mass goes along with the galaxies and does not coincide with the concentration of the intergalatic gases. This can only be explained by the dark matter halos which come along with the galaxies and do not interact when two clusters collide.

I.2 Basic Properties of Dark Matter

Before moving to the searches of the DM candidate experimentally and the DM candidates from theoretical perspective, we want to make a brief summary of the DM properties that we learn from the astrophysical observations. Various results from measurements arguably suggest that suitable candidates for DM should be a neutral, nonrelativistic, non baryonic and yet stable ones. This section represents the simple arguments as well as astrophysical observations that lead to the claimed set of DM properties.

Electrically neutral & Collisionless

Dark matter is (possibly) a set of particles that do not absorb, reflect or emit photon, since such events are detectable at some characteristic forms of radiation. DM particles thus generally believed to be electrically neutral or at least are milicharged particles. Constraints on heavy millicharged particles are inferred from cosmological and astrophysical observations (e.g [7]) give an upper bound on the charge ϵ as

$$\epsilon \leq 3.5 \times 10^{-7} (M_{\rm DM}/\mathrm{GeV})^{0.58}, \quad (M_{\rm DM} > 1\,\mathrm{GeV}) \tag{I.2.1}$$

which is more stringent than the upper bound from CMB [8]

$$\epsilon \leq 7.6 \times 10^{-4} M_{\rm DM}/(\mathrm{TeV})^{1/2}. \tag{I.2.2}$$

The same arguments apply for the interaction between the DM and ordinary matters; in some context this property is referred as "collisionless". The DM particles collide with neutral atoms can lead to excitation or even ionization if the colliding energy is sufficiently high.

Non-baryonic

From the results of CMB, no more than 20% matter content is made of ordinary matters: atoms and molecules, which are bound states of protons, neutrons and electrons. Protons and neutrons carry most of the mass and is collectively named as baryonic matter. The rest 80% of matter content is an unknown non-baryonic form, which refers to the non-baryonic DM problem. More specifically, measurements of anisotropies indicate that the Universe is close to flat, hence the mass-energy density of the Universe must be equal to the critical density $\rho_c = 3H_0^2/(8\pi G)$, where G is the Newton's gravitational constant, and H_0 is the present value of the Hubble constant. It is useful to expressed the abundance of each type of matter and energy in unit of energy density through the density parameter

$$\Omega_i = \rho_i/\rho_c, \tag{I.2.3}$$

where i stands for baryonic matter, non-baryonic matter or energy density.

As mentioned in [9], the combination analysis of three types of observations: supernova measurements of the recent expansion history of the Universe, cosmic microwave background measurements of the degree of spatial flatness, and measurements of the amount of matter in galaxy structures obtained through big galaxy redshift surveys, suggests the matter density $\Omega_m \simeq 0.27$, the energy density $\Omega_\Lambda \simeq 0.73$. The baryon density does not exceed $\Omega_b \simeq 0.05$, then the remaining $\Omega_m - \Omega_b \simeq 0.22$ is of non-baryonic DM.

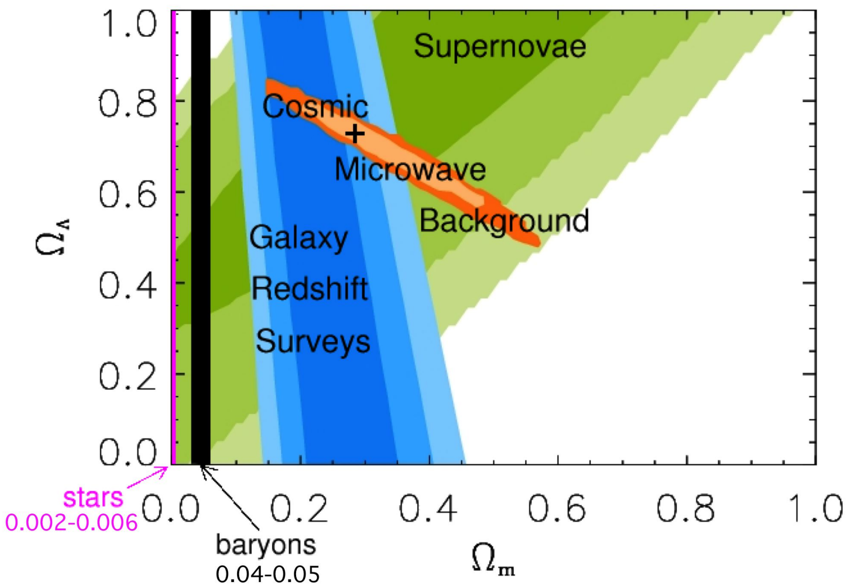


Figure I.3: Three measurements which determine the percentage of energy density, matter density and baryon density. The black plus sign gives the best fit point obtained from the data [10]. Figure taken from Ref. [9].

Planck spacecraft observations of the CMB in 2013 gave a more accurate estimate of 68.3% dark energy, 26.8% dark matter and 4.9% ordinary matter (Ref. [11, Table 10]).

Long-lived

Since the footprint of the DM can still be observed nowadays through the gravitational effects in the cluster of galaxies and its essential role for structure formation, DM is widely believed to be composed of stable or at least long-lived particles with the lifetime exceeding the age of the Universe.

A wide class of stable DM candidates is proposed by models in which new discrete symmetry is imposed to ensure that the lightest particle with an exotic charge is suitable as a DM candidate. For instance, we can assume that all particles are either even or odd with respect to a specific discrete symmetry. The lighest odd particle is stable since it cannot decay into lighter particles of the SM (that are assumed to be evenly charged). Within the scope of the thesis, we will discuss about the supersymmetric extension of the SM with an extra symmetry so called R-parity that distinguish between a SM particle and its superpartner. In the R-parity conserved models, the lightest supersymmetric particle is chosen to be the DM candidate. More details will be discussed in Chapter V and VI.

Massive & non-relativistic

From the simulations of the early Universe, it is soon realized that the DM has to be non-relativistic (cold dark matter, or CDM for short) at the epoch of structure formation (which is not necessarily true at present time for formation of galaxies). Hot dark matter (HDM) has a larger free-streaming length than standard cold dark matter, i.e the HDM particles travel with relativistic velocity, and the distance they travel before becoming non-relativistic is larger than the scale of the density fluctuations required for galaxy formation ([12]; for neutrinos HDM see [13]). One thus conclude that HDM particles cannot constitute the majority of DM.

On the other hand, cold dark matter simulations at the galactic scales can lead to too much structuer in DM haloes. A possible solution for this situation is warm dark matter (WDM). These particles are relativistic when freezing-out, but becomes non-relativistic sufficiently quick after. With the WDM scenarios, the sub-galactic structure formation is suppressed. A studying about the satellite galaxies in the Milky Way (Ref. [14]) shows that WDM masses somewhat larger than 1 keV would yield a minimum dark matter halo mass consistent with the mass scale being observed.

I.3 Freeze-out Mechanism & WIMPs

Decoupling & Freeze-out of particles

Astrophysical data have shown that the DM exists and constitues four fifth of the total matter in the Universe at the present time. A question one can ask is how it was produced in the early Universe. There are many proposals of the DM genesis mechanism, together with the DM candidates; for a review we refer the readers to [7; 15]. In our study the DM candidate is the lighest supersymmetric particle, we therefore adopt the most relevant production mechanism so-called *freeze-out*. All detailed calculations of DM relics will be presented in Chapter II and Chapter VI; here we give a short introduction that motivates our study. Due to its simplicity yet robustness, freeze-out mechanism is considered in many contexts as a very likely one to be realized in nature. Let us go a little more detailed about decouplings and freeze-out of a particle species before the discussion of the corresponding DM candidate so-called WIMPs as below.

The basic ideas are proceeded as follows. We consider a homogeneous and isotropic Universe which is well-described by the FRW cosmological model (see Appendix D). Within this framework, our Universe expands with the rate of expansion determined by the Hubble parameter $H \equiv \dot{R}/R$ with R is the scale factor corresponds to the spatial coefficient of the FLRW metric. The DM species are assumed to be abundant and in the thermal equilibrium (i.e both in kinetic and chemical equilibrium) with the thermal bath containing all other types of particles in the early Universe, when the equilibrium temperature exceeds the mass m_χ of the DM particle. The *kinetic equilibrium* is maintained by the elastic scattering processes between the DM species and other type of particles. A system of particles in kinetic equilibrium has the phase space occupancy given by the Bose-Einstein or Fermi-Dirac distributions depends on its statistical nature:

$$f(\mathbf{p},T) = \frac{1}{\frac{E(\mathbf{p})-\mu(T)}{T} \pm 1}, \tag{I.3.1}$$

with the $+$ sign for bosons and $-$ sign for fermions; μ is the chemical potential of the given particles collection and generally depends on temperature T. On the other hand, *chemical equilibrium* is achieved when the rate of forward and backward reactions of a reversible process are at equal. With the given distribution one can derive a full set of evolution equations for the density of each species i, which we will represent in the next chapter. It is reasonable and intuitively to guess that the change of number density depends on two main factor: the dilusion of the system due to the expansion of Universe characterized by rate of expansion H, and the collision processes where the number of particles is not conserved, which is characterized by the interaction rate $\Gamma_i = n_i \langle \sigma_i v_i \rangle$. To a sufficiently good approximation, we can distinguish between two regimes based on the comparison of Γ_i and H:

- $\Gamma_i \gtrsim H$: particles of species i are being created and destroyed within a Hubble time; the collision term thus dominates and the collection of particles of the given species evolves while remain in thermal equilibrium.
- $\Gamma_i \lesssim H$: the collision term cannot compensate the expansion of the Universe. The system then departs from thermal equilibrium (i.e decouple from the thermal bath). The abundance of the species i (by abundance here we mean the total number of particles within a comoving volume) remains nearly constant of after decoupling, a phenomenon which is known as freeze-out.

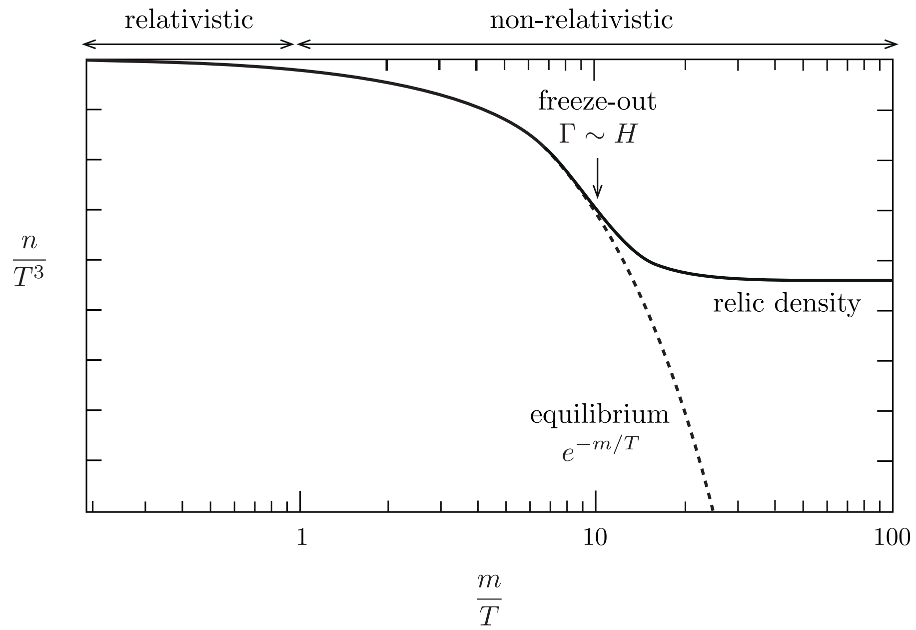


Figure I.4: A schematic illustration of the freeze-out mechanism. Before freeze-out point, the number density traces closely with the equilibrium one. The number density then begin to deviate from the equilibrium and reaches a constant value after decoupling from thermal bath, while the equilibrium density is Boltzman supressed with the factor $e^{-m/T}$. Figure taken from [16].

WIMPs & the WIMP miracle

WIMPs stand for weakly interacting massive particles, which is a type of CDM candidate and is the most frequently considered subject in studying about DM particles, corresponds to the WIMP miracle - a simple mechanism of the dark matter generation in the early Universe. Here we give a brief review about this mechanism: assuming the existence of heavy stable and neutral particle χ. Assuming χ interact weakly with other particles, but the $\chi\chi$ annihilation rate is sufficiently large to keep χ in thermal equilibrium at high temperature. When the temperature drops, at some point the species χ decouple from thermal bath and "freeze-out", implying that the quantity n_χ/s approaches a constant value [1] , where s is the entropy density. A complete approximation scheme can be found in e.g [17]. The result on mass-to-entropy ratio yields

$$\frac{\rho_\chi(T_0)}{s_0} = m_\chi \frac{n_\chi(T_0)}{s_0} \simeq \frac{\ln\left(M^*_{\text{Pl}} m_\chi \sigma_0\right)}{\sqrt{g_*(T_f)} M_{\text{Pl}} \sigma_0(T_f)}. \tag{I.3.2}$$

On the other hand, the mass-to-entropy ratio calculated via astronomical observations is

$$\frac{\rho_{\text{DM}}(T_0)}{s_0} = \frac{\Omega_{\text{DM}}\rho_c}{s_0} \approx \frac{0.26 \times 5 \times 10^{-6}\,\text{GeV}\,\text{cm}^{-3}}{3000\,\text{cm}^{-3}} = 4 \times 10^{-10}\text{GeV}. \tag{I.3.3}$$

Comparing Eq. (I.3.2) and (I.3.3) one obtains the estimate on the order of the weighted annihilation cross section:

$$\sigma_0(T_f) \equiv \langle \sigma_\chi v \rangle (T_f) \sim 10^{-36}(\text{cm}^2), \tag{I.3.4}$$

which is of order of weak scale interacion. This "coincidence" is referred as the WIMP miracle.

It is arguably that the most prominent and well-motivated WIMP candidate is the neutralino of a supersymmetric extension of the standard model. These are the mass eigenstates of the basis containing two neutral Higgsino and two neutral gauginos, which is a neutral particle that interact weakly with the rest of the particle content. We will first represent the theoretical background of relic density calculations in Chapter II, next we construct the basis of the NMSSM - a supersymmetric extension of SM, and finally focus on calculations of relic density with the lighest neutralino being the studying objects in Chapter VI and VII.

[1] This can be understand roughly as follows: after freezing out, the number of particles in a comoving volume is conserved; the same holds for the total entropy in that volume. This implies their ratio remains constant after χ freezing-out, so as to the ratio between number density and entropy density. More detailed discussion will be proceeded on the next chapter.

II

Calculation Framework of the Relic Density

Outline

As mentioned in the first chapter, many measurements has put constraints on the relic density of the DM. Our next tasks are to represent a theoretical background for calculating the relic density of a general particle species. The evolution of the number density of such species is described by the Boltzmann transport equation. Using reasonable approximations, one can cast the Boltzmann equation into a non-linear first order differential equation of Ricatti form, and the task becomes solving this first order ordinary differential equation with physical initial condition. We start with the solution of the Boltzmann equation by dividing our work into two parts:

- Chapter II focuses on the derivation and reformulation of the Boltzmann equation, with detailed discussion on the approximation schemes and asymptotic behaviour of the solution in some special cases.
- Chapter VII summarizes the steps on solving the Boltzmann equation numerically, including the cases that approximations can give a rough result.

II.1 The Boltzmann Transport Equation

This chapter is devoted to introduce the key concepts of the Boltzmann equation, which describe the time evolution of the distribution function. Recall that a distribution function $f = f(x^\mu, p^\mu)$ is defined generally on the phase space and is dependent on four-position and four-vector. Note that through the mass-energy equivalence $E^2 = m^2 + \mathbf{p}^2$, and since m is a const for each type of particle, we can write

$$f(x^\mu, p^\mu) = f(\mathbf{x}, \mathbf{p}, t). \tag{II.1.1}$$

The physical meaning of the distribution function is such that

$$\begin{aligned} &\frac{g}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t)\, \mathrm{d}\Gamma = \frac{g}{(2\pi)^3} f(\mathbf{x}, \mathbf{p}, t)\, \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{p} \\ &= \#\text{ of particles in the volume element } \mathrm{d}^3\mathbf{x} \text{ about } \mathbf{x} \text{ and with momenta in a range } \mathrm{d}^3\mathbf{p} \text{ about } \mathbf{p} \text{ at time t,} \end{aligned} \tag{II.1.2}$$

where g is the number of intrinsic degrees of freedom (i.e spin degrees of freedom; other quantum numbers are contained within the calculations of cross sections later) of the particle under consideration. Eventhough both $\mathrm{d}^3\mathbf{x}$ and $\mathrm{d}^3\mathbf{p}$ are not Lorentz-invariant, the phase space element $d\Gamma$ is a Lorentz-scalar so that the number of particles within the given infinitesimal phase-space volume is also a Lorentz-scalar.

The Boltzmann equation attemps to describe the evolution equation of phase-space distribution f, which is expressed abstractly in the form

$$\hat{\mathbf{L}}[f] = \hat{\mathbf{C}}[f], \tag{II.1.3}$$

where the first term is obtained by acting the Liouville operator on the distribution to describe the change of number of particles within a unit phase-space volume. The second term, so-called *collision functional*, describes the change of the number due to collisions between them.

II.1.1 The Liouville Term

In order to distinguish between the phase-space differential and time differential, let us temporarily use the notation Δ for the increment of the phase-space volume. We start by considering the number of particles contained in a volume $\Delta\Gamma(t)$ at a specific time t

$$\Delta N(t) = f(\mathbf{x}, \mathbf{p}, t)\Delta\Gamma(t). \tag{II.1.4}$$

The changes in position and momentum with respect to time are

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\boldsymbol{\tau}}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathbf{p}}{m}\frac{m}{E} = \frac{\mathbf{p}}{E}, \tag{II.1.5}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}, \mathbf{p}, t). \tag{II.1.6}$$

The change in number of particles within this volume after an infinitesimal amount of time $\mathrm{d}t$ is

$$\mathrm{d}\Delta N(t) = \Delta N(t + \mathrm{d}t) - \Delta N(t) = [\mathrm{d}f(\mathbf{x}, \mathbf{p}, t)]\,\Delta\Gamma(t) + f(\mathbf{x}, \mathbf{p}, t)\mathrm{d}\Delta\Gamma(t). \tag{II.1.7}$$

The variation of the distribution with respect to time is obtained by Taylor expanding f around the point $(\mathbf{x}, \mathbf{p}, t)$

$$\mathrm{d}f(\mathbf{x}, \mathbf{p}, t) = f(\mathbf{x} + \mathrm{d}\mathbf{x}, \mathbf{p} + \mathrm{d}\mathbf{p}, t + \mathrm{d}t) - f(\mathbf{x}, \mathbf{p}, t) = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{\partial f}{\partial \mathbf{x}}\mathrm{d}\mathbf{x} + \frac{\partial f}{\partial \mathbf{p}}\mathrm{d}\mathbf{p}. \tag{II.1.8}$$

Next we calculate the variation of the phase-space volume

$$\mathrm{d}\Delta\Gamma(t) = \left\{\det\left[\frac{\partial\left(\mathbf{x} + \mathbf{p}\mathrm{d}t/E, \mathbf{p} + \mathbf{F}dt\right)}{\partial(\mathbf{x}, \mathbf{p})}\right] - 1\right\}\Delta\Gamma(t) = \left(\frac{\partial}{\partial \mathbf{p}}\cdot\mathbf{F}\right)\Delta\Gamma(t)\mathrm{d}t. \tag{II.1.9}$$

Plugging the variations (II.1.8) and (II.1.9) back to (II.1.7) yields

$$\mathrm{d}\Delta N(t) = \left[\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{E}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\partial(\mathbf{F}f)}{\partial \mathbf{p}}\right]\Delta\Gamma(t)\mathrm{d}t. \tag{II.1.10}$$

The LHS describes the change in number of particles within a Lorentz-invariant phase-space volume, thus implies that the RHS is also a Lorentz scalar, and is possibly cast into a manifestly covariant form. We shall show below how to write explicitly such form. Consider the first two terms in Eq. (II.1.10) and noting that $\mathrm{d}t/E = \mathrm{d}\tau/m$, we have

$$\left[\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{E}\cdot\frac{\partial f}{\partial \mathbf{x}}\right]\mathrm{d}t = \left[E\frac{\partial f}{\partial t} + \mathbf{p}\cdot\frac{\partial f}{\partial \mathbf{x}}\right]\frac{\mathrm{d}t}{E} = \frac{p^\mu}{m}\frac{\partial f}{\partial x^\mu}\mathrm{d}\tau. \tag{II.1.11}$$

This means that the last term $\partial(\mathbf{F}f)\partial\mathbf{p}$ can also be written in an covariant manner. Note that the "classical" force vector $\mathbf{F}$ defined in Eq. (II.1.6) is not the spatial part of some four-vector. For convenience, let us introduce a new vector which can be interpret as spatial part of a four-force vector by rescaling $\mathbf{F}$ as

$$\mathbf{K} \equiv \gamma\mathbf{F} = \frac{p^0\mathbf{F}}{m}. \tag{II.1.12}$$

Obviously, the covariant form containing $\mathbf{K}$ is

$$K^{\mu} = \frac{\mathrm{d}p^{\mu}}{\mathrm{d}\tau} \tag{II.1.13}$$

Similar to the Minkowski orthogonality relation between four-position x^{μ} and four-velocity $v^{\mu} \equiv \mathrm{d}x^{\mu}/\mathrm{d}\tau$, we shows here a similar relation between four-momentum p^{μ} and four-force K^{μ} using the on-shell condition:

$$\begin{aligned} p^{\mu}p_{\mu} = m^2 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left(p^{\mu}p_{\mu}\right) = 0 \Rightarrow 2p_{\mu}\frac{\partial p^{\mu}}{\partial \tau} = 0 \\ \Rightarrow K^{\mu}p_{\mu} = 0 \Rightarrow K^0 p_0 = \mathbf{K}\cdot\mathbf{p}. \end{aligned} \tag{II.1.14}$$

The proof of the invariance of the third term in Eq. (II.1.10) can be proceed as follows

$$\begin{aligned} \frac{\partial(\mathbf{F}f)}{\partial\mathbf{p}}\mathrm{d}t &= \frac{E}{m}\frac{\partial(\mathbf{F}f)}{\partial\mathbf{p}}\mathrm{d}\tau = \frac{E}{m}\left[\frac{\mathbf{p}}{E}\frac{\partial}{\partial p^0} + \frac{\partial}{\partial\mathbf{p}}\right]\cdot\left(\frac{m\mathbf{K}f}{p^0}\right)\mathrm{d}\tau \\ &= \frac{\partial}{\partial p^0}\left(\frac{\mathbf{p}\cdot\mathbf{K}f}{p^0}\right) + \frac{\partial}{\partial\mathbf{p}}\cdot(\mathbf{K}f)\,\mathrm{d}\tau \\ &\overset{(\text{II.1.14})}{=} \left[\frac{\partial}{\partial p^0}\left(K^0 f\right) + \frac{\partial}{\partial\mathbf{p}}(\mathbf{K}f)\right]\mathrm{d}\tau = \frac{\partial}{\partial p^{\mu}}\left(K^{\mu}f\right)\mathrm{d}\tau. \end{aligned} \tag{II.1.15}$$

Note that in the calculations above, E is the on-shell energy while p^0 is consider as a variable independent of $\mathbf{p}$ and we only set $p^0 = E$ after all of the derivatives have been taken. From Eq. (II.1.11), (II.1.15) and (II.1.10) we obtain the covariant form of the change in particle number

$$\begin{aligned} \mathrm{d}\Delta N(t) = \left[\frac{p^{\mu}}{m}\frac{\partial f}{\partial x^{\mu}} + \frac{\partial}{\partial p^{\mu}}\left(K^{\mu}f\right)\right]\Delta\Gamma\mathrm{d}\tau \\ \Rightarrow \hat{\mathbf{L}}[f] \equiv \frac{\mathrm{d}\Delta N(t)}{\mathrm{d}t}\Delta\Gamma = \frac{1}{E}\left[p^{\mu}\frac{\partial f}{\partial x^{\mu}} + m\frac{\partial(K^{\mu}f)}{\partial p^{\mu}}\right]. \end{aligned} \tag{II.1.16}$$

The last attempt we make to generalize the Liouville term by taking into account the effect of gravitational field. Following the work in [18, Section 12.3], the modification is made to the four-gradient term

$$p^{\mu}\frac{\partial f}{\partial x^{\mu}} \longrightarrow p^{\mu}\frac{\partial f}{\partial x^{\mu}} - \Gamma^{\mu}_{\rho\sigma}p^{\rho}p^{\sigma}\frac{\partial f}{\partial p^{\mu}}, \tag{II.1.17}$$

where the components of Christoffel symbol $\Gamma^{\mu}_{\rho\sigma}$ can be calculated via derivatives of the metric tensor

$$\Gamma^{\mu}{}_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\left(g_{\nu\rho,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu}\right). \tag{II.1.18}$$

Let us consider a homogenous and isotropic space. It is suitable to work with the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological model with the corresponding metric reads

$$(\mathrm{d}s)^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \mathrm{d}t^2 - R(t)^2\mathrm{d}\mathbf{\Sigma}^2, \tag{II.1.19}$$

where the dimensionless quantity $R(t)$ represents the scale factor of the Universe, and the homogenous and isotropic spatial part can be cast into the form

$$\mathrm{d}\mathbf{\Sigma} = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{II.1.20}$$

The results are greatly simplified since most of the components of the metric as well as the Christoffel symbols (calculated using Eq. (II.1.18)) vanishes, leaving only the following non-zero ones

$$g_{00} = 1, \quad g_{ij} = -R^2\delta_{ij}, \quad \Gamma^0_{ij} = \dot{R}R\delta_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{R}}{R}\delta^i_j = H\delta^i_j, \tag{II.1.21}$$

where $H \equiv \dot{R}/R$ is the Hubble parameter; the dot indicates a derivative with respect to time. Plugging these components into Eq. (II.1.17) yields

$$\boxed{\hat{\mathbf{L}}[f] = E\left[\frac{\partial f}{\partial t} - H|\mathbf{p}|\frac{\partial f}{\partial|\mathbf{p}|}\right] = E\left[\frac{\partial f}{\partial t} - H\frac{|\mathbf{p}|^2}{E}\frac{\partial f}{\partial E}\right].}\,^{1} \tag{II.1.22}$$

The number density is obtained by taking integral of the phase-space distribution over momentum

$$n = g \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, f. \tag{II.1.23}$$

To obtain the evolution of the number density, let us consider the following integral

$$\begin{aligned} g \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\hat{\mathbf{L}}[f]}{E} &= g \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \left[\frac{\partial f}{\partial t} - H\,|\mathbf{p}|\,\frac{\partial f}{\partial\,|\mathbf{p}|}\right] = g\frac{\partial}{\partial t}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, f - gH \int \frac{\mathrm{d}\,|\mathbf{p}|}{4\pi^2}\,|\mathbf{p}|^3\,\frac{\partial f}{\partial\,|\mathbf{p}|} \\ &= \frac{\mathrm{d}n}{\mathrm{d}t} + 3gH \int \frac{\mathrm{d}\,|\mathbf{p}|}{4\pi^2}\,|\mathbf{p}|^2\, f = \boxed{\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn}, \end{aligned} \tag{II.1.24}$$

where we perform the integration by parts on both terms containing derivative over time and over the norm of momentum, and dropping the boundary terms. [1] In a collisionless process or a system which are in chemical equilibrium where the rate of production of particles is identical with the rate of annihilation, the Liouville functional must evaluate to zero, indicating

$$\dot{n} + 3Hn = 0 \tag{II.1.25}$$

Furthermore, a collisionless or equilibrium system implies the conservation of number of particles within a comoving box. Indeed, we can rewrite the expression (II.1.24) in collisionless system as

$$g \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{\hat{\mathbf{L}}[f]}{E} = \dot{n} + 3Hn = \frac{1}{R^3}\frac{\mathrm{d}\left(nR^3\right)}{\mathrm{d}t} = 0. \tag{II.1.26}$$

II.1.2 The Collision Term

Taking the integral over momentum both sides of the relativistic Boltzmann equation, and applying the explicit form of the Liouville operator in Eq. (II.1.24) gives

$$\frac{\mathrm{d}n}{\mathrm{d}t} + 3Hn = \frac{g}{(2\pi)^3} \int \hat{\mathbf{C}}[f]\,\frac{\mathrm{d}p^3}{E} \tag{II.1.27}$$

We now consider the creation or annihilation of particles due to collision. The derivation the explicit form of this term is quite lengthy, we refer the reader to Ref. [20, Section 1.3]. Below we use the result in a general form before applying some approximations on this collision term. Labelling the particle under consideration χ, and assuming a collection of general processes for creation and annihilation χ are of the form $(\chi + a + \cdots \leftrightarrow i + j + \dots)$, the integral of the collision term in momentum space is [2]

$$\boxed{\begin{aligned} \frac{g_\chi}{(2\pi)^3}\int \hat{\mathbf{C}}[f_\chi]\,\frac{\mathrm{d}^3\mathbf{p}}{E_\chi} = -\sum_{\substack{\text{processes}\\ \text{spins}}} \int & \mathrm{d}\Pi_\chi\,\mathrm{d}\Pi_a \dots \mathrm{d}\Pi_i\,\mathrm{d}\Pi_j \dots \\ & \times (2\pi)^4 \delta^{(4)}\left(p_\chi + p_a + \cdots - p_i - p_j - \dots\right) \\ & \times \frac{1}{S_{i,j,\dots}}\Big[\left|\mathcal{M}_{\chi+a+\dots\to i+j+\dots}\right|^2 f_\chi f_a \dots (1+\eta_i f_i)(1+\eta_j f_j)\dots \\ & \quad - \left|\mathcal{M}_{i+j+\dots\to\chi+a+\dots}\right|^2 f_i f_j \dots (1+\eta_\chi f_\chi)(1+\eta_a f_a)\dots\Big], \end{aligned}} \tag{II.1.28}$$

with a sum over all possible processes where the number of χ is not conserved, and over all spin degrees of freedom of both inital and final states. The Lorentz invariant phase-space element is defined as (taking into

[1] It should be noted that the momentum vector $\mathbf{p}$ is so-called the *local momentum*, obtained by scaling the usual momentum with a factor $R(t)$. See Ref. [19, p. 15] for the discussion on this newly introduced quantity.
[1] It is often assumed that the phase-space distribution and its derivative vanishes at the integration boundary at infinity.
[2] This collision term belongs to Uehling-Uhlenbeck equation, a quasi-classical Boltzmann equation that incorporates the quantum statistics. These modification can be ignored in some cases of interest as we mentioned below.

account the spin degrees of freedom g)

$$\mathrm{d}\Pi = \frac{1}{(2\pi)^3}\frac{\mathrm{d}^3 p}{2E}. \tag{II.1.29}$$

$|\mathcal{M}|^2$ is the unpolarized square matrix element. The symmetry factor $S_{i,j,\ldots}$ is included to account for identical final state particles, such that if there are N identical particles $\Rightarrow S_{i,j,\ldots} = N!$. Finally, the factor η characterizes the statistics of the type of particle under consideration, given by

$$\eta = \begin{cases} +1 & \text{for Bose-Einstein statistics} \\ 0 & \text{for Maxwell-Boltzmann statistics} \\ -1 & \text{for Fermi-Dirac statistics} \end{cases}. \tag{II.1.30}$$

It is necessary to take a look at the collision term Eq. (II.1.28) and see if there is any approximation can be done to simplify this expression further. Some notes should be taken at this step:

- Compared to the classical version of Boltzmann equation, the collision term (II.1.28) has no factor 1/2 ahead. This fact also valid when the two incoming particles are identical (e.g in the case of Majorana fermions; we will consider later the Majorana fermion pair-annihilation with our study object is the lightest neutralino in Chapter VI.)

- In the scope of this thesis, the coannihilations would be neglected and would be consider in future plan. This makes sense when the mass difference between the DM particle and the coannihilation particles are small enough so that the contribution of these processes are negligible. For the sake of completeness, we will represent the general approach of calculating the DM relic density including the coannihilations effects at the end of this chapter.

- Assuming the equilibrium distribution follows the relativistic Boltzmann distribution in the cosmic co-moving frame (the frame where the collection of particles is considered to be at rest, i.e there is no bulk flow of particles).

$$f^{\mathrm{eq}}(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1} \approx e^{-(E(p)-\mu)/T} \approx e^{-E(p)/T}. \tag{II.1.31}$$

  This is a good approximation for temperatures $T \lesssim 3m_\chi$. Note that from now on, our calculations stick mainly to the Boltzmann distribution and thus the comoving frame is always assumed, unless explicitly stated. At early Universe, the chemical potentials of all particles are presumably negligible.

- The quantum statistical factor can be ignored, i.e both of the Pauli-blocking term $(1-f)$ and Bose-enhancement $(1+f)$ term ~ 1. These mechanical factors can be neglected if we consider the massive particles which decouple in the early Universe while they are a non-degenerate gas, as mentioned in [**21**, **Section 2**]. This happens due to the fact that f is negligible if we already apply the condition of the Boltzmann statistics above, i.e when $T \lesssim 3m$.

- The annihilation products $(i, j, \ldots)$ go quickly to equilibrium with the thermal background[1]. Thus the phase-space distribution can be replaced by the corresponding equilibrium distribution

$$f_i \equiv f_i^{\mathrm{eq}}, f_j \equiv f_j^{\mathrm{eq}}, \ldots \tag{II.1.32}$$

- With the assumption that all annihilation products is in equilibrium with the thermal bath, the principle of detailed balance can be applied: each elementary process is in equilibrium with its reverse process. This principle can be represented by the equality between the phase-space of the equilibrium system of the initial states and final states, i.e

$$f_i^{\mathrm{eq}} f_j^{\mathrm{eq}} \cdots = f_\chi^{\mathrm{eq}} f_a^{\mathrm{eq}} \ldots \tag{II.1.33}$$

- Assuming no CP violation (equivalently T invariance) in the DM sector, hence the unitarity of the scattering matrix implies

$$\int |\mathcal{M}_{i+j+\cdots\to\chi+a+\ldots}|^2 (2\pi)^4 \delta^{(4)} (p_\chi + p_a + \cdots - p_i - p_j - \ldots)\, \mathrm{d}\Pi_i\, \mathrm{d}\Pi_j \ldots$$

[1] This is straightforward if the products are electrically charged since they interact with the thermal photons present. Ref. [**21**] also mention this is true in most cases of interests for neutral particles.

$$= \int |\mathcal{M}_{\chi+a+\dots\to i+j+\dots}|^2 (2\pi)^4\delta^{(4)}(p_\chi + p_a + \dots - p_i - p_j - \dots)\, \mathrm{d}\Pi_i\, \mathrm{d}\Pi_j \dots \tag{II.1.34}$$

To simplify the expression, we use here the definition of the unpolarized cross section

$$\begin{aligned}&\frac{1}{S_{i,j,\dots}} \sum_{\text{spins}} \int |\mathcal{M}_{\chi+a+\dots\to i+j+\dots}|^2 (2\pi)^4\delta^{(4)}(p_\chi + p_a + \dots - p_i - p_j - \dots)\, \mathrm{d}\Pi_i\, \mathrm{d}\Pi_j \dots \\ &= 4F(g_\chi g_a \dots)\sigma_{\chi+a+\dots\to i+j+\dots},\end{aligned} \tag{II.1.35}$$

where the products inside the parentheses are over spins of all incoming states, and the invariant flux F is defined as

$$F = \sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}. \tag{II.1.36}$$

It is obvious that the cross section contains all the information about the collision process. Since there is a sum over all possible processes where the number of particles χ is non-conserving, we introduce a new notation of total cross section σ_χ for saving space

$$\sigma_\chi \equiv \sum_{\text{processes}} \sigma_{\chi+a+\dots\to i+j+\dots}. \tag{II.1.37}$$

Using Eq. (II.1.32), (II.1.33), (II.1.34), (II.1.35) and (II.1.37), The collision term (II.1.28) can be rewritten as

$$\frac{g_\chi}{(2\pi)^3}\int \hat{\mathbf{C}}[f_\chi]\frac{\mathrm{d}^3\mathbf{p}}{E_\chi} = -\int 4F\sigma_\chi \left(f_\chi f_a \dots - f_\chi^{\text{eq}} f_a^{\text{eq}}\dots\right)(g_\chi \mathrm{d}\Pi_\chi)(g_a\Pi_a)\dots \tag{II.1.38}$$

where the RHS is being integrated over the products of momenta spaces of incoming particles. At this point, the expression of the collision term is remarkably simplified compared to the original form, but is not in a practical form for numerical computation. Let us keep working on further approximations.

- We shall assume that the main contribution to the total cross section comes from 2 to 2 scattering processes: $\chi_1\chi_2 \to \psi_1\psi_2$, where ψ_1 and ψ_2 are SM-like particles. The collision term (II.1.38) is explicitly written as

$$\begin{aligned}\frac{g_\chi}{(2\pi)^3}\int \hat{\mathbf{C}}[f_{\chi_1}]\frac{\mathrm{d}^3\mathbf{p}_1}{E_1} &= -\int 4F\sigma_{\chi_1}\left(f_{\chi_1}f_{\chi_2} - f_{\chi_1}^{\text{eq}}f_{\chi_2}^{\text{eq}}\right)(g_{\chi_1}\mathrm{d}\Pi_{\chi_1})(g_{\chi_2}\mathrm{d}\Pi_{\chi_2}) \\ &= -\int \sigma_{\chi_1}\frac{F}{E_1E_2}\left(f_{\chi_1}f_{\chi_2} - f_{\chi_1}^{\text{eq}}f_{\chi_2}^{\text{eq}}\right)\frac{g_\chi \mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{g_\chi \mathrm{d}^3\mathbf{p}_2}{(2\pi)^3} \\ &= -\int \sigma_{\chi_1}v_{\text{Mø}l}\left(\mathrm{d}n_1\,\mathrm{d}n_2 - \mathrm{d}n_1^{\text{eq}}\,\mathrm{d}n_2^{\text{eq}}\right),\end{aligned} \tag{II.1.39}$$

with the Møller velocity is defined as [1]

$$v_{\text{Mø}l} \equiv \frac{F}{E_1E_2} = \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}{E_1E_2}, \tag{II.1.40}$$

such that the invariant interaction rate per unit volume per unit time can be written in an arbitrary frame of reference as [2]

$$\frac{\mathrm{d}N}{\mathrm{d}V\mathrm{d}t} = \sigma v_{\text{Mø}l} n_1 n_2. \tag{II.1.41}$$

[1] Note that the Møller velocity only coincides with the relative velocity in cases the velocities of incoming particles are colinear. Indeed

$$\begin{aligned}v_{\text{Mø}l} \equiv \frac{\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}{E_1E_2} &= \sqrt{1 - \frac{2\mathbf{p}_1\cdot\mathbf{p}_2}{E_1E_2} + \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{E_1^2E_2^2} - \frac{\left(E_1^2-\mathbf{p}_1^2\right)\left(E_2^2-\mathbf{p}_2^2\right)}{E_1^2E_2^2}} \\ &= \sqrt{\mathbf{v}_1^2 + \mathbf{v}_2^2 - 2\mathbf{v}_1\mathbf{v}_2 + (\mathbf{v}_1\cdot\mathbf{v}_2)^2 - \mathbf{v}_1^2\mathbf{v}_2^2} = \sqrt{(\mathbf{v}_1-\mathbf{v}_2)^2 - (\mathbf{v}_1\times\mathbf{v}_2)^2},\end{aligned}$$

thus $|v_{\text{Mø}l}| = |\mathbf{v}_1 - \mathbf{v}_2|$ only when $\mathbf{v}_1\times\mathbf{v}_2 = \mathbf{0}$.

[2] The RHS of this equation is indeed invariant and is easily checked: plugging the definition (II.1.40) into (II.1.41), and note that the quantity n/E is invariant under a Lorentz transformation:

$$\frac{n}{E} \xrightarrow{\Lambda} \frac{n'}{E'} = \frac{\gamma n}{\gamma E} = \frac{n}{E}.$$

For more discussion about the Møller flux and related quantities, we refer to [22].

- Assuming that the particle species χ_1 maintains in the kinetic equilibrium even after decoupling (out of chemical equilibrium). Follows the arguments on symmetry in [19, Chapter 6], the distributions in kinetic equilibrium are proportional to that of chemical equilibrium, which allows us to rewrite Eq. (II.1.39) as

$$\boxed{\frac{g_{\chi_1}}{(2\pi)^3}\int \hat{\mathbf{C}}[f_\chi^{(1)}]\,\frac{\mathrm{d}^3\mathbf{p}_1}{E_1} = -\left\langle \sigma_{\chi_1} v_{\text{Møl}}\right\rangle \left(n_1 n_2 - n_1^{\text{eq}} n_2^{\text{eq}}\right),} \tag{II.1.42}$$

with the thermal averaged of the total cross section times Møller velocity is taken with the assumed Boltzmann distribution

$$\left\langle \sigma_{\chi_1} v_{\text{Møl}}\right\rangle = \frac{\int \sigma_{\chi_1} v_{\text{Møl}}\,\mathrm{d}n_1^{\text{eq}} n_2^{\text{eq}}}{\int \mathrm{d}n_1^{\text{eq}} n_2^{\text{eq}}}. \tag{II.1.43}$$

Together with Eq. (II.1.24) and (II.1.42), the Boltzmann equation can now be cast into the form

$$\begin{aligned}
&\frac{g_\chi}{(2\pi)^3}\int \hat{\mathbf{L}}[f_\chi^{(1)}]\,\frac{\mathrm{d}^3\mathbf{p}_1}{E_1} = \frac{g_\chi}{(2\pi)^3}\int \hat{\mathbf{C}}[f_\chi^{(1)}]\,\frac{\mathrm{d}^3\mathbf{p}_1}{E_1} \\
&\Rightarrow \dot{n}_1 + 3Hn_1 = -\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \left(n_1 n_2 - n_1^{\text{eq}} n_2^{\text{eq}}\right),
\end{aligned} \tag{II.1.44}$$

which is applicable both before and after decoupling.

As mentioned earlier, the total cross section is summed over all final states and averaged over initial spins without the symmetric factor for identical final states, which is the case of our interest since we will consider the annihilation of a Majorana fermion so-called neutralino later. Let $n = n_{\chi_1} = n_{\chi_2}$, Eq. (II.1.44) in such cases with identical initial states becomes [1]

$$\boxed{\dot{n}_\chi + 3Hn_\chi = -\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \left[n_\chi^2 - (n_\chi^{\text{eq}})^2\right].} \tag{II.1.46}$$

In the scope of this thesis, only the annihilation of the lightest neutralino will be investigated, thus we will simply ignore the non-identical initial particles above (i.e no co-annihilation processes will be covered in the context).

The asymptotic value of the number density must approach 0 when t tends to ∞. One can easily verified that the temperature drops when t increases, thus the collision term significantly drops, leaving behind the asymptotic collisionless Boltzmann equation. The total number of a species "freeze-out" and tends to a constant value while the volume of space keep increasing due to the expansion of the Universe, leading to the behaviour of n_χ at large t (or small T). It is practical to define a new quantities which should be independent of the expansion of the space. Introduce the comoving number density by scaling n_χ with a factor of entropy density s: $Y_\chi = n_\chi/s$. Assuming the entropy conservation in a comoving volume, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(sR^3\right) = 0 \Rightarrow \dot{s}R^3 + 3R^2\dot{R}s = 0 \Rightarrow \dot{s} = -3Hs. \tag{II.1.47}$$

Taking the time derivative of Y_χ gives

$$\begin{aligned}
\frac{\mathrm{d}Y_\chi}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_\chi}{s}\right) = \frac{\dot{n_\chi}}{s} - \frac{n_\chi \dot{s}}{s^2} = \frac{\dot{n}_\chi + 3Hn_\chi}{s} \\
&= -\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \frac{\left[n_\chi^2 - (n_\chi^{\text{eq}})^2\right]}{s} = -s\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \left[Y_\chi^2 - (Y_\chi^{\text{eq}})^2\right].
\end{aligned} \tag{II.1.48}$$

[1] It should be noted that there is a factor 1/2 in front of total cross section if we have to deal with non-identical initial particles. For instance the reactions between a charged particle χ and its anti-partner $\overline{\chi}$, the density $n = n_\chi + n_{\overline{\chi}} = 2n_\chi$; the evolution of each species χ and $\overline{\chi}$ is the same as Eq. (II.1.46), meaning

$$\begin{aligned}
&\dot{n}_\chi + 3Hn_\chi = -\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \left[n_\chi^2 - (n_\chi^{\text{eq}})^2\right], \\
&\Rightarrow \dot{n} + 3Hn = 2\left(\dot{n}_\chi + 3Hn_\chi\right) = -\frac{1}{2}\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \left[(2n_\chi)^2 - (2n_\chi^{\text{eq}})^2\right] = -\frac{1}{2}\left\langle \sigma_\chi v_{\text{Møl}}\right\rangle \left[n^2 - (n^{\text{eq}})^2\right].
\end{aligned} \tag{II.1.45}$$

It is customary to express Y_χ as a function of dimensionless parameter $x \equiv m_\chi/T$ where T is the temperature of thermal bath:

$$\frac{\mathrm{d}Y_\chi}{\mathrm{d}t} = \frac{\mathrm{d}Y_\chi}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = -\frac{\mathrm{d}Y_\chi}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s}(3Hs)$$

$$\Rightarrow \frac{\mathrm{d}Y_\chi}{\mathrm{d}x} = -\frac{1}{3Hs}\frac{\mathrm{d}s}{\mathrm{d}x}\frac{\mathrm{d}Y_\chi}{\mathrm{d}t} = \frac{1}{3H}\frac{\mathrm{d}s}{\mathrm{d}x}\left\langle\sigma_\chi v_{\mathrm{Møl}}\right\rangle\left[Y_\chi^2 - (Y_\chi^{\mathrm{eq}})^2\right]. \tag{II.1.49}$$

The final modification on the Boltzmann equation is to express H and s in terms of *degrees of freedom parameters*. Recall that from one of the Friedmann equation, Eq. (D.2.1a), shows a relation between the Hubble parameter and the energy density:

$$H = \sqrt{\frac{8\pi G\rho}{3}}, \tag{II.1.50}$$

where G is the gravitational constant. The energy density ρ and entropy density s are functions of T and can be written in the form

$$\rho = g_{\mathrm{eff}}(T)\frac{\pi^2}{30}T^4, \quad s = h_{\mathrm{eff}}(T)\frac{2\pi^2}{45}T^3, \tag{II.1.51}$$

where the temperature-dependent g_{eff} and h_{eff} are respectively the effective internal and spin degrees of freedom. Substituting Eq. (II.1.51) and (II.1.50) back to (II.1.49), we finally arrive at the equation of the evolution of the comoving density Y_χ with respect to x

$$\frac{\mathrm{d}Y_\chi}{\mathrm{d}x} = -\sqrt{\frac{\pi}{45G}}\frac{g_*^{1/2}m}{x^2}\left\langle\sigma_\chi v_{\mathrm{Møl}}\right\rangle\left[Y_\chi^2 - (Y_\chi^{\mathrm{eq}})^2\right], \tag{II.1.52}$$

$$g_*^{1/2} = \frac{h_{\mathrm{eff}}}{g_{\mathrm{eff}}^{1/2}}\left(1 + \frac{1}{3}\frac{T}{h_{\mathrm{eff}}}\frac{\mathrm{d}h_{\mathrm{eff}}}{\mathrm{d}T}\right), \tag{II.1.53}$$

with the degrees of freedom parameter g_* accounts for the temperature dependence of the relativistic degrees of freedom of energy density and entropy density.

Note that the effective energy and spin degrees of freedom include two contributions: one from the species in thermal equilibrium which share the same temperature T, the other from the decoupled species which no longer stay in equilibrium with the photon bath and thus having a different temperature T_i, with i stands for the species (both bosons and fermions) we are considering.

$$g_{\mathrm{eff}} = g_{\mathrm{eff}}^{\mathrm{th}}(T) + g_{\mathrm{eff}}^{\mathrm{dec}}(T) = \sum_{i\in\mathrm{th}} g_i(T) + \sum_{i\in\mathrm{dec}} g_i(T)\frac{T_i^4}{T^4}. \tag{II.1.54}$$

Similarly for the calculations of effective entropy degrees of freedom

$$h_{\mathrm{eff}} = h_{\mathrm{eff}}^{\mathrm{th}}(T) + h_{\mathrm{eff}}^{\mathrm{dec}}(T) = \sum_{i\in\mathrm{th}} h_i(T) + \sum_{i\in\mathrm{dec}} h_i(T)\frac{T_i^3}{T^3}, \tag{II.1.55}$$

with the effective degrees of freedom of each species at temperature T is calculated numerically via the following integrals

$$\begin{aligned} g_i(T) &= \frac{30}{\pi^2 T^4}\rho_i(T) = \frac{30}{\pi^2 T^4}\frac{g_i}{(2\pi)^3}\int \frac{E_i(\mathbf{p})\mathrm{d}^3p}{e^{E_i(\mathbf{p})/T} + \eta_i} \\ &= \frac{15 g_i}{\pi^4}x_i^4\int_1^\infty \frac{y\sqrt{y^2-1}}{e^{x_i y} + \eta_i}y\,\mathrm{d}y, \end{aligned} \tag{II.1.56}$$

$$\begin{aligned} h_i(T) &= \frac{45}{2\pi^2 T^3}s_i(T) = \frac{45}{2\pi^2 T^3}\frac{g_i}{(2\pi)^3}\int \frac{3m_i^2 + 4P_i^2\mathrm{d}^3p}{3E_i(\mathbf{p})T_i}\frac{1}{e^{E_i(\mathbf{p})/T} + \eta_i} \\ &= \frac{45 g_i}{4\pi^4}x_i^4\int_1^\infty \frac{y\sqrt{y^2-1}}{e^{x_i y} + \eta_i}\frac{4y^2-1}{3y}\,\mathrm{d}y, \end{aligned} \tag{II.1.57}$$

with g_i, m_i and η_i are the internal (spin) degrees of freedom, mass and statistical factor of species i; $x_i = m_i/T$.

In addition, the x-dependence of the equilibrium comoving density $Y_\chi^{\rm eq}$ is derived directly from its definition:

$$\begin{cases} n_\chi^{\rm eq} = \dfrac{g_\chi}{(2\pi)^3}\int d^3\mathbf{p}\, e^{-E/T} = \dfrac{g_\chi T^3}{2\pi^2}x^2 K_2(x), \\ s = h_{\rm eff}(T)\dfrac{2\pi^2}{45}T^3 \end{cases} \Rightarrow \quad Y_\chi^{\rm eq} = \frac{n_\chi^{\rm eq}}{s} = \frac{45 g_\chi}{4\pi^4}\frac{x^2 K_2(x)}{h_{\rm eff}(m_\chi/x)}, \tag{II.1.58}$$

with $K_2(x)$ is a modified Bessel function of the second kind. The derivation of $n_\chi^{\rm eq}$ in terms of $K_2(x)$ is represented in Appendix E.

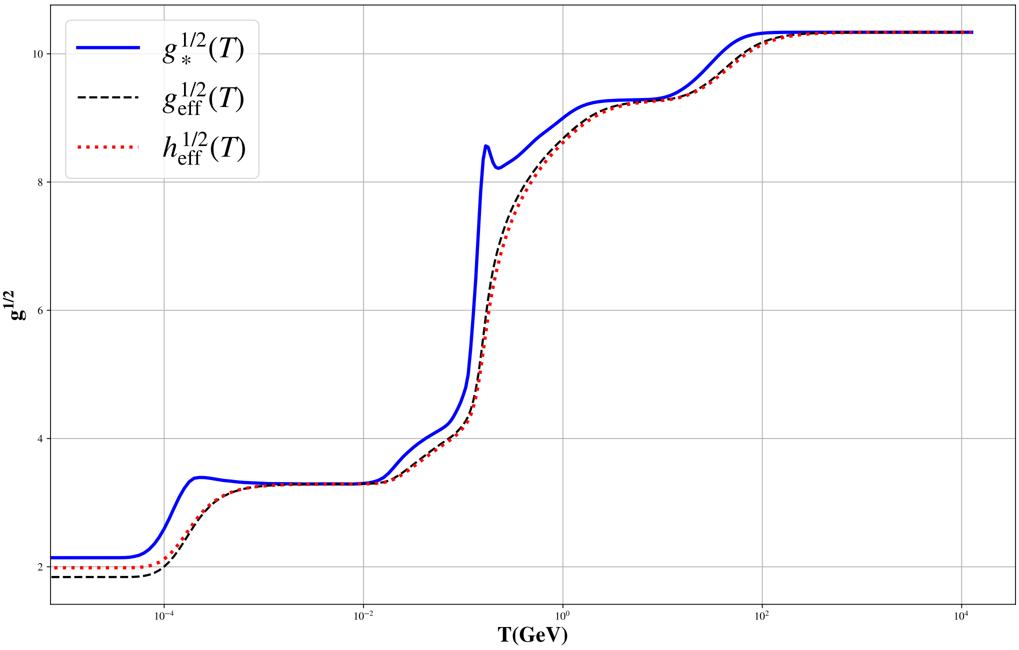


Figure II.1: Degrees of freedom parameter as a function of temperature. This parameter remains roughly constant away from the mass threshold $T \sim m_i$. At temperature roughtly 1 TeV, all the SM particles are relativistic and in thermal equilibrium with $g_* = 106.75$ [1]. With low temperature $T < 0.5MeV$, only photons and three neutrinos have significant contribution with $g_* \approx 2$.Input data is imported from the input file dsdofDHS.dat of the package **DarkSUSY**. More detailed discussion on these degrees of freedom can be found in e.g [23].

In practice, we import the numerical values of these degrees of freedom from input of the package **DarkSUSY** (Ref. [24, Chapter 23]).

II.2 Solution to the Boltzmann Equation

II.2.1 Thermal average

Old treatment: expansion solution

In old context, the thermal averaging is usually done by expanding $\sigma_\chi v_{\rm Møl}$ in terms of energy (or relative velocity), and the classical Maxwell-Boltzmann is applied. To be clear, one writes

$$\langle\sigma_\chi v_{\rm Møl}\rangle_{\rm n.r} = \frac{\int \sigma_\chi v_{\rm lab} e^{-|\mathbf{p_1}|^2/2m_\chi T} e^{-|\mathbf{p_2}|^2/2m_\chi T} d^3p_1\, d^3p_2}{\int e^{-|\mathbf{p_1}|^2/2m_\chi T} e^{-|\mathbf{p_2}|^2/2m_\chi T} d^3p_1\, d^3p_2}. \tag{II.2.1}$$

[1] The degrees of freedom of relativistic particles is simply $g_* = \sum_{i\in \rm bosons} g_i + 7/8\sum_{i\in \rm fermions}$. With totally 28 degrees of freedom of bosons (photons: 2, $W^\pm, Z$: 3×3, gluons: 8×2, Higgs: 1) and 90 from fermions (quarks: $6\times 3\times 4$, charged leptons: 3×4, neutrinos: 3×2), we have

$$g_*(T \le 1TeV) = g_b + \frac{7}{8}g_f = 28 + \frac{7}{8}\times 90 = 106.75 \tag{II.1.59}$$

We make a change of integration variables from $\mathbf{p}_1, \mathbf{p}_2$ to total momentum $\mathbf{p}_T = \mathbf{p}_1 + \mathbf{p}_2$ and relative momentum $\mathbf{p}_R = \mathbf{p}_1 - \mathbf{p}_2$. Note that $|\mathbf{p}|_1^2 + |\mathbf{p}_2|^2 = \frac{|\mathbf{p}_T|^2 + |\mathbf{p}_R|^2}{2}$, and that the differential volume in momentum space is

$$\mathrm{d}^3 p = 4\pi \left|\mathbf{p}\right|^2 \mathrm{d}\left|\mathbf{p}\right|, \tag{II.2.2}$$

Eq. (II.2.1) can be expressed as

$$\begin{aligned}
\langle \sigma_\chi v_{\mathrm{Møl}} \rangle_{\mathrm{n.r}} &= \frac{\int \sigma_\chi v_{\mathrm{lab}} e^{-\left(|\mathbf{p}_T|^2 + |\mathbf{p}_R|^2\right)/4m_\chi T} \left|\mathbf{p}_T\right|^2 \left|\mathbf{p}_R\right|^2 \mathrm{d}\left|\mathbf{p}_T\right| \, \mathrm{d}\left|\mathbf{p}_R\right|}{\int e^{-\left(|\mathbf{p}_T|^2 + |\mathbf{p}_R|^2\right)/4m_\chi T} \left|\mathbf{p}_T\right|^2 \left|\mathbf{p}_R\right|^2 \mathrm{d}\left|\mathbf{p}_T\right| \, \mathrm{d}\left|\mathbf{p}_R\right|} \\
&= \frac{\int \sigma_\chi v_{\mathrm{lab}} e^{-|\mathbf{p}_R|^2/4m_\chi T} \left|\mathbf{p}_R\right|^2 \mathrm{d}\left|\mathbf{p}_R\right|}{\int e^{-|\mathbf{p}_R|^2/4m_\chi T} \left|\mathbf{p}_R\right|^2 \mathrm{d}\left|\mathbf{p}_R\right|} = \frac{\int \sigma_\chi v_{\mathrm{lab}} e^{-x\epsilon} \sqrt{\epsilon} \mathrm{d}\epsilon}{\int e^{-x\epsilon} \sqrt{\epsilon} \mathrm{d}\epsilon} \\
&= \frac{2x^{2/3}}{\sqrt{\pi}} \int_0^\infty \sigma_\chi v_{\mathrm{lab}} \sqrt{\epsilon}\, e^{-x\epsilon} \, \mathrm{d}\epsilon,
\end{aligned} \tag{II.2.3}$$

where at the second equality we cancel all terms related to $|\mathbf{p}_T|$ since $\sigma_\chi v_{\mathrm{lab}}$ depends only on p_R, thus the integration over $|\mathbf{p}_T|$ is trivial. Furthermore, we have made another change of integration of variable from $|\mathbf{p}_R|$ to ϵ via the relation

$$\left|\mathbf{p}_R\right|^2 = 4m_\chi^2 \epsilon \quad \Rightarrow \quad \epsilon = \left(\frac{|\mathbf{p}_R|}{2m_\chi}\right)^2. \tag{II.2.4}$$

Note that in rest frame of one incoming particle, ϵ is interpreted as the total kinetic energy per mass unit.

Assumes $\sigma_\chi v_{\mathrm{lab}}$ is a smooth function of energy and thus can be expressed as a Taylor series with respect to ϵ [1]

$$\sigma_\chi v_{\mathrm{lab}} = \sum_{n=0}^\infty \frac{a^{(n)}}{n!} \epsilon^n, \tag{II.2.5}$$

with $a^{(n)} = \mathrm{d}^n(\sigma_\chi v_{\mathrm{lab}})/\mathrm{d}\epsilon^n|_{\epsilon=0}$.

$$\langle \sigma_\chi v_{\mathrm{Møl}} \rangle_{\mathrm{n.r}} = a^{(0)} + \frac{3}{2} a^{(1)} x^{-1} + \frac{15}{8} a^{(2)} x^{-2} + \mathcal{O}(x^{-3}). \tag{II.2.6}$$

This expansion formula can be used as a fast-calculation method for dealing with thermal averaging. As an example, let us consider the lowest order coefficient $a^{(0)}$ whose expression can be directly obtained from Eq. (C.1.12)

$$\begin{aligned}
a^{(0)} = \sigma_\chi v_{\mathrm{lab}}\big|_{\epsilon=0} &= \left[\frac{\beta_f(s, m_i, m_j)}{64\pi^2 \left(s - 2m^2\right) S_{i,j}} \sum_{i,j} \int \mathrm{d}\Omega \overline{|\mathcal{M}_{\chi\chi\to i+j}|^2}\right]_{s=4m_\chi^2} \\
&= \frac{1}{64\pi m_\chi^2} \left[\frac{\beta_f(s, m_i, m_j)}{S_{i,j}} \sum_{i,j} \int_{-1}^1 \mathrm{d}(\cos\theta) \, \overline{|\mathcal{M}_{\chi\chi\to i+j}|^2}\right]_{s=4m_\chi^2},
\end{aligned} \tag{II.2.7}$$

where $\bar{\mathcal{M}}$ has been summed over final states and averaged over initial states. Below we represent a rough solution of the Boltzmann equation (II.1.52) with s-wave approximation (i.e considering only the leader term $a^{(0)}$; the thermal average $\langle \sigma_\chi v_{\mathrm{Møl}} \rangle$ is then obviously independent of temperature).

[1] In most cases the cross section is "well-behaved" and such expansion is possible. Exceptions occurs when expanding near resonances or threshold energy where we must other methods for approximating $\langle \sigma_\chi v_{\mathrm{Møl}} \rangle$.

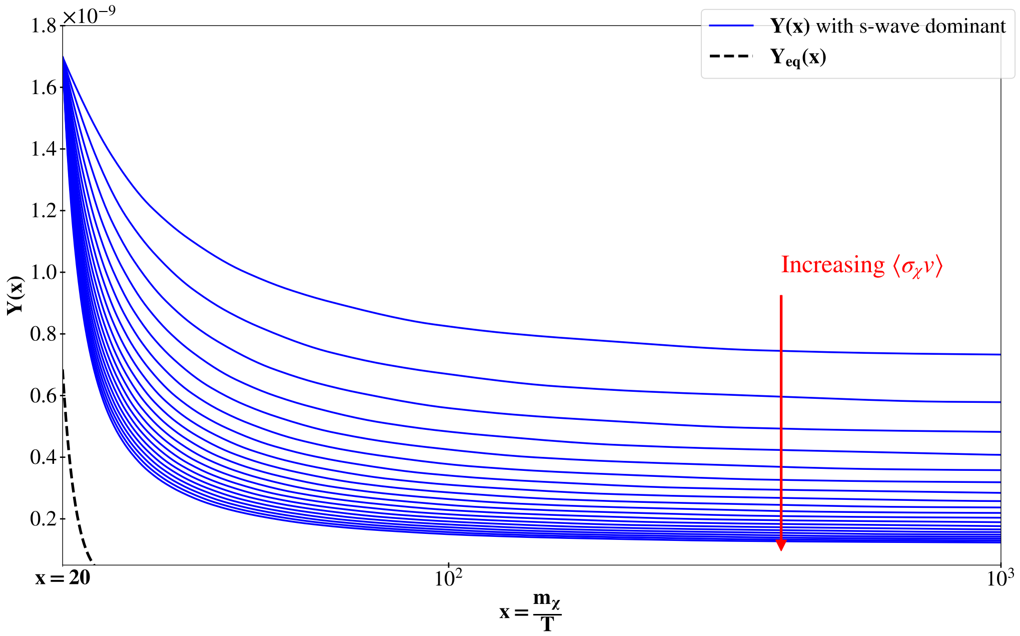


Figure II.2: An illustration of $Y_\chi(x)$ when consider the lowest order of expansion over ϵ. The initial condition is loosely set: $Y_\chi(x=20) = 2.5Y_\chi^{\mathrm{eq}}(x=20)$. The above solution is generated using `integrate.odeint` method of package `scipy` of **`Python`**. Well-below $x = 20$ (or typically below x_f), $Y_\chi(x)$ traces closely with $Y_\chi^{\mathrm{eq}}(x)$, however the `integrate.odeint` produces an oscillatory solution.

Since $\langle\sigma_\chi v_{\mathrm{M\o l}}\rangle$ in this s-wave dominant is temperature independent and assuming the initial condition at $x = 20$, the solution after typical freeze-out point (i.e $x \geq 20$) can be easily obtained using any numerical method for solving first order ODE. This equation with $x < 20$ is stiff, and an explicit method fails to converge.

A better treatment: single-integral formula

We now represent a more modern treatment of taking the thermal average of the cross section times Møller velocity by a single-integral formula. At the end of the calculations, we see that the integral over momentum space is simplified to an integral over total energy. Let us consider the following thermal average formula

$$\langle\sigma_\chi v_{\mathrm{M\o l}}\rangle = \frac{\int \sigma_\chi v_{\mathrm{M\o l}} e^{-E_1/T} e^{-E_2/T} \mathrm{d}^3\mathbf{p}_1\, \mathrm{d}^3\mathbf{p}_2}{\int e^{-E_1/T} e^{-E_2/T} \mathrm{d}^3\mathbf{p}_1\, \mathrm{d}^3\mathbf{p}_2}, \tag{II.2.8}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$ and E_1, E_2 are respectively the three-momenta and energies of incoming particles in the cosmic comoving frame. [1] The normalization factor, i.e the denominator of Eq. (II.2.8), is merely the squared of the equilibrium number density without the factor $g_\chi/(2\pi)^3$, which is represented in Appendix E. Using the expression (E.2.14), we have

$$\int e^{-E_1/T} e^{-E_2/T} \mathrm{d}^3\mathbf{p}_1\, \mathrm{d}^3\mathbf{p}_2 = \left[\frac{(2\pi)^3}{g_\chi} n_\chi^{\mathrm{eq}}\right] = \left[4\pi T^3 x^2 K_2(x)\right]^2, \tag{II.2.9}$$

and what left is to clarify the numerator of (II.2.8). Converting the integral over momenta space by first taking the differential of the relation between energy-momentum

$$\begin{aligned} &E^2 = p^2 + m^2 \Rightarrow E\mathrm{d}E = p\mathrm{d}p \\ &\Rightarrow \mathrm{d}^3\mathbf{p}_1\, \mathrm{d}^3\mathbf{p}_2 = \left(4\pi\,|\mathbf{p}_1|^2\,\mathrm{d}\,|\mathbf{p}_1|\right)\left(4\pi\,|\mathbf{p}_2|^2\,\mathrm{d}\,|\mathbf{p}_2|\right)\mathrm{d}\left(\frac{\cos\theta}{2}\right) = 8\pi^2\,|\mathbf{p}_1|\,|\mathbf{p}_2|\,\mathrm{d}E_1\,\mathrm{d}E_2\,\mathrm{d}\cos\theta \end{aligned} \tag{II.2.10}$$

The Mandelstam variable s can be expressed in terms of the given quantities as

$$s \equiv (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = 2m_\chi^2 + 2E_1E_2 - 2\,|\mathbf{p}_1|\,|\mathbf{p}_2|\cos\theta. \tag{II.2.11}$$

[1] Recall that the Boltzmann formula $f \propto e^{-E/T}$ is used assuming the isotropy of momentum, i.e we are working within the comoving frame where the system of particles is at rest as a whole.

We consider the transformation of integration variables $(E_1, E_2, \cos\theta) \to (E_+ E_- s)$, with the new variables are defined as

$$\begin{cases} E_+ = E_1 + E_2 \\ E_- = E_1 - E_2 \\ s = 2m_\chi^2 + 2E_1E_2 - 2\,|\mathbf{p}_1|\,|\mathbf{p}_2|\cos\theta \end{cases}, \tag{II.2.12}$$

with the corresponding Jacobian

$$\frac{\partial\,(E_+, E_-, s)}{\partial\,(E_1, E_2, \cos\theta)} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2E_2\left(1 - \dfrac{|\mathbf{p}_1|}{|\mathbf{p}_2|}\cos\theta\right) & 2E_2\left(1 - \dfrac{|\mathbf{p}_1|}{|\mathbf{p}_2|}\cos\theta\right) & -2\,|\mathbf{p}_1|\,|\mathbf{p}_2| \end{pmatrix}, \tag{II.2.13}$$

$$\Rightarrow \det\left[\frac{\partial\,(E_+, E_-, s)}{\partial\,(E_1, E_2, \cos\theta)}\right] = 4\,|\mathbf{p}_1|\,|\mathbf{p}_2|\,. \tag{II.2.14}$$

We now express the differential momenta space (II.2.10) in the new variables

$$\begin{aligned} \mathrm{d}^3\mathbf{p}_1\,\mathrm{d}^3\mathbf{p}_2 &= 8\pi^2\,|\mathbf{p}_1|\,|\mathbf{p}_2|\,\mathrm{d}E_1\,\mathrm{d}E_2\,\mathrm{d}\cos\theta = 2\pi^2\det\left[\frac{\partial\,(E_+, E_-, s)}{\partial\,(E_1, E_2, \cos\theta)}\right]\mathrm{d}E_1\,\mathrm{d}E_2\,\mathrm{d}\cos\theta \\ &= 2\pi^2\mathrm{d}E_+\,\mathrm{d}E_-\mathrm{d}s. \end{aligned} \tag{II.2.15}$$

The original integration boundary $\{E_1 \geq m_\chi, E_2 \geq m_\chi, -1 \leq \cos\theta \leq 1\}$ transforms into

$$\begin{cases} s \geq 4m_\chi^2 \\ E_+ \geq \sqrt{s} \\ |E_-| \leq \sqrt{1 - \dfrac{4m_\chi^2}{s}}\sqrt{E_+^2 - s} \end{cases}. \tag{II.2.16}$$

Let us calculate the numerator of (II.2.8) in these new variables:

$$\begin{aligned} \int \sigma_\chi v_{\mathrm{M\o l}} e^{-E_1/T} e^{-E_2/T}\mathrm{d}^3\mathbf{p}_1\,\mathrm{d}^3\mathbf{p}_2 &= 2\pi^2\int_{4m_\chi^2}^{\infty}\mathrm{d}s\int_{\sqrt{s}}^{\infty}\mathrm{d}E_+\int_{-\sqrt{1-4m_\chi^2/s}\sqrt{E_+^2-s}}^{\sqrt{1-4m_\chi^2/s}\sqrt{E_+^2-s}}\mathrm{d}E_-\,\sigma_\chi v_{\mathrm{M\o l}} E_1E_2e^{-E_+/T} \\ &= 4\pi^2\int_{4m_\chi^2}^{\infty}\mathrm{d}s\,\sigma_\chi F\sqrt{1 - \frac{4m_\chi^2}{s}}\int_{\sqrt{s}}^{\infty}\mathrm{d}E_+\,e^{-E_+/T}\sqrt{E_+^2 - s}, \end{aligned} \tag{II.2.17}$$

where at the second equality we note that $\sigma_\chi v_{\mathrm{M\o l}}E_1E_2 = \sigma_\chi F$ is a function of s only. Indeed, σ_χ is a function of total energy while the Møller flux factor can be expressed as a function of s as

$$F = \left[(p_1\cdot p_2)^2 - m_\chi^4\right]^{1/2} = \left[\left(\frac{s - p_1^2 - p_2^2}{2}\right)^2 - m_\chi^4\right]^{1/2} = \left[\frac{\left(s - 2m_\chi^2\right)^2}{4} - m_\chi^4\right]^{1/2} = \frac{1}{2}\sqrt{s\left(s - 4m_\chi^2\right)}. \tag{II.2.18}$$

The integrand is thus independent of E_- and is easily calculated. Consider the integral over E_+, we continue to change the integration variable to a dimensionless $y \equiv E_+/\sqrt{s}$. With the help of the integral representation of modfied Bessel function (E.1.2), we proceed our calculations as

$$\int_{\sqrt{s}}^{\infty}\mathrm{d}E_+\,e^{-E_+/T}\sqrt{E_+^2 - s} = (T\sqrt{s})\frac{\sqrt{s}}{T}\int_1^{\infty}\mathrm{d}y\,\sqrt{y^2-1}e^{-y(\sqrt{s}/T)} = T\sqrt{s}K_1(\sqrt{s}/T). \tag{II.2.19}$$

Plugging (II.2.18), (II.2.19) into (II.2.17), and deviding it by the denominator (II.2.9) gives the important single-integral formula:

$$\langle\sigma_\chi v_{\mathrm{M\o l}}\rangle = \frac{1}{8T^5x^4K_2^2(x)}\int\mathrm{d}s\,(s - 4m_\chi^2)\sqrt{s}K_1(\sqrt{s}/T)\sigma_\chi. \tag{II.2.20}$$

As claimed in [21, **Section 3**], taking the thermal average in the comoving frame is identical to taking the thermal average in the laboratory frame: $\langle\sigma_\chi v_{\mathrm{M\o l}}\rangle = \langle\sigma_\chi v_{\mathrm{lab}}\rangle^{\mathrm{lab}}$; the latter is easier to evaluate. It is therefore practical to change the integration variable from total energy s to the kinetic energy per unit mass in the lab frame ϵ, defined by

$$\epsilon = \frac{(E_{1,\mathrm{lab}} - m_\chi) + (E_{2,\mathrm{lab}} - m_\chi)}{2m_\chi}. \tag{II.2.21}$$

In the laboratory frame, one of the two incoming particles is at rest. Denoting the energy and momentum of the moving particle E_{lab} and p_{lab} respectively, we rewrite ϵ as

$$\begin{aligned}\epsilon &= \frac{E_{\mathrm{lab}} - m_\chi}{2m_\chi} = \frac{2m_\chi(E_{\mathrm{lab}} + m_\chi) - 4m_\chi^2}{4m_\chi^2} = \frac{\left[(E_{\mathrm{lab}} + m_\chi)^2 - \left(E_{\mathrm{lab}}^2 - m_\chi^2\right)\right] - 4m_\chi^2}{4m_\chi^2} \\ &= \frac{(E_{1,\mathrm{lab}} + E_{2\mathrm{lab}})^2 - \left(\mathbf{p}_{1,\mathrm{lab}} + \mathbf{p}_{2\mathrm{lab}}\right)^2 - 4m_\chi^2}{4m_\chi^2} = \frac{(p_{1,\mathrm{lab}} + p_{2\mathrm{lab}})^2 - 4m_\chi^2}{4m_\chi^2} = \frac{s - 4m_\chi^2}{4m_\chi^2}.\end{aligned} \tag{II.2.22}$$

The relative velocity in the laboratory frame can be expressed in terms of the kinetic energy density. Note that the definition of velocity appears in the formula of momentum:

$$p_{\mathrm{lab}}^2 = \frac{m_\chi^2 v_{\mathrm{lab}}^2}{1 - v_{\mathrm{lab}}^2} = E_{\mathrm{lab}}^2 - m_\chi^2, \tag{II.2.23}$$

and the energy of the incoming particle can be calculated via the kinetic energy density ϵ from its definition: $E_{\mathrm{lab}} = m_\chi(2\epsilon + 1)$. Thus

$$\begin{aligned}&\frac{m_\chi^2 v_{\mathrm{lab}}^2}{1 - v_{\mathrm{lab}}^2} = m_\chi^2\,(2\epsilon + 1)^2 - m_\chi^2 = 4m_\chi^2\epsilon(\epsilon + 1) \\ &\Rightarrow v_{\mathrm{lab}} = \sqrt{\frac{4\epsilon\,(1+\epsilon)\,m_\chi^2}{4\epsilon\,(1+\epsilon)\,m_\chi^2 + m_\chi^2}} = \sqrt{\frac{4\epsilon\,(\epsilon+1)}{(1+2\epsilon)^2}} = \frac{2\sqrt{\epsilon(\epsilon+1)}}{1+2\epsilon}.\end{aligned} \tag{II.2.24}$$

Eq. (II.2.20) together with (II.2.22) and (II.2.24) gives us the following single-integral formula for calculating thermal average of interaction rate:

$$\boxed{\langle \sigma_\chi v_{\mathrm{Møl}} \rangle = \int_0^\infty \mathrm{d}\epsilon\, \mathscr{K}(x,\epsilon)\sigma_\chi v_{\mathrm{lab}}} \tag{II.2.25}$$

where the thermal kernel is expressed in terms of ϵ and $x \equiv m_\chi/T$ represent the thermal distribution of ϵ

$$\boxed{\mathscr{K}(x,\epsilon) = \frac{2x}{K_2^2(x)}\sqrt{\epsilon}(1+2\epsilon)K_1(2x\sqrt{1+\epsilon}).} \tag{II.2.26}$$

We can check numerically that $\mathscr{K}(x,\epsilon)$ is normalized for all x

$$\int \mathrm{d}\epsilon\, \mathscr{K}(x,\epsilon) = 1. \tag{II.2.27}$$

This distribution vanishes near the minimum energy (i.e $\sqrt{s} = 2m_\chi$) and at the limit of high energy, and have a global maximum in its range simila to the non-relativistic distribution. At low temperature limit, the species under consideration becomes non-relativistic, thus follows the Maxwell-Boltzmann distribution. At high temperature limit, the plot of $\mathscr{K}(x,\epsilon)$ tends to dispersed toward the large value of ϵ, implies that the higher tempearture of the species, the more kinematic energy it gains and the larger contribution of cross section at high energy to the average. This is the asymptotic behaviour of the given thermal kernel at small x, as being shown in the plot below.

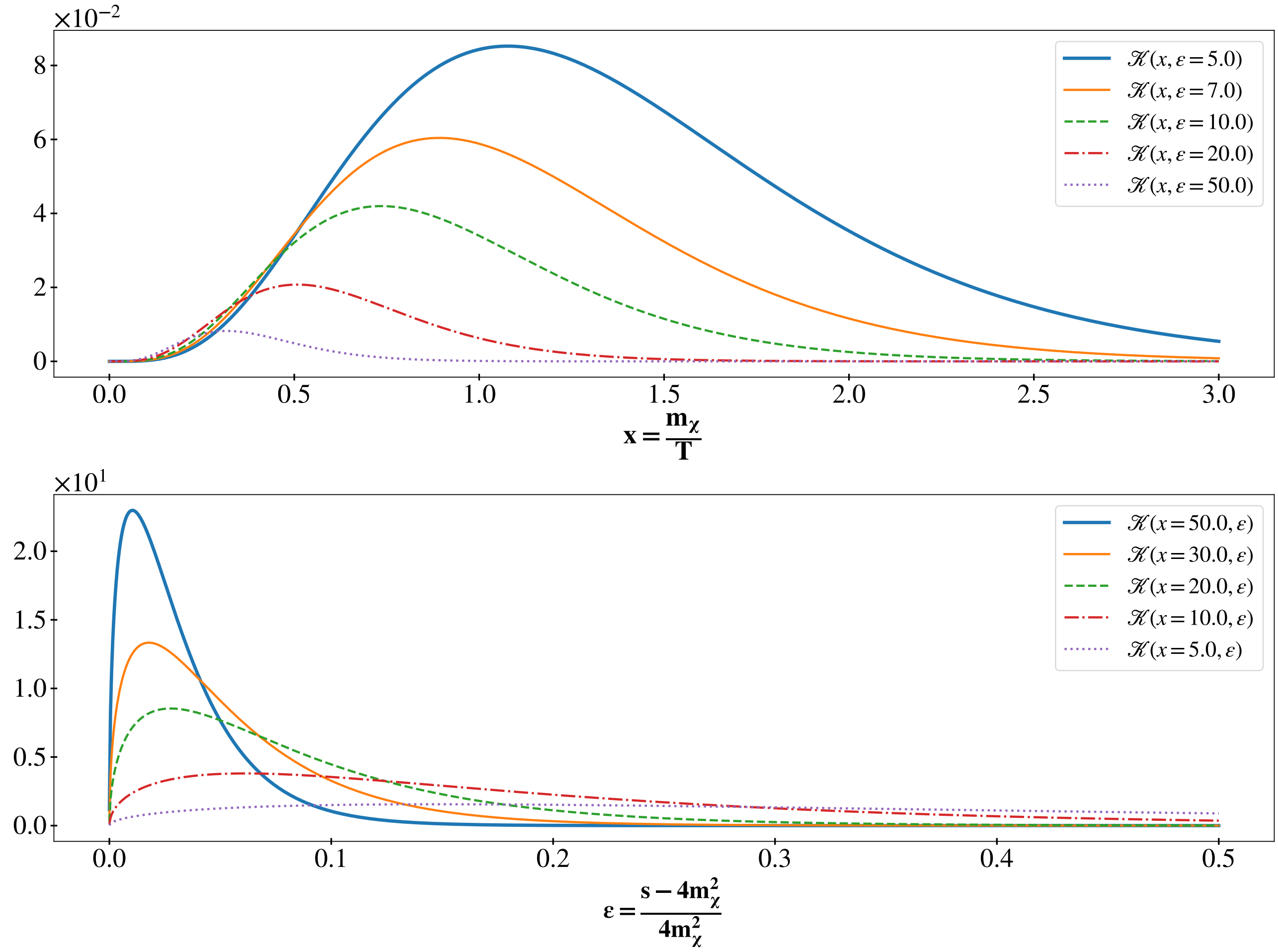


Figure II.3: Plot of the thermal kernel $\mathscr{K}(x, \epsilon)$ with respect to x and ϵ.

With this analytic formula, an numerical integration can be easily done if we know the value of $\langle \sigma_\chi v_{\rm lab} \rangle$. This formula has some the following advantages

- The given single-integral formula overcomes the multi-dimensional integrals.
- An expansion of $\sigma_\chi v_{\rm lab}$ is a treatment in approximating the thermal average in old literature, which fails in case $\sigma_\chi v_{\rm lab}$ varies rapidly with ϵ.

For backward compatibility with Eq. (II.2.6), let us consider again the expansion solution of $\langle \sigma_\chi v_{\rm Møl} \rangle$ in relativistic manner. Recall that the asymptotic expansion of the Bessel functions with large argument z is

$$K_n(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} P_n(z), \tag{II.2.28}$$

Plugging this expansion into Eq. (II.2.25), one obtains

$$\langle \sigma_\chi v_{\rm Møl} \rangle \simeq 2\sqrt{\frac{x^3}{\pi}} \int_0^\infty {\rm d}\epsilon\, \sqrt{\epsilon} \frac{1+2\epsilon}{(1+\epsilon)^{1/4}} e^{-2x\left(\sqrt{1+\epsilon}-1\right)} \frac{P_1\left(2x\sqrt{1+\epsilon}\right)}{P_2^2(x)} \sigma_\chi v_{\rm lab}. \tag{II.2.29}$$

To obtain the series of $\langle \sigma_\chi v_{\rm Møl} \rangle$ with respect to x, consider again the expansion Eq. (II.2.5). Let $y \equiv 2x\left(\sqrt{1+\epsilon}-1\right)$, and using the series (E.1.8), one can calculate analytically all integrals of each power of ϵ in terms of Gamma function:

$$\int_0^\infty {\rm d}y\, y^{n+1/2} e^{-y} = \Gamma\left(\frac{3}{2}+n\right) = \frac{(2n+1)!!}{2^n} \frac{\sqrt{\pi}}{2}. \tag{II.2.30}$$

The thermal average can then be expressed in terms of power of negative x as follows (see [**21**, **Eq. (3.30)**])

$$\langle \sigma_\chi v_{\rm Møl} \rangle = a^{(0)} + \frac{3}{2} a^{(1)} x^{-1} + \left[\frac{9}{2} a^{(1)} + \frac{15}{8} a^{(2)}\right] x^{-2} + \mathcal{O}(x^{-3}). \tag{II.2.31}$$

Relic density with coannihilations

As already clarified in [25], one must pay attention when performing standard calculations of DM relic density in three special cases: coannihilations, "forbidden" annihilation channels and annihilation near pole of the cross sections. The coannihilations occurs when we consider a set of similar particles whose masses are nearly degenerate, with the DM candidate being the lightest and stable one. The relic density of the DM candidate then depends significantly on the annihilation of heavior particles in the given set into the lightest one. The second case concerns annihilation into more massive particles than the DM candidate, which is usually claimed as kinematically forbidden. The third case has been mentioned above, with an expansion treatment propsed in [21, Section 6].

In a general scenario, a DM model can contains several DM candidates. For precision calculation of the relic density, one must track the evolution of densities of all DM candidates taking into account all possible coannihilation partners. The rest of the particle content is assumed to be in thermal equilibrium with the heat bath or decay into the DM sector. For ease of representation, let us follow the work of [26]. Consider N DM particles $\chi_1, \cdots, \chi_N$ which is in ascending order of mass: $m_{\chi_1} < m_{\chi_2} < \cdots < m_{\chi_N}$. Assumes further that these particles are differ from SM particles by a multiplicative quantum number. In the next several chapters we will see that in supersymmetric models, the desired quantum number is R-parity, which is also our case of interest. The lighest particle is protected by this symmetry from decay further into SM particles, thus being stable. We then consider all reactions which change χ_i number densities:

$$\begin{cases} \chi_i \chi_j \leftrightarrow X \\ \chi_i X \leftrightarrow \chi_j X' \, . \\ \chi_i \leftrightarrow \chi_j X \end{cases} \tag{II.2.32}$$

The set of coupled equations for evolution of number density of χ_i is

$$\begin{aligned} \frac{\mathrm{d}n_i}{\mathrm{d}t} =& -3Hn_i - \sum_{j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{\mathrm{eq}} n_j^{\mathrm{eq}} \right) \\ & - \sum_{j \neq i} \left[\langle \sigma'_{Xij} v_{ij} \rangle \left(n_i n_X - n_i^{\mathrm{eq}} n_X^{\mathrm{eq}} \right) - \langle \sigma'_{Xji} v_{ji} \rangle \left(n_j n_X - n_j^{\mathrm{eq}} n_X^{\mathrm{eq}} \right) \right] \\ & - \sum_{j \neq i} \left[\Gamma_{ij} \left(n_i - n_i^{\mathrm{eq}} \right) - \Gamma_{ji} \left(n_j - n_j^{\mathrm{eq}} \right) \right] , \end{aligned} \tag{II.2.33}$$

where X, Y denotes the set of relevant SM particles. The relative velocity between particles χ_i and χ_j is defined as

$$v_{ij} = \frac{\sqrt{(p_i \cdot p_j)^2 - m_i^2 m_j^2}}{E_i E_j}, \tag{II.2.34}$$

which is clearly the generalization of the Møller velocity defined in Eq. (II.1.40). The RHS of Eq. (II.2.33) contains the thermal average of all processes mentioned in (II.2.32). The first term describes the dillusion due to the expansion of the Universe. The second term corresponds to the $\chi_i \chi_j$ annihilation with total annihilation cross section

$$\sigma_{ij} = \sum_X \sigma \left(\chi_i \chi_j \to X \right) . \tag{II.2.35}$$

The third term describes the interaction between the class of DM particles with the thermal bath, with

$$\sigma'_{Xij} = \sum_Y \sigma \left(\chi_i X \to \chi_j Y \right) \tag{II.2.36}$$

is the inclusive scattering cross section. The final term describes the decay of χ_i into the lightest one in the DM sector χ_1, with inclusive decay rates

$$\Gamma_{ij} = \sum_X \Gamma \left(\chi_i \to \chi_j X \right) . \tag{II.2.37}$$

The final abundance is simply the sum of all number densities of χ_i: $n = \sum_i \chi_i$. With the two final terms being canceled, we are left with the evolution equation for total abundance

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn - \sum_{i,j=1}^{N} \langle \sigma_{ij} v_{ij} \rangle \left(n_i n_j - n_i^{\mathrm{eq}} n_j^{\mathrm{eq}} \right)$$

$$= -3Hn - \sum_{i,j=1}^{N} \langle \sigma_{\rm eff} v \rangle \left[n^2 - (n^{\rm eq})^2 \right], \tag{II.2.38}$$

with the effective annihilation rate

$$\langle \sigma_{\rm eff} v \rangle = \frac{\int_0^\infty {\rm d}p_{\rm eff}\, p_{\rm eff}^2 W_{\rm eff} K_1 \left(\frac{\sqrt{s}}{T} \right)}{m_1^4 T \left[\sum_i \frac{g_i}{g_1} \frac{m_i^2}{m_1^2} K_2 \left(\frac{m_i}{T} \right) \right]}, \tag{II.2.39}$$

and the annihilation rate per unit volume

$$W_{\rm eff} \equiv \sum_{ij} \frac{p_{ij}}{p_{11}} \frac{g_i g_j}{g_1^2} W_{ij} = \sum_{ij} \sqrt{\frac{\lambda\left(s, m_i^2, m_j^2\right)}{s(s-4m_1^2)}} \frac{g_i g_j}{g_1^2} W_{ij},$$
$$W_{ij} \equiv \frac{1}{g_i g_j S_f} \sum_{\text{internal d.o.f}} \int \prod_f \frac{{\rm d}^3 p_f}{(2\pi)^3 2E_f} \left| \mathcal{M} \right|^2 (2\pi)^4 \delta^{(4)}(p_i + p_j - \sum_f p_f), \tag{II.2.40}$$

with S_f is the symmetry factor of final states. We can easily check that the general expression for thermal average of annihilation rate (II.2.39) coincides with the formula (II.2.25) if no coannihilation is taken into the consideration.

II.2.2 Freeze-out approximation

Generally, the Boltzmann equation is of the form of the Ricatti equation, which has no known analytic solution, and has to be solved numerically to give the yield at specific x. We have to integrate (II.1.52) to the point $x = x_0 = m/T_0$ at current temperature $T_0 \approx 2.725K$. The initial condition, however, is an asymptotic condition based on the fact that at early time ($x \to 0$), the comoving density is of the same order of $Y_\chi^{\rm eq}$. Since (II.1.52) has a singularity at $x = 0$, we must set the initial condition at some x and assuming that $Y(x_0) \sim Y_\chi^{\rm eq}(x_0)$. The value of x at which we choose to be the initial condition is known at freeze-out point, where the actual density of particles depart from the equilibrium density as we represented in Section I.3. This subsection aims to represent a way to obtain the asymptotic initial condition based on solving the freeze-out condition.

An approximation method for calculating the relic density known as *freeze-out approximation* gives a quite accurate result of the relic density, and is the main method we implemented in our code for now. This approximation scheme has the advantages that we can control the calculated time through the number of sampled points of the integration, thus achieving a rough estimation of the result of the relic density $\Omega h^2 \theta^{-3}$. Another pros is by generating first a table of $\langle \sigma_\chi v_{\rm Møl} \rangle$, we can obtain the full solution of $Y_\chi(T)$ as long as T is in the range of generated data, thus roughly providing the shape of $Y_\chi(T)$.

The basics of freeze-out approximation based on the fact that at early times, the density of a given species traces closely with its equilibrium density, i.e $Y_\chi(x \ll 1) \sim Y_\chi^{\rm eq}(x)$. The freeze-out value is defined as the point where the rate of the annihilation is of the same order of the rate of expansion. At later stage after freezing-out, the equilibrium particle density is supressed by the Boltzmann factor $\sim e^{-m/T}$ while the density of species that left the thermal bath approaches a constant value, and thus the contribution of the equilibrium term in Eq. (II.1.52) can be safely ignored.

To formulate the above physical picture, let us consider the deviation of the actual comoving density from its equilibrium: $\Delta_Y = Y_\chi - Y_\chi^{\rm eq}$. Taking derivative with respect to x and plugging in Eq. (II.1.52) gives

$$\frac{{\rm d}\Delta_Y}{{\rm d}x} = -\left(\frac{\pi}{45G}\right) \frac{g_*^{1/2} m}{x^2} \langle \sigma_\chi v_{\rm Møl} \rangle \Delta_Y \left(\Delta_Y + 2Y_\chi^{\rm eq} \right) - \frac{{\rm d}Y_\chi^{\rm eq}}{{\rm d}x}. \tag{II.2.41}$$

As claimed, before the decoupling Y_χ follows closely with $Y_\chi^{\rm eq}$ and thus the change in deviation ${\rm d}\Delta_Y/{\rm d}x$ can be ignored. Define the freeze-out x_f is the point at which $\Delta = \delta Y_\chi^{\rm eq}$, which gives the condition for freezing-out:

$$\boxed{-\left(\frac{\pi}{45G}\right)^{1/2} \frac{g_*^{1/2} m}{x^2} \langle \sigma_\chi v_{\rm Møl} \rangle Y_\chi^{\rm eq} \delta(\delta+2) + \frac{{\rm d}\ln Y_\chi^{\rm eq}}{{\rm d}x} = 0,} \tag{II.2.42}$$

with δ is a given number, and is typically chosen in the range $[1.5, 2.5]$ (see e.g Ref. [21, p. 163], [24]). As we shall see in the numerical results of x_f (see Fig. VII.3), the variation of x_f with respect to the chosen δ would be ignorable compared to the integration range, and thus does not give significant impact on $Y_\chi(x)$. A more explicit form of the freeze-out condition is achieved by inserting the formula of equilibrium comoving density in Eq. (II.1.58) into (II.2.42)

$$\left(\frac{\pi}{45G}\right)^{1/2} \frac{45g}{4\pi^4} \frac{K_2(x)}{h_{\rm eff}(T)} g_*^{1/2} m \left\langle \sigma_\chi v_{\rm M\o l} \right\rangle \delta\left(\delta+2\right) = \frac{K_1(x)}{K_2(x)} - \frac{1}{x}\frac{\mathrm{d}\ln h_c(T)}{\mathrm{d}\ln T}. \tag{II.2.43}$$

At the stage after decoupling (i.e $x > x_f$), the contribution of $Y_\chi^{\rm eq}$ is small compared to Y_χ, and can be neglected. Taking integration over two temperature thus yields the freeze-out approximation result for Y_χ^0

$$\begin{aligned} \frac{\mathrm{d}Y_\chi}{\mathrm{d}x} &\approx -\left(\frac{\pi}{45G}\right)^{1/2} \frac{g_*^{1/2} m}{x^2} \left\langle \sigma_\chi v_{\rm M\o l} \right\rangle Y_\chi^2 \Rightarrow \mathrm{d}\left(\frac{1}{Y_\chi}\right) \approx -\left(\frac{G}{45\pi}\right)^{1/2} g_*^{1/2} \left\langle \sigma_\chi v_{\rm M\o l} \right\rangle \\ &\Rightarrow \frac{1}{Y_\chi(T)} = \frac{1}{Y_\chi^f} + \left(\frac{\pi}{45G}\right)^{1/2} \int_T^{T_f} g_*^{1/2} \left\langle \sigma_\chi v_{\rm M\o l} \right\rangle \mathrm{d}T, \end{aligned} \tag{II.2.44}$$

with $Y_\chi^f = (\delta+1) Y_\chi^{\rm eq}(x_f)$ is the value of the density at freeze-out point. This solution as a rule works within $\sim 2\%$ accuracy (Ref. [27]). Due to the "approximated" initial condition at T_f, the above integration gives the meaningful result only in the range of temperature $T \in (T_f, \infty)$. As we shall see, the precision of the numerical solution to $Y_\chi(x)$ generally depends on the initial condition, and there are various other approaches to interpret the problem. We refer the reader to the papers on the public code on DM such as [24; 27; 28], where the authors represent their own approach on solving the dark matter relic density.

Despite the advantages of the freeze-out approximation listed at the beginning of this section, we however experience the problem of estimation of the error comes from the neglection the equilibrium term. Another more accurate solution (taking into account the contribution of $Y_\chi^{\rm eq}$, together with an error estimation) would be solving the ODE (II.1.52) with an iterative methods. However, an explicit method like Runge-Kutta fails to converge due to the stiff nature of the given ODE. In the package **DarkSUSY**(Ref. [24, **Section 23.2**]), an implicit trapezoidal method is applied to overcome the difficulties. Basically, the equation is discretized with the trapezoidal method first, and solved iteratively using the Euler method. The step size will be adapted when the difference between two updated solution exceeds a given limit. The detail of the algorithm and implementation can be found in the manual of **DarkSUSY**.

Finally, the numerical results of the relic density of a species in the units of the critical density $\rho_c^0 = 3H_0^2/(8\pi G) = 1.9\times 10^{-29} h^2\, g\, cm^{-3}$ is given by

$$\begin{aligned} \Omega_\chi &= \frac{\rho_\chi^0}{\rho_c} = \frac{m_\chi s_0 Y_\chi^0}{\rho_c} \\ &\Rightarrow \Omega_\chi h^2 = 2.755\times 10^8 \frac{m_\chi}{GeV} \chi 0, \end{aligned} \tag{II.2.45}$$

where Y_χ^0 is calculated from Eq. (II.2.44) at the temperature of current background radiation $T = T_0 \approx 2.726K$

$$\boxed{Y_\chi^0 \approx \frac{1}{\dfrac{1}{Y_\chi^f} + \left(\dfrac{\pi}{45G}\right)^{1/2} \displaystyle\int_{T_0}^{T_f} g_*^{1/2} \left\langle \sigma_\chi v_{\rm M\o l} \right\rangle \mathrm{d}T}} \tag{II.2.46}$$

The first term $1/Y_\chi^f$ can be further discarded due to its small contribution. This approximation corresponds to an infinite DM relic density at T_f, and works within $\sim 20\%$ precision as mentioned in [27].

III

A Brief Review of the Standard Model

Outline

This chapter is a detour through the Standard Model of physics, where we want to emphasize the fundamentals from which this model is built to describe known elementary particles as well as their interactions. The outcome of this model has been tested with thousands of measurements and the fact that SM gives a extremely good predictions make it becomes one of the most successful theory ever, and thus the name "Standard".

Nevertheless, the SM is still considered as an effective model as there are several observations where SM cannot provide an acceptable explanation. This is what drive the devolopment of the physics beyond the SM by investigating subtle modifications such that the new model gives predictions consistent with existing data, yet provides plausible answers to the imperfection parts of the SM. One possible way to extend the Standard Model is to enlarge its symmetry group, either the internal symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$ or the spacetime symmetry. Supersymmetry belongs to the second path of extending the symmetry group, and will be represented in detail in the next chapter of the thesis.

III.1 Introduction

With a long history of development, the particle physics is certainly one of the widest and richest branch of science. From the very first discovery of the nuclear structure by the gold foil experiment in the early 20th century, until the current era of high energy particle accelarators with many of these experiments have been set up around the world, producing an extremely rich and detailed data on elementary particles. Alongside with the rapid growth of high energy experiments, theoretical particle physics aims to formulate the fundamental properties of the particles that built up the Universe, and the principles governing the interaction between these small building blocks. The theoretical models thus were developed to uncover the information hidden deeply in the data. The Standard Model of partile physics is one outstanding models, not only it is an extremely compact systhesis of knowledge about the elementary particles and their interactions but also the reliability of this model has been tested through tens of thousands of measurements with high agreements. To name a few:

- SM has predicted the existance of the W and Z bosons, the gluons, the top and charm quarks before their first observations. These are also the last particles being discovered: the W and Z bosons in 1983, the top quark in 1995, the tau neutrino in 2000, and the Higgs boson in 2012.
- SM gives one of the most precision prediction in particle physics, e.g the agreement between measurement of anomalous magnetic dipole moment of the electron and a precise theoretical calculation of the anomalous magnetic dipole moment in terms of structure constant α.
- The prediction of Higgs boson, which is responsible for the symmetry breaking process to generate masses for elementary particles in the SM. It has been half a decade since its first time being theorized by Peter Higgs (Ref. [29]) and five other physicists including Robert Brout and Francois Englert (Ref. [30]); Gerald Guralnik, C. Richard Hagen, and Tom Kibble (Ref. [31]), until this particle is verified by the ATLAS and CMS experiments at CERN's Large Hadron Collider in 2012 (Ref. [32]).

The Standard Model contains twelve named fermions and five named bosons, together with their anti-partner (in some cases, a particle is its own anti-particle). There are quarks and leptons in the class of fermions; the distinction between these two type of particle is if they have strong interaction or not. On the other hand, we have gluons, photon, W, Z and Higgs are all bosons. While the first 4 types of bosons listed above are vector bosons (spin 1) particles, responsible for describing the fundamental interactions (strong, weak and electromagnetic interactions), the scalar Higgs participate in the Higgs mechanism for breaking the symmetry group $SU(2)_L \times U(1)_Y$.

In addition, the fermions are all come into three generations. Same type of particles in each generation differ by their flavour quantum number and mass, but their interactions are identical. Each member of a higher generation also have greater mass compared to the previous generations. The heavier mass a particle have, the more unstable it is, hence particles tend to decay to the first generation, explaining why the most abundant part of matter is made up from members of the first family.

Their basic properties (mass, spin and charges) are summarized in the table below.

Name	Particle	Mass	Electric charge
Fermions (spin 1/2)			
Up-type quarks	u	$2.2^{+0.5}_{-0.4}$ (MeV)	2/3
	c	$1.275^{+0.025}_{-0.035}$ (GeV)	
	t	172.76 ± 0.3 (GeV)	
Down-type quarks	d	$4.7^{+0.5}_{-0.3}$ (MeV)	$-1/3$
	s	95^{+9}_{-3} (MeV)	
	b	4.65 ± 0.3 (GeV)	
Charged leptons	e	0.51099895000(15) (GeV)	-1
	μ	105.6583755(23) (MeV)	
	τ	1776.86 ± 0.12 (MeV)	
Neutrinos	ν_e	$\sum_i m_{\nu_i} < 0.23$(GeV)	0
	ν_μ		
	ν_τ		
Vector bosons (spin 1)			
Gluons	g	0	0
W bosons	$W^{\pm}$	80.379 ± 0.012 (GeV)	± 1
Z boson	Z	91.1876 ± 0.0021 (GeV)	0
Photon	γ	0	0
Scalar bosons (spin 0)			
Higgs boson	H	125.18 ± 0.16 (GeV)	0

Table III.1: Particle content in SM with their basic properties: mass, spin and electric charges.

Experiments show a variaty of properties of these particles, e.g the periodicity of particles over generations, and specific reaction channels each type of particle take part in. These properties are the foundation on which the SM is built. With the internal symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$, the rich contents of particles can be ordered in the corresponding multiplets as shown in Table III.2.

Name	Multiplets	$\mathbf{SU(3)_C \times SU(2)_L \times U(1)_Y}$
	Fermions	
Quarks multiplets	$Q_L = (u_L, d_L)$	$(\mathbf{3}, \mathbf{2}, 1/3)$
	$\overline{u}_R$	$(\overline{\mathbf{3}}, \mathbf{1}, -4/3)$
	$\overline{d}_R$	$(\overline{\mathbf{3}}, \mathbf{1}, 2/3)$
Leptons multiplets	$L = (\nu_L, e_L)$	$(\mathbf{1}, \mathbf{2}, -1)$
	$\overline{e}_R$	$(\mathbf{1}, \mathbf{1}, 1)$
	Vector bosons	
Gluons	G^a_μ	$(\mathbf{8}, \mathbf{1}, 0)$
W vector field	W^i_μ	$(\mathbf{1}, \mathbf{3}, 0)$
B vector field	B_μ	$(\mathbf{1}, \mathbf{1}, 0)$
	Scalar bosons	
Higgs boson	$\Phi = (\Phi^+, \Phi^0)$	$(\mathbf{1}, \mathbf{2}, 1)$

Table III.2: Particle multiplets in the SM. The bold number represent the dimension of the representation of $SU(N)_C$ where the multiplets belong to, with a bar on top indicates the complex conjugate representation. Matter multiplets usually belongs to either **1** (the trivial representation) or **N** (the standard representation) and their complex conjugate representations. Gauge multiplets belongs to the adjoint representation $\mathbf{N^2 - 1}$ of the corresponding group, and is a singlet in **1** otherwise. The unbold number in the triplet represents the hypercharge of each multiplet.

The rest of this chapter is divided into two parts. The first part is meant to be a revision of the basic ingredients of the Standard Model, and the formulation of this theory using the Lagrangian interpretation. The second one lists the issues of the SM, which motivates the development of theories beyond the SM.

III.2 Building Blocks of the Standard Model

III.2.1 Symmetries & the gauge sector

Symmetry is a long-standing concept in the history of physics. Especially in the branch of particle physics, symmetries play a vital role as they becomes dominate the understanding of fundamental principles of physics. In the context of the SM, the internal symmetry group is the foundation of the construction of SM Lagrangian, providing a discription on three of four known fundamental interactions in the nature: the strong, weak and electromagnetic interaction. Specifically, these symmetries were successfully included via the gauge theory developed by Yang and Mills [33] whose main idea is introducing new bosons into the models as interactions carrier particles. The gauge group $SU(3)_C$ is described in the *Quantum Chromodynamics* (QCD) (see e.g Ref. [34]) with 8 massless gauge bosons so-called gluons for mediating the strong interaction. On the other hand, the unified electroweak interaction describing by the $SU(2)_L \times U(1)_Y$ introduces to the model four vector fields, including three W^i_μ and one B_μ. After the electroweak symmetry is simultaneously broken, the SM predicts new massive charged gauge bosons $W^\pm$ and a neutral gauge boson Z, whose existance is verified by the UA1 and UA2 collaboration at CERN in 1983.

The strategy of employing the local symmetry $SU(N)$ into the Lagrangian of matter sector requires adding $N^2 - 1$ vector fields. In order to describe the dynamics of the intermediate gauge bosons, we add the corresponding Lagrangian whose form is generally written as

$$\mathcal{L}_G = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}, \tag{III.2.1}$$

where the flavour index a ranges from $1 \to (N^2 - 1)$ and the tensor of gauge field strength $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, with f^{abc} being the structure constants of group $SU(N)$. The total Lagrangian is locally $SU(N)$ invariant if we make the following modification on the derivative of the fermion fields

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - ig A^a_\mu T^a, \tag{III.2.2}$$

where T^a are generators of $SU(N)$. We further force the vector fields transform under $e^{-ig\alpha^a(x)T^a} \in SU(N)$ as

$$A^a_\mu \longrightarrow A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a(x) - f^{abc}\alpha^b(x)A^c_\mu. \tag{III.2.3}$$

In the context of SM, the Lagrange must be invariant under the local symmetry gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. According to the Yang-Mills theory, we add the following gauge bosons into SM Lagrangian for gauge sector

- One gauge field B^μ corresponds to Abellian gauge group $U(1)_Y$ with the corresponding coupling constant g'.
- Three gauge fields W^i_μ correspond to non-Abellian gauge group $SU(2)_L$ whose three generators are the isospin operators $I^i = \sigma^i/2$, with the three matrices $\sigma^1, \sigma^2, \sigma^3$ being the Pauli matrices. The coupling constants is denoted as g.
- Eight gauge fields G^a_μ correspond to non-Abellian gauge group $SU(3)_C$, whose eight associated generators are half of the Gell-Mann matrices $\{\lambda^a/2, a = 1\ldots 8\}$. The coupling constant with the matter sector is denoted as g_s.

Note that the SM Lagrangian have no mass term since it violates the gauge symmetry group. We know that fermion mass term mixes the left and right chiral fermions [1] . Recall that in SM, the left chiral spinors transform under the fundamental representation of $SU(2)_L$ while the right chiral spinors are singlet of this group. Take lepton sector for example, under the gauge transformation $e_L \leftrightarrow \nu_L$ while e_R remains the same. A fermion mass term hence does not respect the symmetry $SU(2)_L$ of the SM and is not allowed. In a similar manner, a mass term of gauge boson is forbidden since such term is not invariant under the local gauge transformation (III.2.3).

III.2.2 The matter sector

The SM matter sector contains 12 fermions (and their anti-partners), including 6 quarks (each with three colors) and 6 leptons. Except for neutrinos where we have not understand its Majorana or Dirac nature yet, these fermions transforms under the spinor representation $(1/2, 0) \oplus (0, 1/2)$ of the proper orthochronous Lorentz group, hence being described by the four-components Dirac spinor Ψ, whose equation of motion is the well-known Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\,\Psi = 0. \tag{III.2.4}$$

The mathematical framework allows us to decompose the Dirac spinor into a composition of left and right chiralities *Weyl spinors* Ψ_L and Ψ_R; each of those Weyl spinor belongs to either $(0, 1/2)$ or $(1/2, 0)$ representation of Lorentz group:

$$\Psi = P_L\Psi_L + P_R\Psi_R, \tag{III.2.5}$$

where P_L and P_R are the left- and right-chiral projection operators. The Dirac equation for a Dirac spinors can be break down to a system of equations for describing the Weyl spinors $\Psi_L\Psi$ and $\Psi_R\Psi$, with the mass term being the mixing of chirality eigenstates. The importance of this left-right projection of Dirac spinors becomes essential in the formulation of weak interaction where we know that only left-handed fermions will participate in. Note that before the electroweak symmetry is broken and hence all of the particles in the models are massless, the description of chirality and helicity (i.e the projection of the spin onto the direction of momentum) are the same. We therefore can split a type of particle into two sub-types base on the orientation of the spins.

The last piece about representing a fermion we want to mention is *Majorana fermion*, described by a spinor satisfying the Majorana condition:

$$\Psi^C_M = \Psi_M, \tag{III.2.6}$$

i.e this type of fermion is left invariant under a charge-conjugation. From the definition, it follows that the $U(1)$ current vanishes: $\overline{\Psi}^C\gamma^\mu\Psi^C = -\overline{\Psi}_M\gamma^\mu\Psi_M$. This implies that a Majorana fermion is electrically neutral since its $U(1)$ current vanishes, meaning that Majorana fermions cannot couple directly to electromagnetic field. Only neutrinos are neutral fermions in the SM and are the only candidates for Majorana fermions. However, the fact that these particles are Dirac-fermions or Majorana-fermions remains inconclusive.

[1] Recall a usual Dirac mass term can be written in terms of left and right chiral spinors as $-m\overline{\Psi}\Psi = -m(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L)$.

Leptons

Leptons are fermions that do not undergo strong interactions, and thus do not appear in the combination form like quarks. Mathematically speaking, leptons are singlet under the $SU(3)_C$ group, thus have no color charge. The six leptons over three generations can be divided further into two subgroup: one are electrically neutral, extremely light weight [1] . This is the subgroup of neutrinos, which have three flavours corresponding to their partner leptons: ν_e, ν_μ and ν_τ. On the other hand, electron e^-, muon μ^- and tau τ^- are all negatively charged (with the same unit electric charge), hence much easier to measured. Electron is the lightest of all electron-like leptons, which is the stable and most abundant charged lepton in the Universe since the muon and tau unstable, and rapidly decay into electrons and neutrinos. In fact, the identification of electron as a particle by J. J. Thomson and his team in 1897 is one of the earliest discovery in the particle physics history.

Before the symmetry group $SU(2)_L \times U(1)_Y$ is broken by the Higgs mechanism, all of the leptons (and their anti-partners) are massless and can be cast into the following multiplets:

$$L = \left\{ \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \right\}, \quad e = \{e_R, \mu_R, \tau_R\}\,. \tag{III.2.7}$$

We know that the left and right chiral fermions behave differently under weak interaction. The left-handed fermions (and right-handed anti-fermions) participate in the weak interaction and thus being ordered into a doublet. The right-handed one do not interact weakly with other particles, and is a singlet under $SU(2)_L$ group. Note further that these right chiral leptons are negatively charged and can interact electrically. Unlike $\{e_R, \mu_R, \tau_R\}$, the right-handed neutrinos and the left-handed anti-neutrinos have no known interaction with other elementary particles in the context of the SM, and thus are not included in the SM.

Quarks

Quarks are spin-1/2 particles that participate in all three interactions in the SM. Similar to leptons, we have six type of quarks, each of which is a triplet of $SU(3)_C$ (with the corresponding quantum numbers so called *color* including red, blue and green). Due to the confinement property of the strong interaction, the quarks are always detected in the bound states; all of these quarks compositions are known to be colorless and have integer electric charge. A *baryon* is a type of composite subatomic particle which contains an odd number of valence quarks (at least 3) while a *meson* is made of one quark and one antiquark. Collectively the quarks bound states are called as *hadrons*.

As for the multiplets of the leptons, we also have quarks being ordered into multiplets under the internal symmetry group of SM as shown in Table III.2. All quarks are electrically charged, with up-type quarks contains a fractionary electric charge of 2/3 unit charge whereas the down-type quarks have a charge $-1/3$.

The quark flavours can be mixed to form the mass eigenstates, which is characterized by the 3×3 Cabibbo-Kobayashi-Maskawa matrix (CKM) matrix. Specifically, from the unitary transformation of the flavour basis u_i, d_i to mass eigenbasis $\hat{u}_i, \hat{d}_i$

$$\hat{u}_i = U^u_{ij} u_j, \quad \hat{d}_i = U^d_{ij} d_j, \tag{III.2.8}$$

the bilinear terms of up- and down-type quarks is rotated into diagonal form

$$\mathcal{L} = h^u_{ij} \overline{u}_i u_j + h^d_{ij} \overline{d}_i d_j \longrightarrow \hat{\mathcal{L}} = M^u_i \overline{u}_i u_i + M^d_i \overline{d}_i d_i, \tag{III.2.9}$$

where M^u_i, M^d_i are the masses of up-type and down-type quarks in generation space. The charged current is tranformed form weak interaction basis to

$$\mathcal{L}_{CC} = \left[\overline{u}_i \gamma^\mu \left(1 - \gamma^5\right) d_i W_\mu + h.c\right] \longrightarrow \hat{\mathcal{L}}_{CC} = \left[\overline{\hat{u}}_i V^{CKM}_{ij} \gamma^\mu \left(1 - \gamma^5\right) \hat{d}_i W_\mu + h.c\right]. \tag{III.2.10}$$

The CKM matrix has one complex phase describing the CP violation in weak interaction in the quark sector.

III.2.3 Breaking symmetries: the Higgs mechanism

The motivation of this mechanism is that the mass terms of fermions and gauge bosons is not allowed in the SM Lagrangian since their appearance explicitly violate the SM symmetry group $\times SU(2)_L \times U(1)_Y$. As

[1] In SM, the neutrinos are considered to be massless. In fact, this mass is so tiny that only recently neutrinos are confirmed to be massive.

we know, most of the fermions and gauge bosons have non zero mass (except photon and gluons), implying the symmetry must be broken somehow. The briliant idea is by adding appropriate fields into the model, the symmetry of the total Lagrangian can be kept, while a non zero vacuum breaks this symmetry to generate masses for particles. That mechanism is known as *electroweak spontaneous symmetry breaking* (EWSB). In the SM, the minimal case of symmetry breaking is by adding an $SU(2)_L$ complex scalar doublets with the corresponding Lagrangian

$$\mathcal{L}_\Phi = \left(D_\mu \Phi\right)^\dagger \left(D^\mu \Phi\right) - V(\Phi), \tag{III.2.11}$$

where the Higgs doublet $\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}$, and

$$D_\mu \Phi = \partial_\mu - \frac{i}{2} g Y B_\mu - \frac{i}{2} g' \sigma^j W_\mu^j, \tag{III.2.12}$$

$$V(\Phi) = -\mu^2 \left|\Phi\right|^2 + \lambda \left|\Phi\right|^4 = -\mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2. \tag{III.2.13}$$

A physical minimum of the Higgs potential requires the positivity of parameter λ. The *vacuum expectation value* (vev) is then the value at which $V(\Phi)$ is minimal:

$$\begin{aligned} \left.\frac{\partial V(\Phi)}{\partial(\left|\Phi\right|^2)}\right|_{\Phi=\langle\Phi\rangle} &= -\mu^2 + 2\lambda(\langle\Phi\rangle^\dagger \langle\Phi\rangle) = 0 \Rightarrow \langle\Phi\rangle^\dagger \langle\Phi\rangle = \frac{\mu^2}{2\lambda} \\ &\Rightarrow \left|\langle\phi^+\rangle\right|^2 + \left|\langle\phi^0\rangle\right|^2 = \frac{\mu^2}{2\lambda}. \end{aligned} \tag{III.2.14}$$

In order to achieve the two minima for spontaneous symmetry breaking, we insists $\mu^2 > 0$.

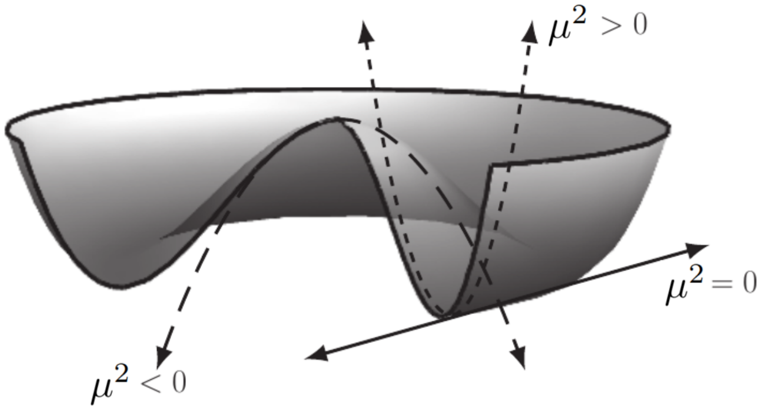


Figure III.1: Illustration of the Higgs potential in three cases: $\mu^2 < 0$, $\mu^2 = 0$ and $\mu^2 > 0$. Source: [**35, Fig. 28.1**].

To preseve the $U(1)_Q$ symmetry, after EWSB the vev of each fields are evaluated as

$$\left\langle\phi^+\right\rangle = 0 \Rightarrow \left\langle\phi^0\right\rangle = \frac{v}{\sqrt{2}} = \sqrt{\frac{\mu^2}{2\lambda}}, \tag{III.2.15}$$

The Higgs field can then be expanded around the chosen minimum

$$\Phi(x) = \begin{pmatrix} \phi^+(x) \\ \frac{1}{2}\left[v + h(x) + i\chi(x)\right] \end{pmatrix}, \tag{III.2.16}$$

with only one massive neutral CP-even h field, and two massless states including one charged $\phi^\pm$ and one neutral CP-odd G^0. These massless fields are unphysical Nambu-Goldstone bosons that are absorbed by the weak gauge bosons to generate their masses. One simple way to see how $W^\pm$ and Z gain their masses is by working in the *unitary gauge* where the Higgs doublet can be expanded around its vacuum in the form

$$\Phi = \begin{pmatrix} 0 \\ \frac{(v+h)}{\sqrt{2}} \end{pmatrix}. \tag{III.2.17}$$

Let us consider the kinetic term of the Higgs doublet after EWSB

$$\begin{aligned} K_\Phi &= (D_\mu\Phi)^\dagger (D^\mu\Phi) \\ &= \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{8}g^2(v+h)^2(W^1_\mu + iW^2_\mu)(W^\mu_1 - W^\mu_2) + \frac{1}{8}(v+h)^2(gW^3_\mu - g'B_\mu)(gW^{3\mu} - g'B^\mu). \end{aligned} \quad \text{(III.2.18)}$$

By defining a new basis [1]

$$W^\pm_\mu \equiv \frac{W^1_\mu \pm iW^2_\mu}{\sqrt{2}}, \quad Z^\mu \equiv \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}}, \quad A^\mu \equiv \frac{g'W^3_\mu + gB_\mu}{\sqrt{g^2+g'^2}}, \quad \text{(III.2.19)}$$

one obtains a massive charged gauge boson (with its antiparticle) W, a massive neutral gauge boson Z and a massless gauge field A with the mass spectrum

$$M_W = \frac{gv}{2}, \quad M_Z = \frac{\sqrt{g^2+g'^2}v}{2}, \quad M_\gamma = 0. \quad \text{(III.2.20)}$$

Similarly, we plug the expansion (III.2.17) into the Yukawa couplings (III.3.8) to diagonalize this Lagrangian, which results in

$$m_f = \frac{y_f v}{\sqrt{2}}, \quad \text{(III.2.21)}$$

where the subscript f represents the leptons and quarks in three generations. There is no such mass term for neutrinos, thus the SM predicts that the neutrinos are massless particles, which has been rejected by the neutrino oscillation experiments such as Super-Kamiokande Observatory and Sudbury Neutrino Observatory.

III.3 The Standard Model Lagrangian

The classical Standard Model Lagrangian is constructed having in mind the basic ideas of the gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$ representing the three fundamental interactions, the particle contents over three generations and the Higgs mechanism which generate masses of these fundamental particles through the process electroweak spontaneous symmetry breaking. This Lagrangian is thus a combination of the kinetic terms for fermions and gauge bosons, the Yang-Mills couplings between the fermion fields via gauge fields, the Lagrange for Higgs sector and the Yukawa couplings: [2]

$$\mathcal{L}_{SM} = \mathcal{L}_G + \mathcal{L}_\Psi + \mathcal{L}_\Phi + \mathcal{L}_Y, \quad \text{(III.3.1)}$$

[1] In this basis, the general covariant derivative (III.3.7) is expressed as follows

$$D_\mu = \partial_\mu - ig_s G^a_\mu T^a - i\frac{g}{\sqrt{2}}\left(W^+_\mu I^+ + W^-_\mu I^-\right) - i\frac{g}{c_W}\left(I_3 - s^2_W Q\right) Z_\mu - ieQA_\mu,$$

with $I^\pm \equiv (\sigma^1 \pm i\sigma^2)/2$, and the charge operator relates to the I^3 and Y_W operators by the Gellmann-Nishijima formula $Q = I^3 + Y_W/2$.

[2] For quantization purposes, a gauge-fixing term and Faddeev-Popov term must be included into this Lagrange.

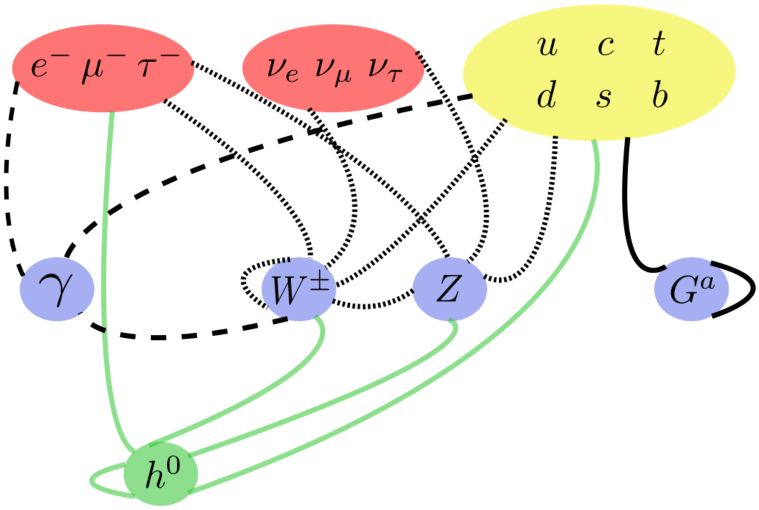


Figure III.2: Tree-level interactions between particles in matter sector by exchanging gauge bosons or Higgs bosons as being described by the SM Lagrangian (III.3.1). Lines represent the bosons being exchanged, with gluons (plain black), γ (dashed), Z and W (dotted) and Higgs boson (green). Source: [36, Fig. 1.2].

Let us examine each term of $\mathcal{L}_{SM}$ explicitly. The first term is the gauge kinetic terms, which describe the kinematics of the three types of gauge fields G, W and B as

$$\mathcal{L}_G = -\frac{1}{2}\operatorname{Tr}\left[G_{\mu\nu}G^{\mu\nu}\right] - \frac{1}{2}\operatorname{Tr}\left[W_{\mu\nu}W^{\mu\nu}\right] - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \quad = -\frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \frac{1}{4}W^{i\mu\nu}W^i_{\mu\nu} - \frac{1}{4}B^{\mu\nu}B_{\mu\nu}, \tag{III.3.2}$$

where the implicit sum is apply on the indices $a = 1, \ldots, 8$ and $b = 1, 2, 3$. Each of the field strength tensor is constructed from the corresponding gauge fields as follows

$$G_{\mu\nu} = G^a_{\mu\nu}T^a, \qquad G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} g^b_\mu g^c_\nu; \tag{III.3.3}$$
$$W_{\mu\nu} = W^i_{\mu\nu}I^i, \qquad W^i_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_s \varepsilon^{ijk} g^j_\mu g^k_\nu; \tag{III.3.4}$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{III.3.5}$$

The fermionic fields Lagrangian can be split into two parts: the kinematic terms governed by the Dirac equation, and the coupling with gauge fields. Specifically

$$\mathcal{L}_\Psi = \mathcal{L}^{(0)}_\Psi + \mathcal{L}^{YM}_{int}. \tag{III.3.6}$$

These two terms can be collected by introducing the gauge covariant derivatives, which generally written as

$$D_\mu = \partial_\mu - ig_s T^a G^a_\mu - igI^j W^j_\mu - ig'\frac{Y_W}{2}B_\mu. \tag{III.3.7}$$

An explicit mass term would break the gauge symmetry of the SM through EWSB with the Lagrange has already given in Eq. (III.2.11). The final part of the SM Lagrangian we mentioned above is the Yukawa couplings between the Higgs boson and other fermions in the particle contents:

$$\mathcal{L}_Y = \overline{L}\Phi Y^L E - \overline{Q}\Phi Y^D D - \overline{Q}\Phi^C Y^U U + h.c \tag{III.3.8}$$

where $L = (\nu_L, e_L)$ is the left-chiral lepton doublet, $Q = (u_L, d_L)$ is the left-chiral quark doublet; the corresponding right-chiral quantities are denoted by E, D, U. The superscript C denotes the charge conjugation of Φ to assure a neutral hypercharge coupling: $\Phi^C = i\sigma^2\Phi^*$. Note that Y^L, Y^D and Y^U are matrices acting on three generations; these 3×3 matrices contains information about Yukawa couplings parameters.

III.4 Challenges to the SM

Despite being the most rigorous and successful theory in particle physics with incredible precision predictions, there are various phenomena that SM remains inadequate to explain, hence SM is widely considered as an incomplete theory, i.e the SM of physics is merely a low-energy approximation of some deeper theory of nature. Such unanswered questions motivate the development of the *Beyond Standard Models*.

The neutrino mass problem: Neutrinos are massless particles in the context of the SM, which has been ruled out by the famous neutrino oscillations experiments. Theoretically one can add the neutrino mass terms into the SM picture, but this gives rise to a theoretical problem about extraordinarily smallness of these by-hand added mass terms. This tininess implies that the neutrino mass terms may not arise by the same mechanism as other masses of elementary particles, open a chance for other BSM theories to propose more reasonable answers.

Gravity: A well-known fact in particle physics is that the SM efficiently describes three over four fundamental interactions in nature, with gravitational interaction does not included. As discussed above, the strong, weak and electromagnetic interactions result from the exchanging of force-carrier particles, e.g gluons carry strong interactions between quarks while W and Z bosons mediating the weak interaction. The idea to incorporate gravity into the SM with the same approach (i.e applying Yang-Mills theory by adding intermediate bosons so-called *graviton*) have not been successful since gauge symmetry for gravity cannot be considered as an internal symmetry. Up to now, there are no existed theory succeeded in unifying this weakest interaction in nature with the other three, proving the difficulties on fitting the gravitational interaction into a bigger picture.

The strong CP problem: This theoretical orientation problem is about the smallness of θ_{QCD} parameter. Specifically, apart from the typical QCD Lagrange there is one CP-violated term that can be added:

$$\mathcal{L}_G^{\theta_{QCD}} = \frac{\theta_{QCD}}{16\pi^2} G^a_{\mu\nu} \tilde{G}^{a\mu\nu}, \tag{III.4.1}$$

with the dual field strength tensor $\tilde{G}^{a\mu\nu} \equiv \dfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$. Such term contributes to the neutron electric dipole moments, and the work [37] put an upper bound on $|\theta_{QCD}| < 10^{-11}$. As there is no known explanation within the SM context that CP needs to be conserved in strong interaction, the above stringent raises a fine-tuning problem.

The hierarchy problem: The SM contains quartic divergence, which can be seen by using the cut-off method to compute one-loop correction to Higgs mass such as

$$\Delta m_H^2 = -\frac{|\lambda|_f}{8\pi^2} \left[\Lambda^2 + 2m_f^2 \log\left(\frac{\Lambda}{m_f}\right) + \dots \right], \tag{III.4.2}$$

where the cut-off scale Λ can be as large as the Planck mass $M_{\rm pl} \sim 10^{19}$ GeV. With the experimental Higgs mass ~ 125 (GeV), one to fine-tune counterterms in such a way that there is a cancellation of the quartic terms so that the quantum correction remains of electroweak scale. A plausible solution to this issue can be found in the context of supersymmetric extension of the SM.

The dark matter: The last issue of SM we want to mention here is that no elementary particle within the SM is compatible with the cosmological observations on the relic density of the dark matter. As discussed at the beginning of this thesis, the only particle satisfying the proposed list of properties of DM (namely massive, neutral, stable, non-baryonic,...) in the context of SM are neutrinos. Unfortunately, the constraints on neutrinos masses makes them to be insufficient to be compatible with the current relic density of DM. Moreover, the tiny value of mass make neutrinos remain relativistic while decoupling from thermal bath; or put another way neutrinos are hot dark matter candidates. This indicates that "at very least, structure formation with neutrino-dominated Universe is more complicated than the standard inflation pictures" as stated and demonstrated with various reasons in [38].

IV
Supersymmetry

Outline

IV.1 History & Motivation for Supersymmetry

In 1967, Coleman and Mandula [39] proved a theorem which says that under certain assumptions (including Lorentz invariance, particle-finiteness, weak elastic analyticity, occurence of scattering, etc.), the only possible symmetries of the S-matrix are:

- CPT symmetry.
- Poincaré symmetry, with generators P^μ and $M^{\mu\nu}$. This is the group of Minkowski spacetime isometries, meaning the interval between events are left invariant under the group action.
- Some internal symmetry group with generators B_ℓ which are Lorentz scalars.

This theorem can be evaded by relaxing some of its assumptions. In 1975, Haag, Lopuszanski and Sohnius [40] generalized the notion of Lie algebra to include anticommutators. This extension allows to include spinor generators Q_α that transform fermions to bosons and vice versa, allows the states of different spin mix with each other. In such theory, the largest possible symmetry group is

$$\text{SuperPoincaré} \times \text{Internal Symmetries}$$

A theory can be systematically extended by modifying the internal symmetry groups, e.g the simplest Grand Unified Theory (GUT) model is built upon the group $SU(5)$ or the 331 model whose gauge group is $SU(3)_C \times SU(3)_L \times U(1)_X$. Such extension is more limited when consider the spacetime symmetries, which can be extended to include the supersymmetry between fermionic and bosonic states under the persuasive assumptions above. Specifically, starting from the usual Poincaré group whose algebra comprised of four generators for translations P^μ and 6 independent generators for rotations and boosts contained in $M^{\mu\nu}$:

$$[P^\mu, P^\nu] = 0, \tag{IV.1.1}$$

$$[M^{\mu\nu}, P^\sigma] = i\left(\eta^{\nu\sigma} P^\mu - \eta^{\mu\sigma} P^\nu\right), \tag{IV.1.2}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\rho} M^{\nu\sigma}\right), \tag{IV.1.3}$$

one can extend the symmetry group by imposing internal symmetries generated by B_l, and adding a set of fermionic generators $Q^I_\alpha, \overline{Q}^J_{\dot\alpha}$. The role of these fermionic generators can be loosely spoken as changing the spin of a states, i.e

$$Q|B\rangle \sim |F\rangle, \qquad Q|F\rangle \sim |B\rangle. \tag{IV.1.4}$$

Let us mention some of the reasons why supersymmetry becomes well-known and widely studied in the context of high energy physics:

- In the minimal case, one consider only a pair of Q and $\overline{Q}$, and thus double the number of particles in the original model due to the equality of bosonic states and fermionic states. This means any particle would have its superpartner that has different spin, and thus the minimal extension of the SM doubles the number of elementary particles, features a rich particle content. This provide a wider range of solutions for phenomenological problems, such as the DM candidate.

- The hierarchy problem of SM addressed in Section III.4 is resolved in SUSY theories by taking into account the one-loop correction of the fermion superpartner. In comparison with (III.4.2), the total mass correction to Higgs from both loop reads

$$\Delta m_H^2 = \frac{1}{8\pi^2}\left(\lambda_S - |\lambda_f|^2\right)\Lambda^2 + \ldots, \tag{IV.1.5}$$

  where λ_S is the scalar coupling. The invariance of supersymmetry requires $\lambda_S = |\lambda|_f$ the mass correction is perfectly canceled. In realistic there are reasons that SUSY must be broken, though the cancellation of quadratic divergences can hold if the soft-term is consider to break SUSY. Such scenarios yield a logarithmically divergent terms

$$\Delta m_H^2 = m_{\text{soft}}^2\left[\frac{\lambda}{16\pi^2}\log\left(\frac{\Lambda}{m_{\text{soft}}}\right) + \ldots\right], \tag{IV.1.6}$$

  where m_{soft} describes the mass difference between the fermion and its superpartner. A sufficiently small m_{soft} implies the small Higgs mass correction, provides a possible solution to the fine-tuning problem.

- Another interesting feature of a supersymmetric model is the existence of multiplicatively conserved quantum number known as R-parity. If supersymmetric is indeed realized in the nature, the conservation of R-parity make the lightest supersymmetric particle becomes stable and thus being a suitable DM candidate. We will come back to the discussion of R-parity later in Section V.1.

IV.2 SuperPoincaré Algebra

A more detail discussion and derivation of the (anti)commutation relations can be found in Appendix A.3. For simplicity, we will investigate the minimal case with $N = 1$ from here and follows in this thesis. Below we represent a list of the algebra that generate the SuperPoincaré group, in the minimal case $N = 1$:

Minimal SuperPoincaré algebra

$$[P^\mu, P^\nu] = 0, \tag{IV.2.1a}$$
$$[M^{\mu\nu}, P^\sigma] = i\left(\eta^{\nu\sigma}P^\mu - \eta^{\mu\sigma}P^\nu\right), \tag{IV.2.1b}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\rho}M^{\nu\sigma}\right), \tag{IV.2.1c}$$
$$[P^\mu, Q_\alpha] = 0, \quad \left[P^\mu, \overline{Q}^{\dot\alpha}\right] = 0, \tag{IV.2.1d}$$
$$[M^{\mu\nu}, Q_\alpha] = -\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta, \tag{IV.2.1e}$$
$$\left[M^{\mu\nu}, \overline{Q}^{\dot\alpha}\right] = -\left(\overline{\sigma}^{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\, \overline{Q}^{\dot\beta}, \tag{IV.2.1f}$$
$$\left\{Q_\alpha, \overline{Q}_{\dot\beta}\right\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu, \tag{IV.2.1g}$$
$$\{Q_\alpha, Q_\beta\} = 0, \quad \left\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} = 0, \tag{IV.2.1h}$$
$$[B_l, B_m] = i f^n_{lm} B_n, \tag{IV.2.1i}$$
$$[Q_\alpha, B_l] = 0, \quad \left[\overline{Q}_{\dot\alpha}, B_l\right] = 0. \tag{IV.2.1j}$$

Note that we also cover the possible internal symmetry groups, generated by the set $\{B_\ell\}$. In the minimal case with only one pair of fermionic operators, the supersymmetry and internal symmetry are unrelated according to the two last identities (IV.2.1j). For a more general consideration of the SuperPoincaré algebra, see Appendix A.3.

IV.3 Superspace & Superfields formalism

IV.3.1 Superfields & Supersymmetric Variations

Consider a superfield built up on superspace coordinates $Y(x, \theta, \overline{\theta})$, we can decompose Y into sum of fields, each corresponds to a specific power of θ and $\overline{\theta}$. Note further that this expansion is finite since θ and $\overline{\theta}$ anticommute implying any product involving more than two θ's or $\overline{\theta}$'s vanishes. Thus the most general scalar superfield is expanded as

$$\boxed{\begin{aligned} Y(x,\theta,\overline{\theta}) &= \varphi(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta\ M(x) + \overline{\theta\theta}\ N(x) \\ &\quad + \theta\sigma^\mu\overline{\theta}\ V_\mu(x) + \theta\theta\ \overline{\theta\lambda}(x) + \overline{\theta\theta}\ \theta\rho(x) + \theta\theta\ \overline{\theta\theta}\ D(x). \end{aligned}} \tag{IV.3.1}$$

The following quantities

$$\varphi, \psi, \overline{\chi}, M, N, V_\mu, \overline{\lambda}, \rho, D \tag{IV.3.2}$$

are called component fields, and their geometric nature are characterized by their transformation properties under Lorentz group, which can be inferred from the Lorentz scalar $Y(x, \theta, \overline{\theta})$:

- φ, M, N are complex scalar/pseudoscalar fields.
- ψ, ρ are left-chiral Weyl spinor fields.
- $\overline{\chi}, \overline{\lambda}$ are right-chiral Weyl spinor fields.
- V_μ is a vector field.
- D is a scalar field.

Thus superfield is a short way to denote a finite multiplet of fields. The space-time translations is generated through momentum operator P^μ, which is parameterized by infinitesimal a^μ as

$$\varphi(x+a) = e^{-iaP}\varphi(x)e^{iaP} = \varphi(x) - ia^\mu\left[P_\mu, \varphi(x)\right]. \tag{IV.3.3}$$

On the other hand, applying Taylor expansion of ϕ around x reads

$$\varphi(x+a) = \varphi(x) + a^\mu\partial_\mu\varphi(x) + \mathcal{O}(a^2). \tag{IV.3.4}$$

Equating both sides gives the following commutation relation

$$[\varphi(x), P_\mu] = -i\partial_\mu \varphi(x) \equiv \mathcal{P}_\mu \varphi(x), \tag{IV.3.5}$$

with $\mathcal{P}_\mu$ is the representation of spacetime translation on field space. Therefore, a translation of field φ by parameter a^μ induces a change on the field as

$$\delta_a \varphi(x) = \varphi(x+a) - \varphi(x) = i a^\mu \mathcal{P}_\mu \varphi(x). \tag{IV.3.6}$$

The same procedure can by applied on supercharge to obtain its differential form. We first define the infinitesimal supersymmetry transformation induced on the general superfield $Y(x, \theta, \overline{\theta})$ by a quantity $(\epsilon_\alpha, \overline{\epsilon}_{\dot{\alpha}})$ is

$$\begin{aligned} Y(x+\delta x, \theta+\delta\theta, \overline{\theta}+\delta\overline{\theta}) &= e^{-i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)} Y(x, \theta, \overline{\theta}) e^{i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)} \\ &= e^{-i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)} e^{-i\left(xP + \theta Q + \overline{\theta}\overline{Q}\right)} Y(0,0,0) e^{i\left(xP + \theta Q + \overline{\theta}\overline{Q}\right)} e^{i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)}. \end{aligned} \tag{IV.3.7}$$

To merge the product of two exponentials, it is useful to employ the Baker-Campbell-Haussdorff formula

$$\begin{aligned} &\exp\left\{i(xP + \theta Q + \overline{\theta Q})\right\} \exp\left\{i(\epsilon Q + \overline{\epsilon}\overline{Q})\right\} \\ &\approx \exp\left\{i(xP + \theta Q + \overline{\theta Q}) + i(\epsilon Q + \overline{\epsilon}\overline{Q}) + \frac{1}{2}\left[i(xP + \theta Q + \overline{\theta Q}), i(\epsilon Q + \overline{\epsilon}\overline{Q})\right]\right\} \\ &\approx \exp\left\{i x^\mu P_\mu + i(\epsilon+\theta)Q + i(\overline{\epsilon}+\overline{\theta})\overline{Q} - \frac{1}{2}\left[\overline{\theta Q}, \epsilon Q\right] - \frac{1}{2}\left[\theta Q, \overline{\epsilon}\overline{Q}\right]\right\} \\ &\approx \exp\left\{i x^\mu P_\mu + i(\epsilon+\theta)Q + i(\overline{\epsilon}+\overline{\theta})\overline{Q} + \epsilon\sigma^\mu\overline{\theta}P_\mu - \theta\sigma^\mu\overline{\epsilon}P_\mu\right\} \\ &\approx \exp\left\{i(x^\mu + i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta})P_\mu + i(\epsilon+\theta)Q + i(\overline{\epsilon}+\overline{\theta})\overline{Q}\right\}, \end{aligned} \tag{IV.3.8}$$

thus induces the supersymmetry variation of coordinates

$$\begin{cases} \delta x^\mu = i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta} \\ \delta\theta^\alpha = \epsilon^\alpha \\ \delta\overline{\theta}^{\dot{\alpha}} = \overline{\epsilon}^{\dot{\alpha}} \end{cases}. \tag{IV.3.9}$$

The supersymmetry variation of superfield Y is obtained using Taylor expansion around $(x, \theta, \overline{\theta})$ as

$$\begin{aligned} \delta_{\epsilon,\overline{\epsilon}} Y(x,\theta,\overline{\theta}) &\equiv Y(x+\delta x, \theta+\delta\theta, \overline{\theta}+\delta\overline{\theta}) - Y(x,\theta,\overline{\theta}) \\ &= \delta x^\mu \partial_\mu Y(x,\theta,\overline{\theta}) + \delta\theta^\alpha \partial_\alpha Y(x,\theta,\overline{\theta}) + \overline{\partial}_{\dot{\alpha}} Y(x,\theta,\overline{\theta}) \delta\overline{\theta}^{\dot{\alpha}} \\ &= i\left(\theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \overline{\epsilon}^{\dot{\alpha}} - \epsilon^\alpha \sigma^\mu_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}}\right) \partial_\mu Y(x,\theta,\overline{\theta}) + \epsilon^\alpha \partial_\alpha Y(x,\theta,\overline{\theta}) + \overline{\partial}_{\dot{\alpha}} Y(x,\theta,\overline{\theta}) \overline{\epsilon}^{\dot{\alpha}} \\ &= i\epsilon^\alpha \left(-i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}} \partial_\mu\right) Y(x,\theta,\overline{\theta}) - i\left(i\overline{\partial}_{\dot{\alpha}} + \theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu\right) \overline{\epsilon}^{\dot{\alpha}} Y(x,\theta,\overline{\theta}). \end{aligned} \tag{IV.3.10}$$

Similar to spacetime translation, we obtain the representation of supercharge operator on field space by equating the above variation with

$$\delta_{\epsilon,\overline{\epsilon}} Y(x,\theta,\overline{\theta}) = \left(i\epsilon Q + i\overline{\epsilon}\overline{Q}\right) Y(x,\theta,\overline{\theta}) = \left[i\epsilon^\alpha Q_\alpha - i\overline{\epsilon}^{\dot{\alpha}} \overline{Q}_{\dot{\alpha}}\right] Y(x,\theta,\overline{\theta}), \tag{IV.3.11}$$

$$\Rightarrow \begin{cases} Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}} \partial_\mu \\ \overline{Q}_{\dot{\alpha}} = +i\overline{\partial}_{\dot{\alpha}} + \theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu \end{cases} \tag{IV.3.12}$$

Having the differential form of supercharge operators, we are able to find the variations of component fields of $Y(x,\theta,\overline{\theta})$:

$$\begin{aligned} \delta_{\epsilon,\overline{\epsilon}} Y = i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right) Y &= i\left[\epsilon^\alpha \left(-i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}}\right) - \overline{\epsilon}^{\dot{\alpha}} \left(i\overline{\partial}_{\dot{\alpha}} + \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu\right)\right] \\ &= \epsilon^\alpha (\partial_\alpha Y) - i\epsilon^\alpha \sigma^\mu_{\alpha\dot{\beta}} \overline{\theta}^{\dot{\beta}} (\partial_\mu Y) + \overline{\epsilon}^{\dot{\alpha}} \left(\overline{\partial}_{\dot{\alpha}} Y\right) - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overline{\epsilon}^{\dot{\alpha}} (\partial_\mu Y), \end{aligned} \tag{IV.3.13}$$

with each term is expanded as

$$\epsilon^\alpha(\partial_\alpha Y) \qquad = \epsilon^\alpha \left(\psi_\alpha + 2\theta_\alpha\ M + \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}}\ V_\mu + 2\theta_\alpha \overline{\theta\lambda} + \overline{\theta\theta}\ \rho_\alpha - 2\theta_\alpha \overline{\theta\theta}\ D\right)$$

$$= \epsilon\psi + 2(\epsilon\theta)M + (\epsilon\sigma^\mu\overline{\theta})V_\mu + 2(\epsilon\theta)(\overline{\theta\lambda}) + (\overline{\theta\theta})(\epsilon\rho) + 2(\epsilon\theta)(\overline{\theta\theta})D, \quad \text{(IV.3.14)}$$

$$-i(\epsilon\sigma^\mu)(\partial_\mu Y) = -i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu\left[\varphi + \theta\psi + \overline{\theta}\overline{\chi} + \theta\theta M + (\theta\sigma^\nu\overline{\theta})V_\nu + (\theta\theta)\overline{\theta\lambda}\right], \quad \text{(IV.3.15)}$$

$$\begin{aligned}\overline{\epsilon}^{\dot{\alpha}}(\overline{\partial}_{\dot{\alpha}}Y) &= -\overline{\epsilon}^{\dot{\alpha}}\left(\overline{\chi}_{\dot{\alpha}} + 2\overline{\theta}_{\dot{\alpha}}N + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}V_\mu + (\theta\theta)\overline{\lambda}_{\dot{\alpha}} + 2\overline{\theta}_{\dot{\alpha}}(\theta\rho) + 2(\theta\theta)\overline{\theta}_{\dot{\alpha}}D\right)\\ &= \overline{\epsilon\chi}2(\overline{\epsilon\theta})N + (\theta\sigma^\mu\overline{\epsilon})V_\mu + (\theta\theta)(\overline{\epsilon\lambda}) + 2(\overline{\epsilon\theta})(\theta\rho) + 2(\theta\theta)(\overline{\epsilon\theta})D,\end{aligned} \quad \text{(IV.3.16)}$$

$$i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\overline{\epsilon}^{\dot{\alpha}}(\partial_\mu Y) = i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu\left[\varphi + \theta\psi + \overline{\theta}\overline{\chi} + (\overline{\theta\theta})N + (\theta\sigma^\nu\overline{\theta})V_\nu + (\overline{\theta\theta})(\theta\rho)\right]. \quad \text{(IV.3.17)}$$

Summing all the terms and rearranging the fields in the appropriate form

$$\begin{aligned}\delta_{\epsilon,\overline{\epsilon}}Y = {}&[\epsilon\psi + \overline{\epsilon\chi}] + [2(\epsilon\theta)M + i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu\varphi + (\theta\sigma^\mu\overline{\epsilon})V_\mu] + \left[(\epsilon\sigma^\mu\overline{\theta})V_\mu - i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu\varphi + 2(\overline{\epsilon\theta})N\right]\\ &+ \left[(\theta\theta)(\overline{\epsilon\lambda}) + i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu(\theta\psi)\right] + \left[(\overline{\theta\theta})(\epsilon\rho) - i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu(\overline{\theta\chi})\right]\\ &+ \left[2(\epsilon\theta)(\overline{\theta\lambda}) - i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu(\theta\psi)\right] + \left[2(\overline{\epsilon\theta})(\theta\rho) + i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu(\overline{\theta\chi})\right]\\ &+ \left[-i(\epsilon\sigma^\mu\overline{\theta})(\theta\theta)\partial_\mu M + 2(\theta\theta)(\overline{\epsilon\theta})D + i(\theta\sigma^\mu\overline{\epsilon})(\theta\sigma^\nu\overline{\theta})\partial_\mu V_\nu\right]\\ &+ \left[2(\epsilon\theta)(\overline{\theta\theta})D - i(\epsilon\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu V_\nu + i(\theta\sigma^\mu\overline{\epsilon})(\overline{\theta\theta})\partial_\mu N\right] + \left[-i(\epsilon\sigma^\mu\overline{\theta})(\theta\theta)(\overline{\theta}\partial_\mu\overline{\lambda}) + i(\theta\sigma^\mu\overline{\epsilon})(\overline{\theta\theta})(\theta\partial_\mu\rho)\right].\end{aligned} \quad \text{(IV.3.18)}$$

Comparing this with the variation of Y from definition

$$\begin{aligned}\delta_{\epsilon,\overline{\epsilon}}Y(x,\theta,\overline{\theta}) = {}&\delta\varphi + \theta\delta\psi + \overline{\theta}\delta\overline{\chi} + \theta\theta\ \delta M + \overline{\theta\theta}\ \delta N\\ &+ \theta\sigma^\mu\overline{\theta}\ \delta V_\mu + \theta\theta\ \overline{\theta}\delta\overline{\lambda} + \overline{\theta\theta}\ \delta\theta\rho + \theta\theta\ \overline{\theta\theta}\ \delta D\end{aligned} \quad \text{(IV.3.19)}$$

gives the variations of component fields:

$$\delta\varphi = \epsilon\psi + \overline{\epsilon\chi}, \quad \text{(IV.3.20)}$$

$$\theta\delta\psi = 2(\epsilon\theta)M + i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu\varphi + (\theta\sigma^\mu\overline{\epsilon})V_\mu, \quad \text{(IV.3.21)}$$

$$\overline{\theta}\delta\overline{\chi} = 2(\overline{\epsilon\theta})N - i(\epsilon\sigma^\mu\overline{\theta})(\partial_\mu\varphi) + (\epsilon\sigma^\mu\overline{\theta})V_\mu, \quad \text{(IV.3.22)}$$

$$\theta\theta\delta M = (\theta\theta)(\overline{\epsilon\lambda}) + \underbrace{i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu(\theta\psi)}_{\equiv T_1}, \quad \text{(IV.3.23)}$$

$$\overline{\theta\theta}\delta N = (\overline{\theta\theta})(\epsilon\rho) - \underbrace{i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu(\overline{\theta\chi})}_{\equiv T_2}, \quad \text{(IV.3.24)}$$

$$(\theta\sigma^\mu\overline{\theta})\delta V_\mu = \underbrace{2(\epsilon\theta)(\overline{\theta\lambda})}_{\equiv T_3} - \underbrace{i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu(\theta\psi)}_{\equiv T_4} + \underbrace{2(\overline{\epsilon\theta})(\theta\rho)}_{\equiv T_5} + \underbrace{i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu(\overline{\theta\chi})}_{\equiv T_6}, \quad \text{(IV.3.25)}$$

$$(\theta\theta)(\overline{\theta}\delta\overline{\lambda}) = -i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu(\theta\theta M) + 2(\theta\theta)(\overline{\epsilon\theta})D + \underbrace{i(\theta\sigma^\mu\overline{\epsilon})(\theta\sigma^\nu\overline{\theta})\partial_\mu V_\nu}_{\equiv T_7}, \quad \text{(IV.3.26)}$$

$$(\overline{\theta\theta})(\theta\rho) = 2(\epsilon\theta)(\overline{\theta\theta})D - \underbrace{i(\epsilon\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu V_\nu}_{\equiv T_8} + i(\theta\sigma^\mu\overline{\epsilon})(\overline{\theta\theta})\partial_\mu N, \quad \text{(IV.3.27)}$$

$$(\theta\theta)(\overline{\theta\theta})D = -\underbrace{i(\epsilon\sigma^\mu\overline{\theta})(\theta\theta)(\overline{\theta}\partial_\mu\overline{\lambda})}_{\equiv T_9} + \underbrace{i(\theta\sigma^\mu\overline{\epsilon})(\overline{\theta\theta})(\theta\partial_\mu\rho)}_{\equiv T_{10}}. \quad \text{(IV.3.28)}$$

The θ and $\overline{\theta}$ in these expressions can be eliminated to give the supersymmetric variation of pure fields. Starting with ψ in Eq. (IV.3.21), (note that $\epsilon\theta = \theta\epsilon$)

$$\theta\delta\psi = 2(\theta\epsilon)M + i(\theta\sigma^\mu\overline{\epsilon})\partial_\mu\varphi + (\theta\sigma^\mu\overline{\theta})V_\mu \Rightarrow \boxed{\delta\psi = 2\epsilon M + \sigma^\mu\overline{\epsilon}(i\partial_\mu\varphi + V_\mu).} \quad \text{(IV.3.29)}$$

Similarly for the variation of $\overline{\chi}$ in Eq. (IV.3.22)

$$\overline{\theta}\delta\overline{\chi} = 2(\overline{\epsilon\theta})N - i(\epsilon\sigma^\mu\overline{\theta})\partial_\mu\varphi + (\epsilon\sigma^\mu\overline{\theta})V_\mu \Rightarrow \boxed{\delta\chi = 2\overline{\epsilon}N - \epsilon\sigma^\mu(i\partial_\mu\varphi - V_\mu).} \quad \text{(IV.3.30)}$$

The terms from T_1 to T_{10} requires a little lengthy calculations to bring similar form to the LHS. To save space, we will use the identities of two-component spinors without explicitly derivation in Appendix A.2, as follows

$$T_1 \equiv i\left(\theta\sigma^\mu\overline{\theta}\right)(\theta\partial_\mu\psi) = -\frac{i}{2}(\theta\theta)\left(\partial_\mu\psi\sigma^\mu\overline{\epsilon}\right), \tag{IV.3.31}$$

which leads to the variation of M

$$\theta\theta M = (\theta\theta)(\overline{\epsilon}\overline{\lambda}) - \frac{i}{2}(\theta\theta)(\partial_\mu\psi\sigma^\mu\overline{\epsilon}) \Rightarrow \boxed{\delta M = \overline{\epsilon}\overline{\lambda} - \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\overline{\epsilon}\right).} \tag{IV.3.32}$$

$$T_2 \equiv i\left(\epsilon\sigma^\mu\overline{\theta}\right)\left(\overline{\theta}\partial_\mu\overline{\chi}\right) = -\frac{i}{2}\left(\overline{\theta\theta}\right)\left(\epsilon\sigma^\mu\overline{\chi}\right). \tag{IV.3.33}$$

The variation of N-field is thus

$$\overline{\theta\theta}\delta N = (\overline{\theta\theta})(\epsilon\rho) + \frac{i}{2}(\overline{\theta\theta})(\epsilon\sigma^\mu\partial_\mu\overline{\chi}) \Rightarrow \boxed{\delta N = \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\overline{\chi}.} \tag{IV.3.34}$$

Similarly, the next four terms corresponding to V_μ can be Fierz transformed into [1]

$$\begin{aligned}
T_3 &\equiv 2(\theta\epsilon)(\overline{\theta\lambda}) = (\theta\sigma^\mu\overline{\theta})(\epsilon\sigma_\mu\overline{\lambda}),\\
T_4 &\equiv i(\epsilon\sigma^\mu\overline{\theta})(\theta\partial_\mu\psi) = -\frac{i}{2}\left(\theta\sigma^\nu\overline{\theta}\right)\left(\epsilon\sigma^\mu\overline{\sigma}_\nu\partial_\mu\psi\right),\\
T_5 &\equiv 2(\overline{\epsilon}\overline{\theta})(\theta\rho) = \left(\theta\sigma^\mu\overline{\theta}\right)\left(\rho\sigma_\mu\overline{\epsilon}\right),\\
T_6 &\equiv i\left(\theta\sigma^\mu\overline{\epsilon}\right)\left(\overline{\theta}\partial_\mu\overline{\chi}\right) = -\frac{i}{2}\left(\theta\sigma^\nu\overline{\theta}\right)\left(\partial_\mu\overline{\chi}\overline{\sigma}_\nu\sigma^\mu\overline{\epsilon}\right).
\end{aligned} \tag{IV.3.36}$$

The supersymmetric variation of vector field is

$$(\theta\sigma^\mu\overline{\theta})\delta V_\mu = 2\left(\theta\sigma^\mu\overline{\theta}\right)\left(\epsilon\sigma_\mu\overline{\lambda}\right) + \frac{i}{2}\left(\theta\sigma^\nu\overline{\theta}\right)\left(\epsilon\sigma^\mu\overline{\sigma}_\nu\partial_\mu\psi\right) + \left(\theta\sigma^\mu\overline{\theta}\right)\left(\rho\sigma_\mu\overline{\epsilon}\right) - \frac{i}{2}\left(\theta\sigma^\nu\overline{\theta}\right)\left(\partial_\mu\overline{\chi}\overline{\sigma}_\nu\sigma^\mu\overline{\epsilon}\right)$$

$$\Rightarrow \boxed{\delta V_\mu = \epsilon\sigma_\mu\overline{\lambda} + \rho\sigma_\mu\overline{\epsilon} + \frac{i}{2}\epsilon\sigma^\nu\overline{\sigma}_\mu\partial_\nu\psi - \frac{i}{2}\partial_\nu\overline{\chi}\overline{\sigma}_\nu\sigma^\mu\overline{\epsilon} = \epsilon\sigma_\mu\overline{\lambda} + \rho\sigma_\mu\overline{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\overline{\sigma}_\nu\epsilon - \overline{\epsilon}\overline{\sigma}_\nu\sigma_\mu\partial^\nu\overline{\chi}\right).} \tag{IV.3.37}$$

The terms T_7 and T_8 require the usage of identity $\left(\theta\sigma^\mu\overline{\theta}\right)\left(\theta\sigma^\nu\overline{\theta}\right) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\overline{\theta\theta})$

$$\begin{aligned}
T_7 &\equiv i\left(\theta\sigma^\mu\overline{\epsilon}\right)\left(\theta\sigma^\nu\overline{\theta}\right)\partial_\mu V_\nu = i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\overline{\epsilon}^{\dot{\alpha}}\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu V_\nu = -i(\theta^\alpha\theta^\beta)\overline{\epsilon}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu V_\nu\\
&= \frac{i}{2}(\theta\theta)\overline{\epsilon}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\beta}\dot{\gamma}}\sigma^\nu_{\beta\dot{\beta}}\overline{\theta}_{\dot{\gamma}}\partial_\mu V_\nu = -\frac{i}{2}(\theta\theta)\overline{\epsilon}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\overline{\sigma}^{\nu\dot{\gamma}\alpha}\overline{\theta}_{\dot{\gamma}}\partial_\mu V_\nu = \frac{i}{2}(\theta\theta)\overline{\theta}_{\dot{\gamma}}\overline{\sigma}^{\nu\dot{\gamma}\alpha}\sigma^\mu_{\alpha\dot{\alpha}}\overline{\epsilon}^{\dot{\alpha}}\partial_\mu V_\nu\\
&= \frac{i}{2}(\theta\theta)(\overline{\theta}\overline{\sigma}^\nu\sigma^\mu\overline{\epsilon})\partial_\mu V_\nu.
\end{aligned} \tag{IV.3.38}$$

$$\begin{aligned}
T_8 &= i(\epsilon\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu V_\nu = i\epsilon^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu V_\nu = -i\epsilon^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\theta^\beta(\overline{\theta}^{\dot{\alpha}}\overline{\theta}^{\dot{\beta}})\partial_\mu V_\nu\\
&= \frac{i}{2}(\overline{\theta\theta})\epsilon_\gamma\varepsilon^{\alpha\gamma}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}}\theta^\beta\partial_\mu V_\nu = \frac{i}{2}(\overline{\theta\theta})\epsilon_\gamma\overline{\sigma}^{\mu\dot{\beta}\gamma}\sigma^\nu_{\beta\dot{\beta}}\theta^\beta\partial_\mu V_\nu = \frac{i}{2}(\overline{\theta\theta})\theta^\beta\sigma^\nu_{\beta\dot{\beta}}\overline{\sigma}^{\mu\dot{\beta}\gamma}\epsilon_\gamma\\
&= \frac{i}{2}(\overline{\theta\theta})(\epsilon\overline{\sigma}^\mu\sigma^\nu\theta)\partial_\mu V_\nu.
\end{aligned} \tag{IV.3.39}$$

The variations of $\overline{\lambda}$ and ρ are expressed as follows:

$$\boxed{\delta\overline{\lambda} = i\overline{\sigma}^\mu\epsilon\partial_\mu M + 2\overline{\epsilon}D + \frac{i}{2}\overline{\sigma}^\nu\sigma^\mu\overline{\epsilon}\partial_\mu V_\nu,} \tag{IV.3.40}$$

$$\boxed{\delta\rho = 2\epsilon D - \frac{i}{2}\overline{\sigma}^\nu\sigma^\mu\epsilon\partial_\mu V_\nu + i\sigma^\mu\overline{\epsilon}\partial_\mu N.} \tag{IV.3.41}$$

[1] The terms T_4 and T_6 could be rewritten simply as

$$\begin{aligned}
T_4 &= -\frac{i}{2}\left(\theta\sigma^\mu\overline{\theta}\right)\left(\epsilon\partial_\mu\psi\right),\\
T_6 &= -\frac{i}{2}\left(\theta\sigma^\mu\overline{\theta}\right)\left(\overline{\epsilon}\partial_\mu\overline{\chi}\right).
\end{aligned} \tag{IV.3.35}$$

Finally we consider the two terms in the variation of D that need to be Fierz transformed

$$\begin{aligned} T_9 &\equiv i(\epsilon\sigma^\mu\overline{\theta})(\theta\theta)(\overline{\theta}\partial_\mu\overline{\lambda}) = -\frac{i}{2}(\theta\theta)(\overline{\theta}\overline{\theta})(\epsilon\sigma^\mu\partial_\mu\overline{\lambda}), \\ T_{10} &\equiv i(\theta\sigma^\mu\overline{\epsilon})(\overline{\theta}\overline{\theta})(\theta\partial_\mu\rho) = -\frac{i}{2}(\theta\theta)(\overline{\theta}\overline{\theta})(\partial_\mu\rho\sigma^\mu\overline{\epsilon}). \end{aligned} \tag{IV.3.42}$$

$$\Rightarrow \boxed{\delta D = \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\overline{\lambda} - \frac{i}{2}\partial_\mu\rho\sigma^\mu\overline{\epsilon}.} \tag{IV.3.43}$$

In summary, we get the change of each terms in the general superfield Y after infinitesimal supersymmetric transformation

Supersymmetric Variations of Component Fields

$$\begin{cases} \delta\varphi = \epsilon\psi + \overline{\epsilon}\overline{\chi} & \text{(IV.3.44a)} \\ \delta\psi = 2\epsilon M + \sigma^\mu\overline{\epsilon}(i\partial_\mu\varphi + V_\mu) & \text{(IV.3.44b)} \\ \delta\overline{\chi} = 2\overline{\epsilon}N - \epsilon\sigma^\mu(i\partial_\mu\varphi - V_\mu) & \text{(IV.3.44c)} \\ \delta M = \overline{\epsilon}\overline{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\epsilon} & \text{(IV.3.44d)} \\ \delta N = \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\overline{\chi} & \text{(IV.3.44e)} \\ \delta V_\mu = \epsilon\sigma_\mu\overline{\lambda} + \rho\sigma_\mu\overline{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\overline{\sigma}_\nu\epsilon - \overline{\epsilon}\overline{\sigma}_\nu\sigma_\mu\partial^\nu\overline{\chi}\right) & \text{(IV.3.44f)} \\ \delta\overline{\lambda} = 2\overline{\epsilon}D + \frac{i}{2}\left(\overline{\sigma}^\nu\sigma^\mu\overline{\epsilon}\right)\partial_\mu V_\nu + i\overline{\sigma}^\mu\epsilon\partial_\mu M & \text{(IV.3.44g)} \\ \delta\rho = 2\epsilon D - \frac{i}{2}\left(\sigma^\nu\overline{\sigma}^\mu\epsilon\right)\partial_\mu V_\nu + i\sigma^\mu\overline{\epsilon}\partial_\mu N & \text{(IV.3.44h)} \\ \delta D = \frac{i}{2}\partial_\mu\left(\epsilon\sigma^\mu\overline{\lambda} - \rho\sigma^\mu\overline{\epsilon}\right) & \text{(IV.3.44i)} \end{cases}$$

IV.4 General Supersymmetric Action

IV.4.1 Basic construction

We can build an action that is automatically invariant under supersymmtric transformation using the superspace and superfield language. First consider the integral of the type [1]

$$\int \mathrm{d}^4x\ \mathrm{d}^4\theta\ Y(x,\theta,\overline{\theta}). \tag{IV.4.1}$$

Since a measure is translational invariant, one has

$$\delta_{\epsilon,\overline{\epsilon}}\int \mathrm{d}^4x\ \mathrm{d}^4\theta\ Y(x,\theta,\overline{\theta}) = \int \mathrm{d}^4x\ \mathrm{d}^4\theta\ \delta_{\epsilon,\overline{\epsilon}}Y(x,\theta,\overline{\theta}), \tag{IV.4.2}$$

and recall that the variation of Y with respect to ϵ and $\overline{\epsilon}$ is

$$\delta_{\epsilon,\overline{\epsilon}}Y(x,\theta,\overline{\theta}) = \epsilon^\alpha\partial_\alpha Y + \overline{\epsilon}_{\dot{\alpha}}\overline{\partial}^{\dot{\alpha}}Y + \partial_\mu\left[-i(\epsilon\sigma^\mu\overline{\theta} - \theta\sigma^\mu\overline{\epsilon})Y\right]. \tag{IV.4.3}$$

Note that integration in $\mathrm{d}^2\theta\ \mathrm{d}^2\overline{\theta}$ kills the first two terms while the final term is a total derivative and is required to be vanished at the spacetime integral boundaries. Hence the proposed integral is supersymmetric invariant

$$\delta_{\epsilon,\overline{\epsilon}}\int \mathrm{d}^4x\ \mathrm{d}^4\theta\ Y(x,\theta,\overline{\theta}) = 0. \tag{IV.4.4}$$

Next, we make a claim that the set of superfields is closed under field multiplication. Let Y_1 and Y_2 be superfields, the variation of their product is

$$\delta(Y_1Y_2) = i\left[Y_1Y_2, \epsilon Q + \overline{\epsilon}\overline{Q}\right] = iY_1\left[Y_2, \epsilon Q + \overline{\epsilon}\overline{Q}\right] + i\left[Y_1, \epsilon Q + \overline{\epsilon}\overline{Q}\right]Y_2$$

[1] For short notation, under integral symbol we will frequently use $\mathrm{d}^4\theta$ to replace $\mathrm{d}^2\theta\ \mathrm{d}^2\overline{\theta}$ whenever there is no confusion.

$$= iY_1 \left[\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right) Y_2\right] + i \left[\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right) Y_1\right] Y_2 = i \left(\epsilon Q + \overline{\epsilon}\overline{Q}\right) \left(Y_1 Y_2\right). \tag{IV.4.5}$$

The sum of two superfields is again a superfield, which follows directly from the linearity of supercharge operators. We want to find the differential operator, when acting on a superfield gives a superfield; such term will appear in the kinetic part of SUSY Lagrangian. The first operator is spacetime derivative ∂_μ, which is easily recognized

$$\begin{aligned}&\left[\partial_\mu, e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}\right] = 0 \Rightarrow \delta_{\epsilon,\overline{\epsilon}}(\partial_\mu) = e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}\partial_\mu e^{i(\epsilon Q+\overline{\epsilon}\overline{Q})} - \partial_\mu = 0\\ &\Rightarrow \delta_{\epsilon,\overline{\epsilon}}(\partial_\mu Y) = \partial_\mu(\delta_{\epsilon,\overline{\epsilon}}Y) = \partial_\mu \left[i(\epsilon Q + \overline{\epsilon}\overline{Q})\right] Y = i(\epsilon Q + \overline{\epsilon}\overline{Q})(\partial_\mu Y).\end{aligned} \tag{IV.4.6}$$

We construct the covariant derivatives of the following form, which are also supersymmetric invariant

$$\boxed{\begin{cases}\mathcal{D}_\alpha \equiv \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu & \text{(IV.4.7a)}\\ \overline{\mathcal{D}}_{\dot{\alpha}} \equiv \overline{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu & \text{(IV.4.7b)}\end{cases}}$$

The structure of each covariant derivative ensure the vanishing of the anticommutation relation

$$\left\{\mathcal{D}_\alpha, \mathcal{D}_\beta \text{ or } Q_\beta \text{ or } \overline{Q}_{\dot{\beta}}\right\} = 0, \tag{IV.4.8}$$

$$\left\{\overline{\mathcal{D}}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}} \text{ or } Q_\beta \text{ or } \overline{Q}_{\dot{\beta}}\right\} = 0, \tag{IV.4.9}$$

implying $\mathcal{D}_\alpha Y$ and $\overline{\mathcal{D}}_{\dot{\alpha}} Y$ are indeed superfields:

$$\begin{aligned}&\left[\mathcal{D}_\alpha, e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}\right] = 0 \Rightarrow \delta_{\epsilon,\overline{\epsilon}}(\mathcal{D}_\alpha) = e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}\mathcal{D}_\alpha e^{i(\epsilon Q+\overline{\epsilon}\overline{Q})} - \mathcal{D}_\alpha = 0\\ &\Rightarrow \delta_{\epsilon,\overline{\epsilon}}(\mathcal{D}_\alpha Y) = \mathcal{D}_\alpha(\delta_{\epsilon,\overline{\epsilon}}Y) = \mathcal{D}_\alpha \left[i(\epsilon Q + \overline{\epsilon}\overline{Q})\right] Y = i(\epsilon Q + \overline{\epsilon}\overline{Q})(\mathcal{D}_\alpha Y).\end{aligned} \tag{IV.4.10}$$

$$\begin{aligned}&\left[\overline{\mathcal{D}}_{\dot{\alpha}}, e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}\right] = 0 \Rightarrow \delta_{\epsilon,\overline{\epsilon}}(\overline{\mathcal{D}}_{\dot{\alpha}}) = e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})}\overline{\mathcal{D}}_{\dot{\alpha}} e^{i(\epsilon Q+\overline{\epsilon}\overline{Q})} - \overline{\mathcal{D}}_{\dot{\alpha}} = 0\\ &\Rightarrow \delta_{\epsilon,\overline{\epsilon}}(\overline{\mathcal{D}}_{\dot{\alpha}} Y) = \overline{\mathcal{D}}_{\dot{\alpha}}(\delta_{\epsilon,\overline{\epsilon}}Y) = \overline{\mathcal{D}}_{\dot{\alpha}} \left[i(\epsilon Q + \overline{\epsilon}\overline{Q})\right] Y = i(\epsilon Q + \overline{\epsilon}\overline{Q})(\overline{\mathcal{D}}_{\dot{\alpha}} Y).\end{aligned} \tag{IV.4.11}$$

The last anticommutator yields

$$\begin{aligned}\left\{\mathcal{D}_\alpha, \overline{\mathcal{D}}_{\dot{\alpha}}\right\} &= \left\{\partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu, \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu\right\} = i\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\left(\partial_\alpha\theta^\beta\right) + i\sigma^\mu_{\alpha\dot{\beta}}\partial_\mu\left(\overline{\partial}_{\dot{\alpha}}\dot{\theta}^{\dot{\beta}}\right)\\ &= 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu = -2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu.\end{aligned} \tag{IV.4.12}$$

To summarize, we can build a superfield based on other superfields using the following properties:

- A product of superfields is a superfield.
- A linear combination of superfields is a superfield. This comes directly from the linearity of supercharge operators.
- Applying each of the following differential operators: ∂_μ, $\mathcal{D}_\alpha$, $\overline{\mathcal{D}}_{\dot{\alpha}}$ on a superfield gives a superfield.

Follow the logic stream, we can build a SuperPoincaré invariant action by taking integral of superfield over superspace

$$\mathcal{S} = \int \mathrm{d}^4x\ \mathcal{L}\left(\varphi(x), \psi(x), A_\mu(x), \dots\right) = \int \mathrm{d}^4x\ \mathrm{d}^4\theta\ \mathcal{A}(x, \theta, \overline{\theta}), \tag{IV.4.13}$$

where $\mathcal{A}$ is a combination of products of superfield rather than Y itself, since general Y can still be reducible, i.e some of the component fields is possibly eliminated. In the next section we will consider a simpler superfields by adding some constraints on the general superfield Y. Here are some promising candidates for irreducible superfields for our theory, and will repeatedly be used on the construction of SUSY Lagrangian:

- Chiral superfield φ such that $\overline{\mathcal{D}}_{\dot{\alpha}}\varphi = 0$.
- Antichiral superfield $\overline{\varphi}$ such that $\mathcal{D}_\alpha\overline{\varphi} = 0$.

- Vector superfield, satisfying the reality condition $\overline{V} = V$.
- Linear superfield L such that $\mathcal{D}\mathcal{D}L = 0$ and $\overline{L} = L$.

To obtain a Lagrangian describing the kinematics of known fields, we need to have superfields that contains spinors for representing matter, and also superfields containing vector fields to describe gauge interaction. Next, we need to combines those fields to obtain a scalar object that is invariant under SuperPoincaré transformation and gauge transformation. Finally, both matter superfields and vector superfields must be coupled to illustrate the matter-radiation interaction.

Of course using the superspace and superfield language requires the supersymmetric Lagrangian must be built out of the general superfields. Such a interpretation is, however, complicated and could lead to the fact that our Lagrangian contains exceeding information. To study the Lagrangian that contains similar information to the original one, and thus more instructive to reduce to the case with no supersymmetry, one needs to consider the similar building unit of the model: the fields. Let us take the SM for instance, it is built out of spin 0 Higgs field, spin 1/2 Dirac fields and spin 1 vector fields. The claimed similarities between these building blocks can be achived by applying the constraints to the general superfields. In the following, we will investigate each type of the constraints in detail.

IV.4.2 Chiral & Antichiral Superfields

A chiral superfield Φ is defined as follows

$$\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0. \tag{IV.4.14}$$

Seemingly, the antichiral superfield Ψ is a superfield such that

$$\mathcal{D}_{\alpha}\Psi = 0. \tag{IV.4.15}$$

By definition, taking the complex conjugation of D yields $\overline{D}$ and vice versa, thus taking the complex conjugation of chiral superfield yields antichiral superfield. This contains the complex nature of chiral superfield: the case Φ happens to be real implies that

$$\begin{cases} \mathcal{D}_{\alpha}\Phi = 0 \\ \overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0 \end{cases} \Rightarrow \begin{cases} \left(\partial_{\alpha} + i\sigma^{\mu}_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_{\mu}\right)\Phi = 0 \\ \left(\overline{\partial}_{\dot{\alpha}} + i\theta^{\beta}\sigma^{\mu}_{\beta\dot{\alpha}}\partial_{\mu}\right)\Phi = 0 \end{cases},$$

implying that Φ is independent of θ and $\overline{\theta}$, and that $\partial_{\mu}\Phi = 0$, which has the constant solution. To deal with the constraints (IV.4.14) and (IV.4.15), it is convenient to use change of variables. We define new coordinates

$$\begin{cases} y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\overline{\theta} & \text{(IV.4.16a)} \\ \overline{y}^{\mu} = x^{\mu} - i\theta\sigma^{\mu}\overline{\theta} & \text{(IV.4.16b)} \end{cases}$$

It is directly realized that [1]

$$\begin{cases} \overline{\mathcal{D}}_{\dot{\alpha}}\theta_{\beta} = \overline{\mathcal{D}}_{\dot{\alpha}}y^{\mu} = 0 \\ \mathcal{D}_{\alpha}\overline{\theta}_{\dot{\beta}} = \mathcal{D}_{\alpha}\overline{y}^{\mu} = 0 \end{cases} \tag{IV.4.19a}$$

In this new supercoordinate system, the chiral superfield is a function of only y and θ, but not on $\overline{\theta}$ explicitly. The solution for the chiral superfield is

$$\boxed{\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y).} \tag{IV.4.20}$$

Taylor-expanding the above solution for Φ around x to find the solution in the original supercoordinate system

$$\Phi(x,\theta,\overline{\theta}) = \phi(x^{\mu} + i\theta\sigma^{\mu}\overline{\theta}) + \sqrt{2}\theta\psi(x^{\mu} + i\theta\sigma^{\mu}\overline{\theta}) + \theta\theta F(x^{\mu} + i\theta\sigma^{\mu}\overline{\theta})$$

[1] A proof for this is straightforward. First consider D acting on y

$$\overline{\mathcal{D}}_{\dot{\alpha}}y^{\mu} = \left(\overline{\partial}_{\dot{\alpha}} + i\sigma^{\nu}_{\beta\dot{\alpha}}\theta^{\beta}\partial_{\nu}\right)(x^{\mu} + i\theta\sigma^{\mu}\overline{\theta}) = \overline{\partial}_{\dot{\alpha}}\left(i\theta\sigma^{\mu}\overline{\theta}\right) + i\sigma^{\nu}_{\beta\dot{\alpha}}\theta^{\beta}\partial_{\nu}x^{\mu} = -i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}} + i\sigma^{\mu}_{\alpha\dot{\alpha}}\theta^{\alpha} = 0. \tag{IV.4.17}$$

Taking the Hermitian conjugation yields the other identity

$$\mathcal{D}_{\alpha}\overline{y}^{\mu} = 0. \quad \square \tag{IV.4.18}$$

$$= \left[\phi(x) + \partial_\mu\phi(x)(i\theta\sigma^\mu\overline{\theta}) - \frac{1}{2}(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu\partial_\nu\phi(x)\right] + \left[\sqrt{2}\theta\psi(x) + \sqrt{2}\theta\partial_\mu\psi(x)(i\theta\sigma^\mu\overline{\theta})\right] + \theta\theta F(x). \tag{IV.4.21}$$

We Fierz transform these following terms for simplicity. The identities we use here will be given explicitly on Appendix A.2.

$$\begin{aligned}(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu\partial_\nu\phi &= \frac{g^{\mu\nu}}{2}(\theta\theta)(\overline{\theta}\overline{\theta})\Box\phi,\\ (\theta\partial_\mu\psi)(\theta\sigma^\mu\overline{\theta}) &= -\frac{1}{2}(\theta\theta)\partial_\mu(\psi\sigma^\mu\overline{\theta}).\end{aligned} \tag{IV.4.22}$$

and rewrite the solution of chiral superfield as

$$\Phi(x,\theta,\overline{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + i(\theta\sigma^\mu\overline{\theta})\partial_\mu\phi(x) + (\theta\theta)F(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \frac{1}{4}(\theta\theta)(\overline{\theta}\overline{\theta})\Box\phi(x). \tag{IV.4.23}$$

In practice, it is sometimes useful to work with variables y^μ instead of x^μ since it shortens many calculations involving chiral superfields. Let us now investigate the form of supercharge operators in this new coordinates:

$$(x,\theta,\overline{\theta}) \longrightarrow (y,\theta',\overline{\theta}'): \quad \begin{cases} y^\mu &= x^\mu + i\theta\sigma^\mu\overline{\theta}\\ \theta' &= \theta\\ \overline{\theta}' &= \overline{\theta}\end{cases}. \tag{IV.4.24}$$

Using chain rules to obtain the differential operators with respect to $(y,\theta',\overline{\theta}')$

$$\begin{cases} \dfrac{\partial}{\partial x^\mu} = \dfrac{\partial y^\nu}{\partial x^\mu}\dfrac{\partial}{\partial y^\nu} = \delta^\nu_\mu\dfrac{\partial}{\partial y^\nu} = \dfrac{\partial}{\partial y^\mu}, & \text{(IV.4.25a)}\\ \dfrac{\partial}{\partial\theta^\alpha} = \dfrac{\partial\theta'^\beta}{\partial\theta^\alpha}\dfrac{\partial}{\partial\theta'^\beta} + \dfrac{\partial y^\mu}{\partial\theta^\alpha}\dfrac{\partial}{\partial y^\mu} = \dfrac{\partial}{\partial\theta'^\alpha} + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}'^{\dot{\beta}}\dfrac{\partial}{\partial y^\mu}, & \text{(IV.4.25b)}\\ \dfrac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} = \dfrac{\partial\overline{\theta}'^{\dot{\beta}}}{\partial\overline{\theta}'^{\dot{\alpha}}}\dfrac{\partial}{\partial\overline{\theta}^{\dot{\beta}}} + \dfrac{\partial y^\mu}{\partial\overline{\theta}^{\dot{\alpha}}}\dfrac{\partial}{\partial y^\mu} = \dfrac{\partial}{\partial\overline{\theta}'^{\dot{\alpha}}} - i\theta'^\beta\sigma^\mu_{\beta\dot{\alpha}}\dfrac{\partial}{\partial y^\mu}, & \text{(IV.4.25c)}\end{cases}$$

and rewrite the supercharges as

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu = -i(\partial'_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}'^{\dot{\beta}}\partial^y_\mu) - \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu = -i\partial'_\alpha, \tag{IV.4.26}$$

$$\overline{Q}_{\dot{\alpha}} = i\overline{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu = i\left(\overline{\partial}'_{\dot{\alpha}} - i\theta'^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial^y_\mu\right) + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial^y_\mu = i\overline{\partial}'_{\dot{\alpha}} + 2\theta'^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial^y_\mu. \tag{IV.4.27}$$

From the definition (IV.4.16a) and (IV.4.16b), the change of variables from $(x,\theta,\overline{\theta})$ to $(\overline{y},\theta'',\overline{\theta}'')$ can be obtained directly from (IV.4.25) by changing the sign of σ^μ. The supercharges in this coordinates system are

$$Q_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu = -i(\partial''_\alpha - i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}''^{\dot{\beta}}\partial^{\overline{y}}_\mu) - \sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu = -i\partial''_\alpha - 2\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}''^{\dot{\beta}}\partial^{\overline{y}}_\mu, \tag{IV.4.28}$$

$$\overline{Q}_{\dot{\alpha}} = i\overline{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu = i\left(\overline{\partial}''_{\dot{\alpha}} + i\theta''^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial^{\overline{y}}_\mu\right) + \theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial^{\overline{y}}_\mu = i\overline{\partial}''_{\dot{\alpha}}. \tag{IV.4.29}$$

If there is no ambiguity, we simply ignore the $'$ on θ' and $\overline{\theta}'$. The representation of these generators in new coordinates system reads

$$(y,\theta,\overline{\theta}) \longrightarrow \begin{cases} Q^y_\alpha = -i\partial_\alpha\\ \overline{Q}^y_{\dot{\alpha}} = i\overline{\partial}_{\dot{\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial^y_\mu\end{cases}, \tag{IV.4.30}$$

$$(\overline{y},\theta,\overline{\theta}) \longrightarrow \begin{cases} Q^{\overline{y}}_\alpha = -i\partial_\alpha - 2\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial^{\overline{y}}_\mu,\\ \overline{Q}^{\overline{y}}_{\dot{\alpha}} = i\overline{\partial}_{\dot{\alpha}}\end{cases}. \tag{IV.4.31}$$

The change of variables for covariant derivatives is similar to that of Q and $\overline{Q}$, with the final results

$$(y,\theta,\overline{\theta}) \longrightarrow \begin{cases} \mathcal{D}^y_\alpha = \partial_\alpha + 2i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu\\ \overline{\mathcal{D}}^y_{\dot{\alpha}} = -\overline{\partial}_{\dot{\alpha}}\end{cases}, \tag{IV.4.32}$$

$$(\overline{y}, \theta, \overline{\theta}) \longrightarrow \begin{cases} \mathcal{D}^{\overline{y}}_{\alpha} = \partial_\alpha \\ \overline{\mathcal{D}}^{\overline{y}}_{\dot{\alpha}} = -\overline{\partial}_{\dot{\alpha}} - 2i\theta^\beta \sigma^\mu_{\beta\dot{\alpha}} \partial_\mu \end{cases} . \tag{IV.4.33}$$

Our first application of these new generators is to calculate the variations of component fields ϕ, ψ and F through the variation of chiral superfield Φ with respect to variable y

$$\begin{aligned} \delta_{\epsilon,\overline{\epsilon}}\Phi(y,\theta) &= \left(\epsilon^\alpha \partial_\alpha + 2i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \overline{\epsilon}^{\dot{\beta}} \partial^y_\mu\right) \Phi(y,\theta) = \sqrt{2}\epsilon\psi + 2\epsilon\theta F + 2i(\theta\sigma^\mu\overline{\epsilon}) \left(\partial^y_\mu \phi + \sqrt{2}\theta\partial^y_\mu \psi\right) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta \left(\sqrt{2}\epsilon F + \sqrt{2} i \sigma^\mu \overline{\epsilon} \partial^y_\mu \phi\right) - \theta\theta \left(-i\sqrt{2}\overline{\epsilon}\overline{\sigma}^\mu \partial^y_\mu \psi\right), \end{aligned} \tag{IV.4.34}$$

which leads to the variations of different field components of chiral superfield Φ as [1]

$$\begin{cases} \delta\phi = \sqrt{2}\epsilon\psi & \text{(IV.4.35a)} \\ \delta\psi_\alpha = \sqrt{2}i(\sigma^\mu\overline{\epsilon})_\alpha \partial_\mu\phi + \sqrt{2}\epsilon_\alpha F & \text{(IV.4.35b)} \\ \delta F = i\sqrt{2}\partial_\mu \left(\psi\sigma^\mu\overline{\epsilon}\right) & \text{(IV.4.35c)} \end{cases}$$

Several notes should be made about the chiral superfields, which will play an integral roles on building the SUSY Lagrangian

- A linear combination of chiral superfields is again a chiral superfield. This fact follows directly from the linearity of covariant derivative $\overline{\mathcal{D}}_{\dot{\alpha}}$.
- A product of chiral superfields produce a chiral superfield. More generally, any holomorphic function (one that satisfies $\partial W/\partial\overline{\Phi} = 0$) of chiral superfield is a chiral superfield as one may easily check

$$\overline{\mathcal{D}}_{\dot{\alpha}} W(\Phi) = \frac{\partial W}{\partial \Phi}\overline{\mathcal{D}}_{\dot{\alpha}}\Phi + \frac{\partial W}{\partial \overline{\Phi}}\overline{\mathcal{D}}_{\dot{\alpha}}\overline{\Phi} = 0. \tag{IV.4.36}$$

- Later when building superpotential for chiral sector, we will face the integral of the type

$$\int \mathrm{d}^4x \; \mathrm{d}^2\theta \; W(\Phi), \tag{IV.4.37}$$

  where $W(\Phi)$ is a chiral superfield that is built upon Φ. In computing, it is more compact and simple to use the variable y^μ instead of x^μ by safely setting $y^\mu = x^\mu$ whenever necessary (one can compare the length of solution for Φ with variable $(y, \theta, \overline{\theta})$ in Eq. (IV.4.20) and that of $(x, \theta, \overline{\theta})$ in Eq. (IV.4.21)). The action remains the same after applying this change of variable, whose contribution vanishes after taking the integral over spacetime.

IV.4.3 Vector Superfields

Another supersymmetric constrain we would like to consider is *reality condition*, which defines the vector superfield V

$$V = \overline{V}. \tag{IV.4.38}$$

The vector superfield can be expressed in terms of component fields as

$$\begin{aligned} V(x,\theta,\overline{\theta}) = {} & C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \theta\sigma^\mu\overline{\theta}V_\mu + \frac{i}{2}\left(M(x) + iN(x)\right) \\ & - \frac{i}{2}\overline{\theta\theta}\left(M(x) - iN(x)\right) + i\theta\theta\overline{\theta}\left(\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)\right) \\ & - i\overline{\theta\theta}\theta\left(\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\right) + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})\left(D(x) - \frac{1}{2}\Box C(x)\right). \end{aligned} \tag{IV.4.39}$$

[1] Another way is by comparing the expression of Φ in (IV.4.20) with the general expression of superfield Y, and employ the general variations (IV.3.44).

Gauge Transformation & Wess-Zumino gauge

Notice that this vector superfield has $8_B + 8_F$ degrees of freedom (2_B of $C(x)$ and $D(x)$, 2_B of $M(x)$ and $N(x)$ and 4_B from V_μ; each of spinor field χ and λ contribute 4_F degrees of freedom). This is too much field than needed, so we impose the gauge fixing to reduce half of degrees of freedom (off-shell), which become $2_B + 2_F$ for an on-shell massless vector supermultiplet. If $\Phi(x)$ is a chiral superfield then $\Phi(x) + \overline{\Phi}(x)$ and $\overline{\Phi}(x)\Phi(x)$ are both vector superfields. We adopt the following gauge transformation

$$\boxed{V \to V + \Phi + \overline{\Phi} = V + i\left(\Lambda - \overline{\Lambda}\right),} \tag{IV.4.40}$$

where $\Phi \equiv i\Lambda$. This choice of gauge transformation is more natural and simpler, as we shall see when consider the transformation of corresponding component fields. Finally, $\overline{\Phi}\Phi$ has the dimension 2 while V has dimension 1, which is not compatible if we use the transformation $V \to V + \overline{\Phi}\Phi$. Expanding in terms of component fields, we get

$$\begin{aligned}
\Phi + \overline{\Phi} &= \phi + \overline{\phi} + \sqrt{2}(\theta\psi + \overline{\theta\psi}) + i(\theta\sigma^\mu\overline{\theta})\partial_\mu(\phi - \overline{\phi}) - (\theta\theta F + \overline{\theta\theta F}) \\
&\quad + \frac{i}{\sqrt{2}}\left(\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi} - \theta\theta\partial_\mu\psi\sigma^\mu\overline{\theta}\right) - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\Box(\phi + \overline{\phi}) \\
&= 2\mathrm{Re}\phi + i\theta(-\sqrt{2}i\psi) - i\overline{\theta}(\sqrt{2}i\overline{\psi}) + \frac{i}{2}\theta\theta(2iF) - \frac{i}{2}\overline{\theta\theta}(-2i\overline{F}) \\
&\quad + i\theta\theta\overline{\theta}\left(\frac{\overline{\sigma}^\mu\partial_\mu\psi}{\sqrt{2}}\right) - i\overline{\theta\theta}\theta\left(\frac{-\sigma^\mu\partial_\mu\overline{\psi}}{\sqrt{2}}\right) + \frac{1}{2}(\theta\theta)(\overline{\theta\theta})(-\Box\mathrm{Re}\phi),
\end{aligned} \tag{IV.4.41}$$

which leads to the gauge transformation of component fields of V

$$\begin{cases}
C \to C + 2\mathrm{Re}\phi & \text{(IV.4.42a)} \\
\chi \to \chi - i\sqrt{2}\psi & \text{(IV.4.42b)} \\
M \to M - 2\mathrm{Im}F & \text{(IV.4.42c)} \\
N \to N + 2\mathrm{Re}F & \text{(IV.4.42d)} \\
D \to D & \text{(IV.4.42e)} \\
\lambda \to \lambda & \text{(IV.4.42f)} \\
V_\mu \to V_\mu - 2\partial_\mu\mathrm{Im}\phi & \text{(IV.4.42g)}
\end{cases}$$

from which one can gauge away C, M, N, χ by choosing

$$\mathrm{Re}\phi = -\frac{C}{2}, \quad \psi = -\frac{i}{\sqrt{2}}\chi, \quad \mathrm{Re}F = -\frac{N}{2}, \quad \mathrm{Im}F = \frac{M}{2}. \tag{IV.4.43}$$

This choice of gauge is called Wess-Zumino gauge, and the gauged vector superfield can be written simply as

$$\boxed{V_{WZ}(x) = \theta\sigma^\mu\overline{\theta}\; V_\mu(x) + i\theta\theta\;\overline{\theta\lambda}(x) - i\overline{\theta\theta}\;\theta\lambda(x) + \frac{1}{2}\theta\theta\;\overline{\theta\theta}D(x).} \tag{IV.4.44}$$

Here $V_\mu(x)$ is the gauge field and λ is its supersymmetric partner. $D(x)$ is the auxiliary field. Several observations could be made on Wess-Zumino gauge:

- We did not impose the gauge choice on gauge field V_μ yet; i.e we still have the freedom to perform a gauge transformation while remaining in the WZ gauge.
- WZ gauge is not supersymmetric. This fact is easy to be recognized, e.g consider the variation of left-chiral spinor ψ in Eq. (IV.3.21), it involves V_μ and hence must be nonzero, while in the WZ gauge ψ is gauged away by the choice as in Eq. (IV.4.43). When working in this choice, after a supersymmetry transformation, one has to do a compensating supersymmetric gauge transformation by properly chosen Φ.
- Each term of V_{WZ} contains at least one θ or $\overline{\theta}$, implying

$$V_{WZ}^2 = (\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})V_\mu V_\nu = \frac{1}{2}(\theta\theta)(\overline{\theta\theta})\; V^\mu V_\mu, \tag{IV.4.45}$$

$$V_{WZ}^{2+n} = 0, \forall n \in \mathbb{N}. \tag{IV.4.46}$$

We will see shortly that this property of WZ gauge is extremely useful, especially in building the radiation-matter interaction since the Taylor expansion of an arbitrary analytic function of V using WZ gauge will be cut-off after second order.

IV.5 Construction of Minimal Supersymmetric Lagrangian

Up to this point, we have the necessary ingredients to build the Lagrangian describing SUSY for $N = 1$. Several requirements, or restrictions on the Lagrangian should be listed before we move on:

- The Lagrangian $\mathcal{L}$ is a real object of mass dimension $[M]^4$. In order to get a renormalizable theory, the scalar superfields cannot enter the Lagrangian to an order higher than 3.
- $\mathcal{L}$ must transform as a total space-time derivative under SuperPoincaré transformation. For this requirement, we look for the variation that is also a total derivative of some fields. The very first candidate is the D-term of a general superfield Y (i.e the $(\theta\theta)(\overline{\theta\theta})$ term), whose variation $\delta D = \frac{i}{2}\partial_\mu\left(\epsilon\sigma^\mu\overline{\lambda} - \rho\sigma^\mu\overline{\epsilon}\right)$ has been derived in Section IV.3.1. The second one is the F-term of chiral superfield (which is the $\theta\theta$ term for chiral field) with the variation $\delta F = i\sqrt{2}\partial_\mu(\psi\sigma^\mu\overline{\epsilon})$, first derived in (IV.4.35c). Taking the Hermitian conjugate yields the F-term of chiral superfield, corresponding to $\overline{\theta\theta}$.

To get rid of non-physical Grassmann coordinates θ and $\overline{\theta}$, we can take the integral over that Grassmann variables noting that integration can be identified as differential for θ and $\overline{\theta}$. The desired Lagrangian is of the form

$$\mathcal{L}_{SUSY} = \mathcal{L}_1|_D + (\mathcal{L}_2|_F + h.c) = \int \mathrm{d}^4\theta\ \mathcal{L}_1 + \left(\int \mathrm{d}^2\theta\ \mathcal{L}_2 + \int \mathrm{d}^2\overline{\theta}\ \overline{\mathcal{L}}_2\right), \tag{IV.5.1}$$

with $\mathcal{L}_1$ being a vector superfield while $\mathcal{L}_2$ being a chiral superfield. Since the mass dimension $[\theta] = -1/2$, we then have $[\mathrm{d}^2\theta] = 1$ and $[\mathrm{d}^2\bar{\theta}] = -1$ [1]. This implies $[\mathcal{L}_1] = 2$ and $[\mathcal{L}_2] = 3$.

We first focus on the matter action and make an attempt to derive the most general supersymmetric action describing the dynamics of a set of (interacting) chiral superfields. To include the gauge interaction, we generalize the Yang-Mills theory to SuperYang-Mills from super field strength W_α and $\overline{W}_{\dot{\alpha}}$. Finally, by coupling the chiral sector and the gauge sector, we achieve the goal of deriving the most general $N = 1$ supersymmetric action (and thus supersymmetric Lagrangian) describing the interaction of radiation and matter. At some point, the renormalization requirement would be suppressed in order to have more open and wholly insight to the construction processes.

IV.5.1 Chiral Sector

The chiral sector describes the dynamics of matter without gauge interaction, and thus involves only chiral and antichiral superfields. The most general Lagrangian for multiple chiral fields $\mathbf{\Phi} = (\Phi_1, \Phi_2, \dots)$ based on the above restrictions is

$$\mathcal{L}_{chiral} = K(\mathbf{\Phi}, \overline{\mathbf{\Phi}})|_D + (W(\mathbf{\Phi})|_F + h.c)\,, \tag{IV.5.2}$$

where the so called Kähler potential $K(\mathbf{\Phi}, \overline{\mathbf{\Phi}})$ contribute to the kinetic part of the Lagrangian by taking integral over the whole space, and the superpotential $W(\mathbf{\Phi})$ is a holomorphic function of chiral superfield (and is itself a chiral superfield) contribution to the Lagrangian by taking integral over half space, as we noted in Eq. (IV.5.1).

Off-shell Free Lagrangian

First consider the Käler potential, there are number of requirements we want K to satisfy:

- K should be a superfield so as to contain the D-term, which is supersymmetric invariant.
- K should be a real scalar function. This is the property that the Lagrangian inherits after taking integral over Grassmann variables.
- We want a local field theory, hence our Lagrangian cannot have terms of derivatives of order higher than 2. Thus $K(\mathbf{\Phi}, \overline{\mathbf{\Phi}})$ is a function of $\mathbf{\Phi}$ and $\overline{\mathbf{\Phi}}$, but not of $\mathcal{D}_\alpha\mathbf{\Phi}$ and $\overline{\mathcal{D}}_{\dot{\alpha}}\overline{\mathbf{\Phi}}$, since applying covariant derivatives on chiral superfields would provide $(\theta\theta)(\overline{\theta\theta})$-term contributions having higher order derivatives than needed.

The most general expression for K based on those properties would be

$$K(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) = \sum_{m,n=1}^{\infty} c_{mn}\mathbf{\Phi}^m\overline{\mathbf{\Phi}}^n, \tag{IV.5.3}$$

[1] Consider the general superfield Y in (IV.3.1) for example, we know that $[\phi] = 1$ and $[\psi] = 3/2$, and that $[\theta\psi] = [\phi]$ implies $[\theta] = -1/2$.

with the reality condition are guaranteed by the relation $c_{mn} = c^*_{nm}$. All coefficients c_{mn} but c_{11} have negative dimension, and in a renormalizable theory those terms should not appear, and the Kähler potential have such a simple form as follows

$$K(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) = \overline{\mathbf{\Phi}}\mathbf{\Phi}. \tag{IV.5.4}$$

For multiple independent chiral superfields, an implicit sum over different superfields is needed:

$$\boxed{K(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) = \overline{\Phi}_i \Phi_i.} \tag{IV.5.5}$$

The reason why we did not consider $i = 0$ or $j = 0$ is because two Kähler potentials defined upon a chiral superfields as

$$K'(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) = K(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) + \Lambda(\mathbf{\Phi}) + \overline{\Lambda}(\overline{\mathbf{\Phi}}) \tag{IV.5.6}$$

would not contribute to the action after taking integrals, and thus gives no contribution to the equation of motions:

$$\int \mathrm{d}^4x\ \mathrm{d}^4\theta\ K'(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) = \int \mathrm{d}^4x\ \mathrm{d}^4\theta\ K(\mathbf{\Phi}, \overline{\mathbf{\Phi}}). \tag{IV.5.7}$$

Next we consider the superpotential, which contains information about non-derivative scalar interactions and Yukawa interactions. As noted, $W(\mathbf{\Phi})$ is a chiral superfield built upon component chiral superfields Φ_i. The most general superpotential satisfying those requirements is thus

$$W(\mathbf{\Phi}) = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} \Phi_{i_1}\Phi_{i_2}\ldots\Phi_{i_n}, \tag{IV.5.8}$$

where the first sum over n corresponds to the order of power of chiral superfields, and the sum over $i's$ runs through the different chiral superfields that are considered. For renormalizable theory $[W] = 3$, thus the sum over n must be cut at $n = 3$

$$\boxed{W(\mathbf{\Phi}) = g_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k.} \tag{IV.5.9}$$

The general Lagrangian describing chiral sector is then

$$\boxed{\mathcal{L}_{chiral} = \int \mathrm{d}^4\theta\ \overline{\Phi}_i\Phi_i + \left[\int \mathrm{d}^2\theta \left(g_i\Phi_i + \frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k\right) + h.c\right],} \tag{IV.5.10}$$

where the different terms have the following interpretations:

- $\mathcal{L}_{kin} \equiv \int \mathrm{d}^4\theta\ \overline{\Phi}_i\Phi_i$ is the kinetic energy term.
- $\mathcal{L}_{self} \equiv \int \mathrm{d}^2\theta\ g_i\Phi_i$ is the self-energy term.
- $\mathcal{L}_{mass} \equiv \frac{1}{2}\int \mathrm{d}^2\theta\ m_{ij}\Phi_i\Phi_j$ is the mass term.
- $\mathcal{L}_{int} \equiv \mathrm{d}^2\theta\ \lambda_{ijk}\Phi_i\Phi_j\Phi_k$ is the interaction term (including both non-derivative scalar inteaction and Yukawa interaction.)

Now let us expand the Lagrangian in terms of component fields. Starting with the $(\theta\theta)(\overline{\theta\theta})$ terms of Kähler potential

$$\begin{aligned}\overline{\Phi}_i\Phi_i\big|_{\theta^2\overline{\theta}^2} = &-\frac{1}{4}(\theta\theta)(\overline{\theta\theta})\phi_i\Box\overline{\phi}_i + i(\overline{\theta\theta})\left(\theta\sigma^\mu\partial_\mu\overline{\psi}_i\right)(\theta\psi) + (\theta\theta)(\overline{\theta\theta})\overline{F}_iF_i + (\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})(\partial_\mu\overline{\phi}_i)(\partial_\nu\phi_i)\\ &- i(\overline{\theta\psi}_i)(\theta\theta)(\partial_\mu\psi_i\sigma^\mu\overline{\theta}) - \frac{1}{4}(\theta\theta)(\overline{\theta\theta})\overline{\phi}_i\Box\phi_i.\end{aligned} \tag{IV.5.11}$$

Performing some calculations to bring every terms into the form of mutual $(\theta\theta)(\overline{\theta\theta})$:

$$i(\overline{\theta\theta})\left(\theta\sigma^\mu\partial_\mu\overline{\psi}_i\right)(\theta\psi) = -\frac{i}{2}(\theta\theta)(\overline{\theta\theta})\left(\psi_i\sigma^\mu\partial_\mu\overline{\psi}_i\right), \tag{IV.5.12}$$

$$\left(\theta\sigma^\mu\overline{\theta}\right)\left(\theta\sigma^\nu\overline{\theta}\right)(\partial_\mu\overline{\phi_i})(\partial_\nu\phi_i) = \frac{1}{2}(\theta\theta)(\overline{\theta\theta})g^{\mu\nu}(\partial_\mu\overline{\phi}_i)(\partial_\nu\phi_i) = \frac{1}{2}(\theta\theta)(\overline{\theta\theta})(\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i), \tag{IV.5.13}$$

$$-i(\overline{\theta\psi}_i)(\theta\theta)\left(\partial_\mu\psi_i\sigma^\mu\overline{\theta}\right) = -i(\theta\theta)(\overline{\theta\psi}_i)\left(\partial_\mu\psi_i\sigma^\mu\overline{\theta}\right) = \frac{i}{2}(\theta\theta)(\overline{\theta\theta})\left(\partial_\mu\psi_i\sigma^\mu\overline{\psi}_i\right), \tag{IV.5.14}$$

$$\phi_i\Box\overline{\phi}_i + \overline{\phi}_i\Box\phi_i = \Box(\overline{\phi}_i\phi_i) - 2(\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i). \tag{IV.5.15}$$

Putting all terms together, we get

$$\begin{aligned}\int \mathrm{d}^4\theta\ \overline{\Phi}_i\Phi_i &= (\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i) + \frac{i}{2}\left(\partial_\mu\psi_i\sigma^\mu\overline{\psi}_i - \psi_i\sigma^\mu\partial_\mu\overline{\psi}_i\right) + \overline{F}_iF_i - \frac{1}{4}\Box(\overline{\phi}_i\phi_i)\\ &= (\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i) + \frac{i}{2}\left(\partial_\mu\psi_i\sigma^\mu\overline{\psi}_i - \psi_i\sigma^\mu\partial_\mu\overline{\psi}_i\right) + \overline{F}_iF_i + \text{total der.}\end{aligned} \tag{IV.5.16}$$

The $(\theta\theta)-$terms of Φ, $\Phi\Phi$ and $\Phi\Phi\Phi$ is fairly easy, and we list the final results below

$$\int \mathrm{d}^2\theta\ \Phi_i = F_i(y), \tag{IV.5.17}$$

$$\int \mathrm{d}^2\theta\ \Phi_i\Phi_j = \phi_i(y)F_j(y) + \phi_j(y)F_i(y) - \psi_i(y)\psi_j(y), \tag{IV.5.18}$$

$$\begin{aligned}\int \mathrm{d}^2\theta\ \Phi_i\Phi_j\Phi_k =& \phi_i(y)\phi_j(y)F_k(y) + \phi_i(y)F_j(y)\phi_k(y) + F_i(y)\phi_j(y)\phi_k(y)\\ &- \psi_i(y)\psi_k(y)\phi_j(y) - \psi_i(y)\psi_j(y)\phi_k(y) - \psi_j(y)\psi_k(y)\phi_i(y).\end{aligned} \tag{IV.5.19}$$

Note further that m_{ij} and λ_{ijk} are symmetric under arbitrary permutation of its indices, and that we can use the variable x and y interchangeably, we can rewrite the off-shell matter Lagrangian (modulo total spacetime derivatives) as

$$\boxed{\begin{aligned}\mathcal{L}_{chiral} =& (\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i) + \frac{i}{2}\left(\partial_\mu\psi_i\sigma^\mu\overline{\psi}_i - \psi_i\sigma^\mu\partial_\mu\overline{\psi}_i\right) + \overline{F}_iF_i\\ &+ \left\{g_iF_i(x) + m_{ij}\left[\phi_i(x)F_j - \frac{1}{2}\psi(x)\psi_j(x)\right] + \lambda_{ijk}\left[\phi_i(x)\phi_j(x)F_k(x) - \psi_i(x)\psi_j(x)\phi_k(x)\right] + h.c\right\}.\end{aligned}} \tag{IV.5.20}$$

On-shell Free Lagrangian

As mentioned, this off-shell Lagrangian contains the auxiliary field $F_i(x)$ whose derivatives do not enter the action. Eliminating this field requires the usage of equation of motions:

$$\left\{\begin{aligned}\frac{\partial\mathcal{L}_{chiral}}{\partial F_i(x)} - \partial_\mu\frac{\partial\mathcal{L}_{chiral}}{\partial(\partial_\mu F_i(x))} = 0\\ \frac{\partial\mathcal{L}_{chiral}}{\partial\overline{F}_i} - \partial_\mu\frac{\partial\mathcal{L}_{chiral}}{\partial(\partial_\mu\overline{F}_i)} = 0\end{aligned}\right. \Rightarrow \left\{\begin{aligned}\overline{F}_i(x) + g_i + m_{ij}\phi_j(x) + \lambda_{ijk}\phi_j(x)\phi_k(x) = 0\\ F_i(x) + g_i^* + m_{ij}^*\overline{\phi}_j(x) + \lambda_{ijk}^*\overline{\phi}_j(x)\phi_k(x) = 0\end{aligned}\right., \tag{IV.5.21}$$

which gives rise to the scalar potential

$$V\left(\boldsymbol{\phi},\overline{\boldsymbol{\phi}}\right) \equiv \overline{F}_iF_i = \left(g_i + m_{ij}\phi_j + \lambda_{ijk}\phi_j\phi_k\right)\left(g_i^* + m_{ir}^*\overline{\phi}_r + \lambda_{irs}^*\overline{\phi}_r\phi_s\right). \tag{IV.5.22}$$

This scalar potential is indeed a function of scalar fields ϕ_i of power from 0 up to 4.

It is sometimes convenient to write the expansion of superpotential $W(\boldsymbol{\Phi})$ around $\Phi_i = \phi_i$. Since $\Phi_i - \phi_i = -\sqrt{2}\theta\psi - \theta\theta F$ contains at least one θ in each of its terms, the Taylor expansion around ϕ contains nonzero terms up to $|\boldsymbol{\Phi} - \boldsymbol{\phi}|^2$. [1]

$$W(\boldsymbol{\Phi}) = W(\boldsymbol{\phi}) + \frac{\partial W}{\partial\phi_i}(\Phi_i - \phi_i) + \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(\Phi_i - \phi_i)(\Phi_j - \phi_j)$$

[1] We adopt the notation for partial derivatives $\frac{\partial W}{\partial\phi_i} \equiv \left.\frac{\partial W}{\partial\Phi_i}\right|_{\boldsymbol{\Phi}=\phi}$. Similarly for second order derivatives $\frac{\partial W}{\partial\phi_i\phi_j} \equiv \left.\frac{\partial W}{\partial\Phi_i\Phi_j}\right|_{\boldsymbol{\Phi}=\phi}$.

$$
\begin{aligned}
&= W(\boldsymbol{\phi}) + \left(\sqrt{2}\theta\psi_i + \theta\theta F_i\right)\frac{\partial W}{\partial \phi_i} + \frac{1}{2}\left(\sqrt{2}\theta\psi_i + \theta\theta F_i\right)\left(\sqrt{2}\theta\psi_i + \theta\theta F_i\right)\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} \\
&= W(\boldsymbol{\phi}) + \sqrt{2}(\theta\psi_i)\frac{\partial W}{\partial \phi_i} + (\theta\theta)\left(F_i\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right).
\end{aligned} \tag{IV.5.23}
$$

The Lagrangian for chiral sector is thus

$$
\mathcal{L}_{chiral} = (\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i) + \frac{i}{2}\left(\partial_\mu\psi_i\sigma^\mu\overline{\psi}_i - \psi_i\sigma^\mu\partial_\mu\overline{\psi}_i\right) + \overline{F}_iF_i + \left(F_i\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\psi_i\psi_j\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} + h.c\right). \tag{IV.5.24}
$$

The Euler-Lagrange equation for F_i now reads

$$
\overline{F}_i = -\frac{\partial W}{\partial\phi^i}, \quad F_i = -\frac{\partial\overline{W}}{\partial\overline{\phi}_i}, \tag{IV.5.25}
$$

which gives rise to the on-shell Lagrangian

$$
\boxed{
\begin{aligned}
\mathcal{L}_{matter} =& (\partial_\mu\overline{\phi}_i)(\partial^\mu\phi_i) + \frac{i}{2}\left(\partial_\mu\psi_i\sigma^\mu\overline{\psi}_i - \psi_i\sigma^\mu\partial_\mu\overline{\psi}_i\right) \\
&- \sum_i\left|\frac{\partial W}{\partial\phi_i}\right|^2 - \frac{1}{2}(\psi_i\psi_j)\frac{\partial^2 W}{\partial\phi_i\partial\phi_j} - \frac{1}{2}(\overline{\psi}_i\overline{\psi}_j)\frac{\partial^2 \overline{W}}{\partial\overline{\phi}_i\partial\overline{\phi}_j}.
\end{aligned}} \tag{IV.5.26}
$$

The scalar potential is thus

$$
V(\boldsymbol{\phi},\overline{\boldsymbol{\phi}}) \equiv \overline{F}_iF_i = \sum_i\left|\frac{\partial W}{\partial\phi_i}\right|^2. \tag{IV.5.27}
$$

IV.5.2 SuperYang-Mills Theory

Abelian Supersymmetric Gauge Theory

To find an expression for the field strength tensor, we first introduce

$$
\mathcal{W}_\alpha \equiv -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}_\alpha V, \quad \overline{\mathcal{W}}_{\dot\alpha} \equiv -\frac{1}{4}\mathcal{D}\mathcal{D}\overline{\mathcal{D}}_{\dot\alpha}V, \tag{IV.5.28}
$$

where $\mathcal{W}_\alpha$ and $\overline{\mathcal{W}}_{\dot\alpha}$ are respectively a chiral and antichiral superfield, and are supersymmetric gauge invariant. The chirality (or antichirality) comes from the fact that $\overline{\mathcal{D}}^3 = \mathcal{D}^3 = 0$ identically (since either $\mathcal{D}$ or $\overline{\mathcal{D}}$ contains one Grassmann variable). The gauge invariance is proceeded as follows

$$
\begin{aligned}
\mathcal{W}_\alpha \to \mathcal{W}_\alpha &- \frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}_\alpha(\Phi+\overline{\Phi}) = \mathcal{W}_\alpha + \frac{1}{4}\overline{\mathcal{D}}^{\dot\beta}\overline{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha\Phi \\
&= \mathcal{W}_\alpha + \frac{1}{4}\overline{\mathcal{D}}^{\dot\beta}\left\{\overline{D}_{\dot\beta},\mathcal{D}_\alpha\right\} = \mathcal{W}_\alpha + \frac{1}{4}\overline{\mathcal{D}}^{\dot\beta}2i\sigma^\mu_{\ \alpha\dot\beta}\partial_\mu\Phi = \mathcal{W}_\alpha + \frac{i}{2}\sigma^\mu_{\ \alpha\dot\beta}\partial_\mu\overline{\mathcal{D}}^{\dot\beta}\Phi = \mathcal{W}_\alpha.
\end{aligned} \tag{IV.5.29}
$$

The proof for gauge invariant property of $\overline{\mathcal{W}}_{\dot\alpha}$ is totally similar. Thus we can work with these quantities without bothering the gauge we are using. Let us find the expression for $\mathcal{W}_\alpha$ and $\overline{\mathcal{W}}_{\dot\alpha}$ by plugging in the vector field in WZ-gauge, note that we are working with the coordinates $(y,\theta,\overline{\theta})$

$$
\begin{aligned}
\mathcal{D}_\alpha V_{WZ} &= \left(\partial_\alpha + 2i\sigma^\mu_{\ \alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\right)\left[(\theta\sigma^\mu\overline{\theta})V_\mu + i(\theta\theta)\overline{\theta}\overline{\lambda} + \frac{1}{2}(\theta\theta)(\overline{\theta}\overline{\theta})\left(D - i\partial_\mu V^\mu\right)\right] \\
&= \sigma^\mu_{\ \alpha\dot\beta}\overline{\theta}^{\dot\beta}V_\mu + i\theta_\alpha\overline{\theta}\overline{\lambda} - i\overline{\theta}\overline{\theta}\lambda_\alpha + \theta_\alpha\overline{\theta}\overline{\theta}\left(D - i\partial_\mu V^\mu\right) + 2i\sigma^\mu_{\ \alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\left(\theta\sigma^\nu\overline{\theta}V_\nu + i\theta\theta\overline{\theta}\overline{\lambda}\right).
\end{aligned} \tag{IV.5.30}
$$

We rewrite some terms to group the θ and $\overline{\theta}$

$$
\begin{aligned}
&2i\xi^\alpha\sigma^\mu_{\ \alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu(\theta\sigma^\nu\overline{\theta}V_\nu) = -2i(\xi\sigma^\mu\overline{\theta})(\overline{\theta}\overline{\sigma}^\nu\theta)\partial_\mu V_\nu = i(\overline{\theta}\overline{\theta})(\xi\sigma^\mu\overline{\sigma}^\nu\theta) \\
&\Rightarrow 2i\sigma^\mu_{\ \alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu(\theta\sigma^\nu\overline{\theta}V_\nu) = i(\overline{\theta}\overline{\theta})\left(\sigma^\mu\overline{\sigma}^\nu\theta\right)_\alpha\partial_\mu V_\nu,
\end{aligned} \tag{IV.5.31}
$$

$$i(\overline{\theta\theta})(\sigma^\mu\overline{\sigma}^\nu\theta)_\alpha\partial_\mu V_\nu - i(\overline{\theta\theta})\theta_\alpha\partial_\mu V^\mu = i(\overline{\theta\theta})(\partial_\mu V_\nu)\left(\sigma^\mu\overline{\sigma}^\nu\theta - \eta^{\mu\nu}\theta\right)_\alpha = 2(\overline{\theta\theta})(\partial_\mu V_\nu)(\sigma^{\mu\nu}\theta)_\alpha{}^{1}$$
$$= (\overline{\theta\theta})(\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu V_\nu - \partial_\nu V_\mu) = (\overline{\theta\theta})(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}, \tag{IV.5.32}$$

$$2i\xi^\alpha\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\left(i\theta\theta\overline{\theta\lambda}\right) = -2(\theta\theta)(\xi\sigma^\mu\overline{\theta})(\overline{\theta}\partial_\mu\overline{\lambda}) = (\theta\theta)(\overline{\theta\theta})(\xi\sigma^\mu\partial_\mu\overline{\lambda})$$
$$\Rightarrow 2i\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\left(i\theta\theta\overline{\theta\lambda}\right) = (\theta\theta)(\overline{\theta\theta})\sigma^\mu_{\alpha\dot\beta}\partial_\mu\overline{\lambda}^{\dot\beta}. \tag{IV.5.33}$$

Hence acting Grassmannian derivative $\mathcal{D}$ on WZ vector superfield gives

$$\mathcal{D}_\alpha V_{WZ} = \sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}V_\mu + 2i\theta_\alpha\overline{\theta\lambda} - i\overline{\theta\theta}\lambda_\alpha + \theta_\alpha\overline{\theta\theta}D + (\sigma^{\mu\nu}\theta)_\alpha(\overline{\theta\theta})F_{\mu\nu} + \theta\theta\overline{\theta\theta}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\overline{\lambda}^{\dot\beta}. \tag{IV.5.34}$$

Next, acting $\overline{\mathcal{D}\mathcal{D}} = \overline{\mathcal{D}}_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha}$ (where $\overline{\mathcal{D}}_{\dot\alpha} = -\overline{\partial}_{\dot\alpha}$) on the above expression gives us the field strength $\mathcal{W}_\alpha$

$$\mathcal{W}_\alpha = -\frac{1}{4}(\overline{\partial}_{\dot\beta}\overline{\partial}^{\dot\beta})\mathcal{D}_\alpha V_{WZ} = +\frac{1}{4}\overline{\partial}_{\dot\beta}\left[-2i\overline{\theta}^{\dot\beta}\lambda_\alpha + 2\theta_\alpha\overline{\theta}^{\dot\beta}D + 2(\sigma^{\mu\nu}\theta)_\alpha\overline{\theta}^{\dot\beta}F_{\mu\nu} + 2\theta\theta\overline{\theta}^{\dot\beta}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\overline{\lambda}^{\dot\beta}\right]$$
$$= \frac{1}{2}\delta^{\dot\beta}_{\dot\beta}\left[-i\lambda_\alpha + \theta_\alpha D + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda})_\alpha\right]$$
$$\boxed{= -i\lambda_\alpha + \theta_\alpha D + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda})_\alpha.} \tag{IV.5.35}$$

As we see, the so-called supersymmetric field strength $\mathcal{W}_\alpha$ is a chiral superfield whose lowest component is not a scalar field as before, but rather a Weyl spinor field λ_α. The usual field strength tensor $F_{\mu\nu}$ shows up as a component of $\mathcal{W}_\alpha$, and thus λ_α is the superpartner of the gauge field: the gaugino. For this reason, W_α is also known as the gaugino superfield. We next calculate the contribution of the chiral field $\mathcal{W}_\alpha$ into the Abelian gauge Lagrangian by taking the θ^2-component of $\mathcal{W}^2$

$$\mathcal{W}^\alpha\mathcal{W}_\alpha|_{\theta^2} = -i\theta\theta\left(\lambda\sigma^\mu\partial_\mu\overline{\lambda}\right) + \theta\theta D^2 + \theta^\alpha(\sigma^{\mu\nu}\theta)_\alpha DF_{\mu\nu} + (\sigma^{\mu\nu}\theta)^\alpha\theta_\alpha DF_{\mu\nu}$$
$$+ (\sigma^{\mu\nu}\theta)^\alpha(\sigma^{\rho\sigma}\theta)_\alpha F_{\mu\nu}F_{\rho\sigma} + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda})^\alpha(-i\lambda_\alpha).$$

$$= -2i\theta\theta\left(\lambda\sigma^\mu\partial_\mu\overline{\lambda}\right) + \theta\theta D^2 + 2(\theta\sigma^{\mu\nu}\theta)DF_{\mu\nu} + (\sigma^{\mu\nu}\theta)^\alpha(\sigma^{\rho\sigma}\theta)_\alpha F_{\mu\nu}F_{\rho\sigma}. \tag{IV.5.36}$$

Applying some chiral spinor identities to simplify the above expression

$$\theta\sigma^{\mu\nu}\theta = \frac{i}{4}\left(\theta\sigma^\mu\overline{\sigma}^\nu\theta - \theta\sigma^\nu\overline{\sigma}^\mu\theta\right) = \frac{i}{4}\left(\theta\sigma^\mu\overline{\sigma}^\nu\theta - \theta\sigma^\mu\overline{\sigma}^\nu\theta\right) = 0. \tag{IV.5.37}$$

$$(\sigma^{\mu\nu}\theta)^\alpha(\sigma^{\rho\sigma}\theta)_\alpha F_{\mu\nu}F_{\rho\sigma} = \varepsilon^{\alpha\beta}(\sigma^{\mu\nu})_\beta^{\ \gamma}\theta_\gamma(\sigma^{\rho\sigma})_\alpha^{\ \delta}\theta_\delta F_{\mu\nu}F_{\rho\sigma} = \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}\theta\theta(\sigma^{\mu\nu})_\beta^{\ \gamma}(\sigma^{\rho\sigma})_\alpha^{\ \delta}F_{\mu\nu}F_{\rho\sigma}$$
$$= \frac{1}{2}\left(-\delta^\alpha_\gamma\delta^\beta_\delta + \delta^\alpha_\delta\delta^\beta_\gamma\right)(\theta\theta)(\sigma^{\mu\nu})_\beta^{\ \gamma}(\sigma^{\rho\sigma})_\alpha^{\ \delta}F_{\mu\nu}F_{\rho\sigma}$$
$$= \frac{1}{2}\theta\theta\left[-(\sigma^{\mu\nu})_\delta^{\ \alpha}(\sigma^{\rho\sigma})_\alpha^{\ \delta} + (\sigma^{\mu\nu})_\beta^{\ \beta}(\sigma^{\rho\sigma})_\alpha^{\ \alpha}\right]F_{\mu\nu}F_{\rho\sigma}$$
$$= -\frac{1}{2}\operatorname{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma})F_{\mu\nu}F_{\rho\sigma} = -\frac{1}{4}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} - i\varepsilon^{\mu\nu\rho\sigma}\right)F_{\mu\nu}F_{\rho\sigma}$$
$$= -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{IV.5.38}$$

Thus the half-space integral is equal to

$$\int \mathrm{d}^2\theta\ \mathcal{W}^\alpha\mathcal{W}_\alpha = -2i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{IV.5.39}$$

The real Lagrangian can be obtained by adding the above integral with its Hermitian conjugate, thus gives

$$\boxed{\mathcal{L}_{gauge} = \int \mathrm{d}^2\theta\ \mathcal{W}^\alpha\mathcal{W}_\alpha + \int \mathrm{d}^2\overline{\theta}\ \overline{\mathcal{W}}_{\dot\alpha}\overline{\mathcal{W}}^{\dot\alpha} = -F_{\mu\nu}F^{\mu\nu} - 4i\lambda\sigma^\mu\partial_\mu\overline{\lambda} + 2D^2.} \tag{IV.5.40}$$

[1] Here we use the identity $\sigma^\mu\overline{\sigma}^\nu - \eta^{\mu\nu} = -2i\sigma^{\mu\nu}$. The proof is straightforward with the use of anticommutation relation $\{\sigma^\mu, \overline{\sigma}^\nu\} = 2\eta^{\mu\nu}$, and the definition $\sigma^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$.

Non-Abelian Supersymmetric Gauge Theory

Now consider the non-abelian case, we can generalize the vector and chiral superfields as

$$V = V^a T^a, \quad \Lambda = \Lambda^a T^a, \quad \mathcal{W}_\alpha = \mathcal{W}_\alpha^a T^a, \tag{IV.5.41}$$

where the generators of gauge transformation is defined as usual

$$\left[T^a, T^b\right] = i f^{abc} T^c, \quad \mathrm{Tr}\left(T^a T^b\right) = C\delta^{ab}. \tag{IV.5.42}$$

The gauge transformation is generalized as

$$\Phi \to e^{-i\Lambda}\Phi, \quad e^V \to e^{-i\overline{\Lambda}} e^V e^{i\Lambda}. \tag{IV.5.43}$$

Using the above transformation we can recover the abelian case:$e^V \to e^{-i\overline{\Lambda}} e^V e^{i\Lambda} \Rightarrow V \to V + i(\Lambda - \overline{\Lambda})$ in Eq. (IV.4.40). Such transformation make the supersymmetric field strength defined in Eq. (IV.5.28) is no longer invariant, motivating a new definition for gauge superfields

$$\mathcal{W}_\alpha \equiv -\frac{1}{4}(\overline{\mathcal{D}}\overline{\mathcal{D}})\left(e^{-V}\mathcal{D}_\alpha e^V\right), \quad \overline{\mathcal{W}}_{\dot\alpha} \equiv -\frac{1}{4}(\mathcal{D}\mathcal{D})\left(e^{V}\overline{\mathcal{D}}_{\dot\alpha} e^{-V}\right). \tag{IV.5.44}$$

Let us show the gauge invariance of this new gauge field strength

$$\begin{aligned}
\mathcal{W}_\alpha \to & -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\left[e^{-i\Lambda}e^{-V}e^{i\overline{\Lambda}}\mathcal{D}_\alpha\left(e^{-i\overline{\Lambda}}e^V e^{i\Lambda}\right)\right] \\
&= -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\left\{e^{-i\Lambda}e^{-V}\left[(\mathcal{D}_\alpha e^V)e^{i\Lambda} + e^V\mathcal{D}e^{i\Lambda}\right]\right\} \\
&= -\frac{1}{4}e^{-i\Lambda}\overline{\mathcal{D}}\overline{\mathcal{D}}\left(e^{-V}\mathcal{D}_\alpha e^V\right)e^{i\Lambda} = e^{-i\Lambda}\mathcal{W}_\alpha e^{i\Lambda},
\end{aligned} \tag{IV.5.45}$$

where we used the fact that Λ and $\overline{\Lambda}$ are chiral and antichiral superfields respectively, thus $\overline{\mathcal{D}}e^{-i\Lambda} = \mathcal{D}e^{i\overline{\Lambda}} = 0$, and that $\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}_\alpha e^{-i\Lambda} = 0$ (by applying the commutation relation of $\mathcal{D}$ and $\overline{\mathcal{D}}$). Similarly for antichiral superfield $\overline{\mathcal{W}}_{\dot\alpha}$

$$\overline{\mathcal{W}}_{\dot\alpha} \to e^{-i\overline{\Lambda}}\overline{\mathcal{W}}_{\dot\alpha}e^{i\overline{\Lambda}}. \tag{IV.5.46}$$

The final result is that $\mathcal{W}_\alpha$ and $\overline{W}_{\dot\alpha}$ transform covariantly under gauge transformation. Using the WZ gauge, the gauge field strength can be rewritten as

$$\begin{aligned}
\mathcal{W}_\alpha &= -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\left[\left(1 + V + \frac{V^2}{2}\right)\mathcal{D}_\alpha\left(1 + V + \frac{V^2}{2}\right)\right] \\
&= -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\left(\mathcal{D}_\alpha V + \frac{1}{2}\mathcal{D}_\alpha V^2 - V\mathcal{D}_\alpha V\right) = -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}_\alpha V - \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}V\mathcal{D}_\alpha V - \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}(\mathcal{D}_\alpha V)V + \frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}V\mathcal{D}_\alpha V \\
&= -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}_\alpha V + \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}V\mathcal{D}_\alpha V - \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}(\mathcal{D}V)V = -\frac{1}{4}\overline{\mathcal{D}}\overline{\mathcal{D}}\mathcal{D}_\alpha V + \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\left[V, \mathcal{D}_\alpha V\right].
\end{aligned} \tag{IV.5.47}$$

The first term is the same as in abelian case that has been represented in Eq. (IV.5.28). The second term can be expressed as

$$\begin{aligned}
\frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\left[V, \mathcal{D}_\alpha V\right] &= \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\left[\theta\sigma^\mu\overline{\theta}V_\mu + i\theta\theta\overline{\theta}\overline{\lambda} + \ldots, \sigma^\nu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}V_\nu + \ldots\right] \\
&= \frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}(\theta\sigma^\mu\overline{\theta})(\sigma^\nu\overline{\theta})_\alpha\left[V_\mu, V_\nu\right] + \frac{i}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\theta\theta\left[\overline{\theta}_{\dot\gamma}\overline{\lambda}^{\dot\gamma}, \sigma^\nu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}V_\nu\right].
\end{aligned} \tag{IV.5.48}$$

The two terms in Eq. (IV.5.48) can be simplified using spinor identities

$$\frac{1}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}(\theta\sigma^\mu\overline{\theta})(\sigma^\nu\overline{\theta})_\alpha\left[V_\mu, V_\nu\right] = -\frac{1}{4}(\sigma^\nu\overline{\sigma}^\mu\theta)_\alpha\left[V_\mu, V_\nu\right] = -\frac{i}{2}\left(\sigma^{\mu\nu}\theta\right)_\alpha\left[V_\mu, V_\nu\right], \tag{IV.5.49}$$

$$\begin{aligned}
\frac{i}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\theta\theta\left[\overline{\theta}_{\dot\gamma}\overline{\lambda}^{\dot\gamma}, \sigma^\nu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}V_\nu\right] &= -\frac{i}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\theta\theta\overline{\theta}_{\dot\gamma}\overline{\theta}^{\dot\beta}\sigma^\mu_{\alpha\dot\beta}\left[\overline{\lambda}^{\dot\gamma}, V_\mu\right] = \frac{i}{8}\varepsilon_{\dot\gamma\dot\delta}\varepsilon^{\dot\beta\dot\delta}\sigma^\mu_{\alpha\dot\beta}\left[\dot\lambda^{\dot\gamma}, V_\mu\right] \\
&= -\frac{i}{8}\overline{\mathcal{D}}\overline{\mathcal{D}}\theta\theta\overline{\theta}\overline{\theta}\sigma^\mu_{\alpha\dot\beta}\left[\overline{\lambda}^{\dot\beta}, V_\mu\right] = -\frac{i}{2}\theta\theta\left[V_\mu, \left(\sigma^\mu\overline{\lambda}\right)_\alpha\right].
\end{aligned} \tag{IV.5.50}$$

Adding all terms gives $\mathcal{W}_\alpha$, which is the same as replacing the ordinary derivatives and the field strength in the abelian gauge field strength with the covariant ones

$$\begin{aligned}\mathcal{W}_\alpha &= -i\lambda_\alpha + \theta_\alpha D + (\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu V_\nu - \partial_\nu V_\mu) + \theta\theta(\sigma^\mu\partial_\mu\overline{\lambda})_\alpha - \frac{i}{2}\theta\theta\left[V_\mu, (\sigma^\mu\overline{\lambda})_\alpha\right] - \frac{i}{2}(\sigma^{\mu\nu}\theta)_\alpha\left[V_\mu, V_\nu\right] \\ &= -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha\left(\partial_\mu V_\nu - \partial_\nu V_\mu - \frac{i}{2}\left[V_\mu, V_\nu\right]\right) + \theta\theta\left(\sigma^\mu\partial_\mu\overline{\lambda} - \frac{i}{2}\left[V_\mu, \sigma^\mu\overline{\lambda}\right]\right)_\alpha \\ &= -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta\left(\sigma^\mu D_\mu\overline{\lambda}\right)_\alpha, \end{aligned} \tag{IV.5.51}$$

where the non-abelian version of field strength tensor and gauge covariant derivatives are

$$F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu - \frac{i}{2}\left[V_\mu, V_\nu\right], \quad D_\mu \equiv \partial_\mu - \frac{i}{2}\left[V_\mu, \star\right]. \tag{IV.5.52}$$

The contribution to Super Yang-Mills Lagrangian of the non-abelian gauge is the same as abelian case by substituting the new definition $F_{\mu\nu}$ and replace the partial derivative by covariant derivative D_μ into Eq. (IV.5.39). To introduce the coupling constant g explicitly in the coupling with matter we are going to derive, it is convenient to make the redefinition of the vector superfields

$$V \to 2gV \Leftrightarrow \begin{cases} V_\mu \to 2gV_\mu \\ \lambda \to 2g\lambda \\ D \to 2gD \end{cases}, \tag{IV.5.53}$$

which results in the change of gaugino superfield $\mathcal{W}_\alpha \to 2g\mathcal{W}_\alpha$, with the new definition of $F_{\mu\nu}$ and D_μ as follows

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ig\left[V_\mu, V_\nu\right], \quad D_\mu = \partial_\mu - ig\left[V_\mu, \star\right]. \tag{IV.5.54}$$

If we want to keep the CP-violation term $\sim \varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, the half-space integral of $\mathcal{W}^\alpha\mathcal{W}_\alpha$ should be multiplied by a complex number so that the complex part corresponding to the coefficient of the dual field strength term survives, as we shall see in the following expression of Super Yang-Mills Lagrangian:

$$\boxed{\begin{aligned}\mathcal{L}_{SYM} &= \mathrm{Tr}\left\{\tau\int \mathrm{d}^2\theta\ \mathcal{W}^\alpha\mathcal{W}_\alpha + h.c\right\} = 4g^2\tau\,\mathrm{Tr}\left\{-2i\lambda\sigma^\mu D_\mu\overline{\lambda} + D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + h.c\right\} \\ &= 4g^2\,\mathrm{Tr}\left\{\mathrm{Re}\tau\left(-4i\lambda\sigma^\mu D_\mu\overline{\lambda} + 2D^2 - F_{\mu\nu}F^{\mu\nu}\right) - \mathrm{Im}\tau\left(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right)\right\} \\ &= -i\lambda^a\sigma^\mu D_\mu\overline{\lambda}^a + \frac{1}{2}D^aD^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{\Theta}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}g^2F^a_{\mu\nu}F^a_{\rho\sigma},\end{aligned}} \tag{IV.5.55}$$

with the constant τ defined as

$$\tau \equiv \frac{1}{16g^2C} - i\frac{\Theta}{64\pi^2}, \tag{IV.5.56}$$

with the complex part containing the CP-violation parameter Θ. Hereafter, especially when constructing the MSSM Lagrangian, we will use the standard normalization of $SU(N)$ generators where $C = 1/2$, that is

$$\begin{aligned}\mathrm{Tr}(T^aT^b) &= \frac{\delta^{ab}}{2} \\ \Rightarrow \tau &= \frac{1}{8g^2} - i\frac{\Theta}{64\pi^2},\end{aligned} \tag{IV.5.57}$$

One more thing to note about the difference between abelian and non-abelian gauge theory is that the D-term in the vector superfield also has the variation a total derivative, and thus this term can arises in the general supersymmetric Lagrangian:

$$\boxed{\mathcal{L}_{FI} = \int \mathrm{d}^4\theta\ \xi 2gV = g\xi D.} \tag{IV.5.58}$$

This so-called Fayet-Iliopoulos term is also $U(1)-$gauge invariant as one can easily show by noting that integration of chiral superfields over $d^4\theta$ vanishes. Since taking the trace of generators of $SU(N)$ yields 0, only in the $U(1)$ case does this term survives.

IV.5.3 Gauge-Matter Action

To couple the gauge fields and the matter fields, it is necessary to investigate the transformation of a chiral superfield Φ. Let us assume that Φ transforms under the representation R of the gauge group G, $T^a \to (T^a_R)^i_j$ where $i, j = 1, 2, \ldots, R$. The chiral fields mix under the gauge transformation defined in Eq. (IV.5.43). To couple the radiation with matter in a supersymmetric manner, we include the vector field (which transform covariantly under the gauge transformation) into the Kähler potential as

$$\overline{\Phi}\Phi \to \overline{\Phi}e^V\Phi = \overline{\Phi}\Phi + \overline{\Phi}V\Phi + \frac{1}{2}\overline{\Phi}V^2\Phi. \tag{IV.5.59}$$

The first term is the free Kähler Lagrangian we have calculated. Recall the definition of $\Phi(x, \theta, \overline{\theta})$ and $V_{WZ}(x, \theta, \overline{\theta})$ to proceed the calculations on the next two terms

$$\begin{cases} \Phi(x,\theta,\overline{\theta}) = \left[\phi(x) + \partial_\mu\phi(x)(i\theta\sigma^\mu\overline{\theta}) - \frac{1}{2}(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta})\partial_\mu\partial_\nu\phi(x)\right] \\ \qquad + \left[\sqrt{2}\theta\psi(x) + \sqrt{2}\theta\partial_\mu\psi(x)(i\theta\sigma^\mu\overline{\theta})\right] + \theta\theta F(x), \\ V_{WZ}(x) = \theta\sigma^\mu\overline{\theta}\ V_\mu(x) + i\theta\theta\ \overline{\theta}\overline{\lambda}(x) - i\overline{\theta}\overline{\theta}\ \theta\lambda(x) + \frac{1}{2}\theta\theta\ \overline{\theta}\overline{\theta}D(x) \Rightarrow \begin{cases} V^2_{WZ} = \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}V^\mu V_\mu \\ V^n_{WZ} = 0, \forall n > 2 \end{cases}. \end{cases}$$

$$\begin{aligned} \Rightarrow \overline{\Phi}V\Phi|_{\theta^2\overline{\theta}^2} &= \overline{\phi}\left(\theta\sigma^\mu\overline{\theta}V_\mu\right)\left(i\theta\sigma^\nu\overline{\theta}\right)\partial_\nu\phi - \partial_\mu\overline{\phi}\left(i\theta\sigma^\mu\overline{\theta}\right)(\theta\sigma^\nu\overline{\theta}V_\nu)\phi + 2(\overline{\theta}\overline{\psi})(\theta\sigma^\mu\overline{\theta}V_\mu)(\theta\psi) \\ &\quad - \sqrt{2}i\overline{\phi}(\overline{\theta}\overline{\theta})(\theta\lambda)(\theta\psi) + \sqrt{2}i\overline{\theta}\overline{\psi}(\theta\theta)(\overline{\theta}\overline{\lambda})\phi + \frac{1}{2}\overline{\phi}(\theta\theta)(\overline{\theta}\overline{\theta})D\phi. \end{aligned} \tag{IV.5.60}$$

With some algebra and spinor identities, the above terms can be simplified as

$$\overline{\phi}\left(\theta\sigma^\mu\overline{\theta}V_\mu\right)\left(i\theta\sigma^\nu\overline{\theta}\right)\partial_\nu\phi = \overline{\phi}V_\mu\partial_\nu\phi(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = \frac{i}{2}\theta\theta\overline{\theta}\overline{\theta}\overline{\phi}V_\mu\partial_\nu\phi g^{\mu\nu} = \frac{i}{2}\theta\theta\overline{\theta}\overline{\theta}\overline{\phi}V^\mu\partial_\mu\phi, \tag{IV.5.61}$$

$$-\partial_\mu\overline{\phi}\left(i\theta\sigma^\mu\overline{\theta}\right)(\theta\sigma^\nu\overline{\theta}V_\nu)\phi = -\frac{i}{2}\theta\theta\overline{\theta}\overline{\theta}\partial_\mu\overline{\phi}V_\nu\phi g^{\mu\nu} = -\frac{i}{2}\theta\theta\overline{\theta}\overline{\theta}\partial_\mu\overline{\phi}V^\mu\phi, \tag{IV.5.62}$$

$$2(\overline{\theta}\overline{\psi})(\theta\sigma^\mu\overline{\theta}V_\mu)(\theta\psi) = -2(\overline{\theta}\overline{\psi})(\overline{\theta}\overline{\sigma}^\mu\theta)(\theta\psi)V_\mu = \overline{\theta}\overline{\theta}(\overline{\psi}\overline{\sigma}^\mu\theta)(\psi\theta)V_\mu = -\frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}(\overline{\psi}\overline{\sigma}^\mu\psi)V_\mu, \tag{IV.5.63}$$

$$-\sqrt{2}i\overline{\phi}(\overline{\theta}\overline{\theta})(\theta\lambda)(\theta\psi) = -\frac{i}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\theta}(\lambda\psi), \tag{IV.5.64}$$

$$\sqrt{2}i\overline{\theta}\overline{\psi}(\theta\theta)(\overline{\theta}\overline{\lambda})\phi = -\frac{i}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\theta}(\overline{\psi}\overline{\lambda})\phi. \tag{IV.5.65}$$

Similarly for the quadratic term

$$\frac{1}{2}\overline{\Phi}V^2\Phi = \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\overline{\phi}V^\mu V_\mu\phi. \tag{IV.5.66}$$

Putting all terms together, and taking full-space integral of the gauge invariant Kähler potential gives

$$\begin{aligned} \int \mathrm{d}^4\theta\ \overline{\Phi}e^V\Phi &= (\partial_\mu\overline{\phi})(\partial^\mu\phi) + \frac{i}{2}\left(\partial_\mu\psi\sigma^\mu\overline{\psi} - \psi\sigma^\mu\partial_\mu\overline{\psi}\right) + \overline{F}F + \frac{i}{2}\overline{\phi}V^\mu\partial_\mu\phi - \frac{i}{2}(\partial_\mu\overline{\phi})V^\mu\phi \\ &\quad - \frac{1}{2}\overline{\psi}\overline{\sigma}^\mu V_\mu\psi + \frac{i}{\sqrt{2}}\overline{\phi}\lambda\psi - \frac{i}{\sqrt{2}}\overline{\psi}\overline{\lambda}\phi + \frac{1}{2}\overline{\phi}D\phi + \frac{1}{4}\overline{\phi}V^\mu V_\mu\phi + \text{ Total dev.} \\ &= \left[(\partial_\mu\overline{\phi})(\partial^\mu\phi) + \frac{i}{2}\overline{\phi}V^\mu\partial_\mu\phi - \frac{i}{2}(\partial_\mu\overline{\phi})V^\mu\phi + \frac{1}{4}\overline{\phi}V^\mu V_\mu\phi\right] + \left[i(\partial_\mu\psi)\sigma^\mu\overline{\psi} - \frac{1}{2}\overline{\psi}\overline{\sigma}^\mu V_\mu\psi\right] \\ &\quad + \overline{F}F + \frac{i}{\sqrt{2}}\overline{\phi}\lambda\psi - \frac{i}{\sqrt{2}}\overline{\psi}\overline{\lambda}\phi + \frac{1}{2}\overline{\phi}D\phi + \text{ Total dev.} \\ &= (\overline{D_\mu\phi})(D^\mu\phi) - i\overline{\psi}\overline{\sigma}^\mu D_\mu\psi + \overline{F}F + \frac{i}{\sqrt{2}}\overline{\phi}\lambda\psi - \frac{i}{\sqrt{2}}\overline{\psi}\overline{\lambda}\phi + \frac{1}{2}\overline{\phi}D\phi + \text{ Total dev.}, \end{aligned} \tag{IV.5.67}$$

where the covariant derivatives $D_\mu = \partial_\mu - \frac{i}{2}V_\mu^a T_R^a$. Rewriting $\overline{\psi}\overline{\sigma}^\mu D_\mu \psi = \psi\sigma^\mu D_\mu \overline{\psi}$, performing the rescaling $V \to 2gV$ and taking the sum over the participated chiral superfields, one finally gets

$$\boxed{\mathcal{L}_{\text{Kähler}} = \int \mathrm{d}^4\theta\ \overline{\Phi}_i e^{2gV}\Phi_i = (\overline{D_\mu \phi_i})(D^\mu \phi_i) - i\psi\sigma^\mu D_\mu\overline{\psi} + \overline{F}_i F_i + i\sqrt{2}g\overline{\phi}_i\lambda\psi_i - i\sqrt{2}g\overline{\psi}_i\overline{\lambda}\phi_i + g\overline{\phi}_i D\phi_i.} \tag{IV.5.68}$$

IV.5.4 Full Minimal Supersymmetric Lagrangian

In summary, we have derived all the Lagrangian terms that is SuperPoincaré invariant. The most general $N = 1$ renormalizable supersymmtric Lagrangian with $i, j = 1, \ldots, \mathcal{N}$ chiral fields, $A = 1, \ldots, \mathcal{A}$ abelian factors (shows up in the Fayet-Iliopoulos term) and $n = 1, \ldots, \mathcal{G}$ gauge fields is

$$\begin{aligned}
\mathcal{L}_{SUSY} &= \mathcal{L}_{SYM} + \mathcal{L}_{matter} + \mathcal{L}_{FI}\\
&= \mathrm{Tr}\left\{\tau\int \mathrm{d}^2\theta\ \mathcal{W}^\alpha \mathcal{W}_\alpha + h.c\right\} + \int \mathrm{d}^4\theta\ \overline{\Phi}_i e^{2gV}\Phi_i + \int \mathrm{d}^2\theta\ W(\mathbf{\Phi}) + \int \mathrm{d}^2\overline{\theta}\ \overline{W}(\overline{\mathbf{\Phi}}) + 2g\xi\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} V\\
&= -\frac{1}{4}F^a_{n\mu\nu}F^{a\mu\nu}_n + \frac{\Theta}{32\pi^2}g_n^2\varepsilon^{\mu\nu\rho\sigma}F^a_{n\mu\nu}F^a_{n\rho\sigma} - i\lambda^a_n\sigma^\mu D^{ab}_\mu\overline{\lambda}^b_n + \frac{1}{2}D^a_n D^a_n\\
&\quad + (\overline{D_\mu\phi})(D^\mu\phi) - i\psi\sigma^\mu D_\mu\overline{\psi} + \overline{F}_i F_i + i\sqrt{2}g_n\overline{\phi}_i\lambda_n\psi_i - i\sqrt{2}g_n\overline{\psi}_i\overline{\lambda}_n\phi_i + g_n\overline{\phi}_i D_n\phi_i\\
&\quad - \frac{\partial W}{\partial\phi_i}F_i - \frac{\partial\overline{W}}{\partial\overline{\phi}_i}\overline{F}_i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\psi}_i\overline{\psi}_j + g_n\xi^A D^A_n.
\end{aligned} \tag{IV.5.69}$$

As mentioned, D and F are auxiliary fields (meaning they are actually not independent fields, but are functions of other fields) and can be integrated out with by using the equation of motions

$$\overline{F}_i = \frac{\partial W}{\partial\phi_i}, \quad D^a_n = -g_n\overline{\phi}_i T^a_R\phi_i - g_n\xi^a \quad (\xi^a = \delta^{aA}\xi^A). \tag{IV.5.70}$$

Plugging these equations back to the original Lagrangian to obtain

On-shell Minimal Supersymmetry Lagrangian

$$\begin{aligned}
\mathcal{L}_{SUSY} &= \mathcal{L}_{SYM} + \mathcal{L}_{matter} + \mathcal{L}_{FI}\\
&= -\frac{1}{4}F^a_{n\mu\nu}F^{a\mu\nu}_n + \frac{\Theta}{32\pi^2}g_n^2\varepsilon^{\mu\nu\rho\sigma}F^a_{n\mu\nu}F^a_{n\rho\sigma} - i\lambda^a_n\sigma^\mu D^{ab}_\mu\overline{\lambda}^b_n\\
&\quad + (\overline{D_\mu\phi_i})(D^\mu\phi_i) - i\psi\sigma^\mu D_\mu\overline{\psi} + i\sqrt{2}g_n\overline{\phi}_i\lambda_n\psi_i - i\sqrt{2}g_n\overline{\psi}_i\overline{\lambda}_n\phi_i\\
&\quad - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j - \frac{1}{2}\frac{\partial^2\overline{W}}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\psi}_i\partial\overline{\psi}_j - V(\phi_i, \overline{\phi}_i),
\end{aligned} \tag{IV.5.71}$$

with the scalar potential now contains a part resulting from Fayet-Iliopoulos term

$$V(\phi_i, \overline{\phi}_i) = \left[\overline{F}_i F_i + \frac{D^a_n D^a_n}{2}\right]_{\text{on-shell}} = \frac{\partial W}{\partial\phi_i}\frac{\partial\overline{W}}{\partial\overline{\phi}_i} + \sum_{a,n}\frac{g_n^2}{2}\left[\overline{\phi}_i(T^a_{Rn})_{ij}\phi_j + \xi^a\right]^2, \tag{IV.5.72}$$

and clearly $V(\phi_i, \overline{\phi}_i)$ is positive definite, avoiding the infinitely negative energy levels. The covariant derivatives are defined as usual with an extra sum over gauge fields

$$D^{ab}_\mu = \partial_\mu\delta^{ab} - gf^{abc}V^c_\mu \tag{IV.5.73}$$

$$D_\mu = \partial_\mu - ig_n V_{n\mu} = \partial_\mu - ig_n V^a_{n\mu}T^a_{Rn}. \tag{IV.5.74}$$

where f^{abc} is the constant structure of gauge group. From now on, if there is no ambiguity we denote the generators of gauge groups briefly as T^a_n where a is the generator index and n indicates the gauge group. To obtain a physically meaningful Lagrangian, we need to build one that is similar to the Standard Model Lagrangian (i.e containing leptons and quarks for matter sector, gauge fields for interaction and Higgs field) and respect the supersymmetry. The detailed construction will be represented in the next chapter in this thesis.

V

Supersymmetric Extensions of the Standard Model

Outline

There are many ways to construct an extension of the Standard Model embedded with supersymmetry. For instance, one can start from the (anti)commutation relations of the supersymmetric algebra, build up superfield multiplets and construct the Lagrange containing the claimed internal symmetries as in SM. We can add as much fermionic operators pair Q^I_α, $\overline{Q}^{I\dot{\alpha}}$ as we want, there is no limitation on such a way of construction. However, for simplicity and practical purposes, we will consider the simplest non trivial extension of the SM with only one pair of $Q, \overline{Q}$ representing the boson-fermions symmetry; such minimal extension is called the *Minimal Supersymmetric Standard Model* (MSSM) and will be discussed in the beginning of this chapter. The rest of the current chapter is devoted to the *Next-to-minimal Supersymmetric Standard Model* (NMSSM), a modification of the MSSM by introducing a Higgs singlet to the particle content to solve an issue arising from the construction of the MSSM.

V.1 Minimal Supersymmetric Standard Model

V.1.1 MSSM Particle Content

Our tasks in this chapter are to build a realistic supersymmetric model by extending (in the minimal way) the Standard Model (SM), and try to keep as much properties of SM as possible. The gauge symmetry group for MSSM is thus the same as SM, that is $SU(3)_C \times SU(2)_L \times U(1)_Y$. Since each of particles Q, U, D, L, E is the irreducible representation of the gauge group, the extended supersymmetric version of SM must introduce a seperate chiral superfield for each original field. Similar to gauge bosons, we must include one vector superfield for each gauge interaction.

Since the particle content of the MSSM contains many new fermionic and bosonic fields, it may give rise to the chiral gauge anomalies that could make the theory inconsistent. We know that in the SM, the chiral gauge anomalie vanish when summing over contributions of quarks and leptons in each generation. The same scenario happens in the MSSM, i.e the contributions from quarks and leptons cancel out. Gauginos and scalar fields do not contribute to the anomalies. On the other hand, the Higgsinos - superpartner of the Higgs fields - give a non-vanishing contribution if there is only one Higgs doublet corresponding to only one Higgsino doublet. One can walk around this by adding an additional Higgs doublet which has an opposite hypercharge. Apart from the anomalies cancellation requirement, there is one more good reason why the second Higgs doublet is necessary. In order to give masses for both up- and down-type quarks in the SM, one has to use the Higgs doublet and its charge conjugate. This is not allowed in the supersymmetric potential, i.e one cannot consider both the Higgs superdoublet and its charge conjugate simultaneously.

Moreover, in SM matters are divided into left-handed and right-handed multiplets, each interacts differently with gauge fields. The chiral superfields we introduce in the supersymmetric Lagrangian in the above chapter contains only left-handed Weyl spinors, thus we must employed the charge conjugation $\psi^C_\alpha \equiv i(\sigma^2)_{\alpha\dot{\alpha}}\overline{\psi}^{\dot{\alpha}}$ to obtain right-handed Weyl spinors in our model. Finally, the supersymmetric partners of known elementary particles have not been experimentally confirmed yet, which is a clue for supersymmetry breaking. A detailed discussion on the origin of soft-supersymmetry breaking term by including a spurious field will also be covered in this chapter. In short, the basic ingredients of the MSSM (mostly inheriting from SM) are

1. $SU(3)_C \times SU(2)_L \times U(1)_Y$ vector superfields.
2. Chiral superfields for the quarks and leptons families.
3. Two doublet Higgs fields that are responsible for EM breaking.
4. Quarks and leptons masses, which comes from the trilinear superpotential after EW breaking.
5. Soft-breaking term in Lagrangian, in charges of SUSY-breaking.

Thus, there are twice as much particles as in the standard model, plus one extra Higgs multiplet. We summarize the particles content in Minimal Supersymmetric Standard Model in the tables below.

Super Field	**Spin 1/2**	**Spin 1**	$\mathbf{SU(3)_C \times SU(2)_L \times U(1)_Y}$	**Name**	**Coupling Constant**
$\hat{G}$	$\tilde{G}$	G	$(\mathbf{8},\mathbf{1},\mathbf{0})$	gluinos, gluons	g_s
$\hat{W}$	$\tilde{W}$	W	$(\mathbf{1},\mathbf{3},\mathbf{0})$	winos, W-bosons	g
$\hat{B}$	$\tilde{B}$	B	$(\mathbf{1},\mathbf{1},\mathbf{0})$	binos, B-bosons	g'

Table V.1: The gauge multiplets content of MSSM.

Supermultiplets	**Spin 0**	**Spin 1/2**	$SU(3)_C \times SU(2)_L \times U(1)_Y$	**Name**
$\hat{Q} = (\hat{u}_L, \hat{d}_L)^T$	$\tilde{Q} = (\tilde{u}_L, \tilde{d}_L)^T$	$Q = (u_L, d_L)$	$(\mathbf{3},\mathbf{2},\mathbf{1/3})$	squarks, quarks
$\hat{U}^c = \hat{u}^c$	$\tilde{U} = \overline{\tilde{u}}_R$	$\overline{u}_R$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{-4/3})$	
$\hat{D}^c = \hat{d}^c$	$\tilde{D} = \overline{\tilde{d}}_R$	$\overline{d}_R$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2/3})$	
$\hat{L} = (\hat{\nu}, \hat{e}_L)^T$	$\tilde{L} = (\tilde{\nu}, \tilde{e}_L)^T)$	$L = (\nu, e_L)^T$	$(\mathbf{1},\mathbf{2},\mathbf{-1})$	sleptons, leptons
$\hat{E}^c = \hat{e}^c$	$\tilde{E}^c = \overline{\tilde{e}}^c_R$	$E^c = \overline{e}^c_R$	$(\mathbf{1},\mathbf{1},\mathbf{2})$	
$\hat{H}_u = (\hat{H}^+_u, \hat{H}^0_u)^T$	$H_u = (H^+_u, H^0_u)^T$	$\tilde{H}_u = (\tilde{H}^+_u, \tilde{H}^0_u)^T$	$(\mathbf{1},\mathbf{2},\mathbf{1})$	Higgs, Higgsinos
$\hat{H}_d = (\hat{H}^0_d, \hat{H}^-_d)^T$	$H_d = (H^0_d, H^-_u)^T$	$\tilde{H}_d = (\tilde{H}^0_d, \tilde{H}^-_d)^T$	$(\mathbf{1},\mathbf{2},\mathbf{-1})$	

Table V.2: The matter multiplets content of MSSM.

V.1.2 Lagrangian for MSSM

As constructed generally in the above chapter, the Lagrangian for MSSM should be a sum of the Kähler potential for kinetic part of particles, the superpotential that contains interaction of matter with Higgs fields (and later gives rise to Yukawa mass term of leptons, quarks and their superpartner after electroweak is broken), the super Yang-Mills term for gauge kinetics, the scalar potential, and the Fayet-Iliopoulos term contributed by $U(1)_Y$) gauge superfield.

MSSM Kähler potential

Let us begin with the Kähler superfields, which original form is

$$\begin{aligned}
K_{MSSM} = &\overline{\hat{Q}} \exp\left[2g_s\hat{G} + 2g\hat{W} + g'Y(Q)\hat{B}\right]\hat{Q}\\
&+ \overline{\hat{u}}^c \exp\left[2g_s\hat{G} + g'Y(u)\hat{B}\right]\hat{u}^c\\
&+ \overline{\hat{d}}^c \exp\left[2g_s\hat{G} + g'Y(d)\hat{B}\right]\hat{d}^c\\
&+ \overline{\hat{L}} \exp\left[2g\hat{W} + g'Y(L)\hat{B}\right]\hat{L}\\
&+ \overline{\hat{e}}^c \exp\left[g'Y(Q)\hat{B}\right]\hat{e}^c\\
&+ \overline{\hat{H}}_u \exp\left[2g\hat{W} + g'Y(H_u)\hat{B}\right]\hat{H}_u\\
&+ \overline{\hat{H}}_d \exp\left[2g\hat{W} + g'Y(H_d)\hat{B}\right]\hat{H}_d,
\end{aligned} \tag{V.1.1}$$

where $Y(X)$ denotes the hypercharge of the X superfield, where X belongs to the set of superfields listed in Table V.2. Ignoring the terms related to D-field and F-field that contribute to the scalar potential, the general on-shell Kähler potential reads

$$\mathcal{L}_{\text{OS Kähler}} = \overline{(D_\mu\phi)}(D^\mu\phi) - i\psi\sigma^\mu D_\mu\overline{\psi} + (i\sqrt{2}g_n\overline{\phi}_i\lambda_n\psi_i + h.c), \tag{V.1.2}$$

which after applying to MSSM Kähler potential becomes

$$\begin{aligned}
\mathcal{L}^{\text{MSSM}}_{\text{OS Kähler}} =&\overline{(D^Q_\mu\tilde{Q})}D^{Q\mu}\tilde{Q} - iQ\sigma^\mu D^Q_\mu\overline{Q} + \left[i\sqrt{2}\overline{\tilde{Q}}\left(g_s\tilde{G} + g\tilde{W} + g'\frac{Y(Q)}{2}\tilde{B}\right)Q + h.c\right]\\
&+ \overline{(D^u_\mu\tilde{u}^c)}D^{u\mu}\tilde{u}^c - iu^c\sigma^\mu D^u_\mu\overline{u} + \left[i\sqrt{2}\overline{\tilde{u}}\left(g_s\tilde{G} + g'\frac{Y(u)}{2}\tilde{B}\right)u^c + h.c\right]\\
&+ \overline{(D^d_\mu\tilde{d}^c)}D^{d\mu}\tilde{d}^c - id^c\sigma^\mu D^d_\mu\overline{d} + \left[i\sqrt{2}\overline{\tilde{d}}\left(g_s\tilde{G} + g'\frac{Y(d)}{2}\tilde{B}\right)d^c + h.c\right]\\
&+ \overline{(D^L_\mu\tilde{L})}D^{L\mu}\tilde{L} - iL\sigma^\mu D^L_\mu\overline{L} + \left[i\sqrt{2}\overline{\tilde{L}}\left(g\tilde{W} + g'\frac{Y(L)}{2}\tilde{B}\right)L + h.c\right]\\
&+ \overline{(D^e_\mu\tilde{e}^c)}D^{e\mu}\tilde{e}^c - ie^c\sigma^\mu D^e_\mu\overline{e} + \left[i\sqrt{2}\overline{\tilde{e}}\left(g\tilde{W} + g'\frac{Y(e)}{2}\tilde{B}\right)e^c + h.c\right]\\
&+ \overline{(D^{H_u}_\mu H_u)}D^{H_u\mu}H_u - i\tilde{H}_u\sigma^\mu D^{H_u}_\mu\overline{\tilde{H}}_u + \left[i\sqrt{2}\overline{H}_u\left(g\tilde{W} + g'\frac{Y(H_u)}{2}\tilde{B}\right)\tilde{H}_u + h.c\right]\\
&+ \overline{(D^{H_d}_\mu H_d)}D^{H_d\mu}H_d - i\tilde{H}_d\sigma^\mu D^{H_d}_\mu\overline{\tilde{H}}_d + \left[i\sqrt{2}\overline{H}_d\left(g\tilde{W} + g'\frac{Y(H_d)}{2}\tilde{B}\right)\tilde{H}_d + h.c\right],
\end{aligned} \tag{V.1.3}$$

where the gauge covariant derivative for each chiral field are

$$\begin{aligned}
D^Q_\mu &= \partial_\mu - ig_sT^aG^a_\mu - igI^bW^b_\mu - ig'\frac{Y(Q)}{2}B_\mu,\\
D^u_\mu &= \partial_\mu - ig_sT^aG^a_\mu - ig'\frac{Y(u)}{2}B_\mu,\\
D^d_\mu &= \partial_\mu - ig_sT^aG^a_\mu - ig'\frac{Y(d)}{2}B_\mu,\\
D^L_\mu &= \partial_\mu - igI^bW^b_\mu - ig'\frac{Y(L)}{2}B_\mu,
\end{aligned}$$

$$
\begin{aligned}
D_\mu^e &= \partial_\mu - ig'\frac{Y(e)}{2}B_\mu, \\
D_\mu^{H_u} &= \partial_\mu - igI^bW_\mu^b - ig'\frac{Y(H_u)}{2}B_\mu, \\
D_\mu^{H_d} &= \partial_\mu - igI^bW_\mu^b - ig'\frac{Y(H_d)}{2}B_\mu.
\end{aligned}
$$

MSSM Super Yang-Mills Lagrangian

Next consider the super Yang-Mills term for the kinetics of gauge bosons and gauginos, which has the general form (temporarily ignore the CP-violation part)

$$
\mathcal{L}_{SYM}^{\mathrm{MSSM}} = \int \mathrm{d}^2\theta \left\{ \frac{1}{8g_s^2}\,\mathrm{Tr}\left[\mathcal{W}^\alpha(\hat{G})\mathcal{W}_\alpha(\hat{G})\right] + \frac{1}{8g^2}\,\mathrm{Tr}\left[\mathcal{W}^\alpha(\hat{W})\mathcal{W}_\alpha(\hat{W})\right] + \frac{1}{4g'^2}\mathcal{W}^\alpha(\hat{B})\mathcal{W}_\alpha(\hat{B})\right\} + h.c. \tag{V.1.4}
$$

The free gauge Lagrangian component-wise (ignore terms contributed to scalar potential) reads

$$
\mathcal{L}_{OS\ SYM}^{\mathrm{MSSM}} = \frac{1}{2}\,\mathrm{Tr}\left\{-G_{\mu\nu}G^{\mu\nu} - 4i\tilde{G}\sigma^\mu D_\mu^G\overline{\tilde{G}} - W_{\mu\nu}W^{\mu\nu} - 4i\tilde{W}\sigma^\mu D_\mu^W\overline{\tilde{W}}\right\} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} - i\tilde{B}\sigma^\mu D_\mu^B\overline{\tilde{B}}, \tag{V.1.5}
$$

where $G_\mu \equiv G_\mu^a T^a$, $W_\mu \equiv W_\mu^b I^b$, $\tilde{G} \equiv \tilde{G}^a T^a$, $\tilde{W} \equiv \tilde{W}^b I^b$, and the fields strength and covariant derivatives are defined as usual

$$
\begin{aligned}
G_{\mu\nu} &= \partial_\mu G_\nu - \partial_\nu G_\mu - ig_s\left[G_\mu, G_\nu\right], & D_\mu^G &= \partial_\mu - ig_s\left[G_\mu, \star\right], \\
W_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu - ig\left[W_\mu, W_\nu\right], & D_\mu^W &= \partial_\mu - ig_s\left[W_\mu, \star\right], \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, & D_\mu^B &= \partial_\mu.
\end{aligned}
$$

R-Parity & MSSM superpotential

For minimal construction of the superpotential that respects the baryon and lepton number conservation, only interaction terms with Higgs fields is considered. That means the superpotential should contains only Yukawa couplings, and coupling between two Higgs superfields. The MSSM superpotential is thus constructed from the following superpotential

$$
W^{\mathrm{MSSM}} = -\hat{u}^c\mathbf{Y_u}\hat{Q}\cdot\hat{H}_u + \hat{d}^c\mathbf{Y_d}\hat{Q}\cdot\hat{H}_d + \hat{e}^c\mathbf{Y_e}\hat{L}\cdot\hat{H}_d - \mu\hat{H}_u\cdot\hat{H}_d, \tag{V.1.6}
$$

with the dot product indicates the Weyl spinor product, i.e $\psi\cdot\chi = \psi\chi = \psi^\alpha\chi_\alpha$. The above compact notation should be understand as, for example

$$
\hat{e}^c\mathbf{Y_e}\hat{L}\cdot\hat{H}_d = \hat{e}_i^c\mathbf{Y_e}^{ij}(\hat{L}_j)^\alpha\hat{H}_{d\alpha}, \tag{V.1.7}
$$

with the sum over $SU(2)$ index $\alpha = 1, 2$, and over generation indices i, j. Taking the $SU(2)$ dot product explicitly gives the expression for the superpotential

$$
\begin{aligned}
W^{\mathrm{MSSM}} =& -\hat{u}^c\mathbf{Y_u}\left(\hat{u}_L\hat{H}_u^0 - \hat{d}_L\hat{H}_u^+\right) + \hat{d}^c\mathbf{Y_d}\left(\hat{u}_L\hat{H}_d^- - \hat{d}_L\hat{H}_d^0\right) + \hat{e}^c\mathbf{Y_e}\left(\hat{\nu}\hat{H}_d^- - \hat{e}_L\hat{H}_d^0\right) \\
& - \mu\left(\hat{H}_u^+\hat{H}_d^- - \hat{H}_u^0\hat{H}_d^0\right).
\end{aligned} \tag{V.1.8}
$$

Due to the mass differece between third family fermions and those from the first two families, in the numerical calculations we made the following approximation

$$
\mathbf{Y_{u,d,e}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y_{t,d,\tau} \end{pmatrix}. \tag{V.1.9}
$$

Note that when writing down the MSSM superpotential (V.1.6), not all gauge invariant terms have been included: one neglect bilinear or trilinear terms that violate the lepton and/or baryon numbers conservation.

The presence of such terms can be generally omitted by forcing the MSSM Lagrangian to satisfy additional discrete symmetry so-called R-parity, defined as

$$P_R = (-1)^{3(B-L)+2S} = (-1)^{2S} P_M, \tag{V.1.10}$$

where L, B and S are respectively the lepton number, baryon number and the spin of the particle under consideration; $P_M = (-1)^{3(B-L)}$ thus being the matter parity. All of the SM fields have $P_R = +1$ and their superpartners have $P_R = -1$. In addition, every interaction vertices in an R-parity conserving theory must contains an even number of $P_R = -1$ sparticles. More discussion on R-parity and the R-parity violation MSSM can be found in Ref. [41].

R-parity conserving models have the following important phenomenological consequences

- Sparticles must always appear in even numbers at every interaction vertices, implies sparticles are only produced in pairs.
- The lightest sparticle with odd R-parity, so-called the *lighest supersymmetric particle* (LSP) is stable. A electrically neutral LSP only interact weakly with other particles. This is important since stability is one of the main properties to make the LSP (the lightest neutralino in the our calculations in the next several chapters) is a DM candidate.
- All sparticles must be ended up by decaying into a state that contains an odd number of LSPs.

Without this assumed symmetry, the following R-parity violated superpotential are allowed alongside with (V.1.8)

$$W_{\rm RPV}^{\rm MSSM} = \lambda_{ijk}\hat{L}_i\hat{L}_j\hat{e}^c_k + \lambda'_{ijk}\hat{L}_i\hat{Q}_j\hat{d}^c_k + \kappa_i\hat{L}_i\hat{H}_u, + \lambda''_{ijk}\hat{u}^c_i\hat{d}^c_j\hat{d}^c_k \tag{V.1.11}$$

where the first three terms in Eq. (V.1.11) contain lepton number violating interactions while the last term violates baryon number. To illustrate the fact that R-parity conservation wipe out all of B- and L- violating terms in the MSSM superpotential, we note that the products $(-1)^{2S}$ in all interaction vertices are always 1, leaving behind the products of matter parity P_M. Since $P_M = -1$ for all lepton and quark supermultiplets, while $P_M = 1$ for Higgs supermultiplets, all of the terms in (V.1.11) also violate the R-parity.

In the context of this thesis, the models MSSM and NMSSM is assumed to respect the R-parity conservation. [1]

As derived before, the on-shell superpotential (again ignore the terms contributed to scalar potential) is $-\frac{1}{2}\dfrac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_i\psi_j + h.c.$ Each Yukawa term is a product of three different superfields, after taking sum gives three terms contribute to the superpotential, while each product of two Higgs fields gives two terms, thus there are $6\times 3 + 2\times 2 = 22$ fields product terms in the MSSM potential. That is

$$\begin{aligned}
\mathcal{L}^{\rm MSSM}_{OS\ W} &= -\frac{1}{2}\frac{\partial W^{\rm MSSM}}{\partial^2\phi_i\partial\phi_j}\psi_i\psi_j + h.c \\
&= \frac{1}{2}\Big\{\overline{u}_R\mathbf{Y_u}u_L H^0_u + \overline{u}_R\mathbf{Y}_u\tilde{u}_L\tilde{H}^0_u + \overline{\tilde{u}}_R\mathbf{Y_u}u_L\tilde{H}^0_u - \overline{u}_R\mathbf{Y_u}d_L H^+_u - \overline{u}_R\mathbf{Y_u}\tilde{d}_L\tilde{H}^0_u - \overline{\tilde{u}}_R\mathbf{Y_u}u_L\tilde{H}^0_u \\
&- \overline{d}_R\mathbf{Y_d}u_L H^-_d - \overline{d}_R Y_d\overline{\tilde{u}}_L\tilde{H}^-_d - \overline{\tilde{d}}_R Y_d u_L\tilde{H}^-_d + \overline{d}_R\mathbf{Y_d}d_L H^0_d + \overline{d}_R\mathbf{Y_d}\tilde{d}_L\tilde{H}^0_d + \overline{\tilde{d}}_R\mathbf{Y_d}d_L\tilde{H}^0_d \\
&- \overline{e}_R\mathbf{Y_e}\nu H^-_d - \overline{e}_R Y_e\tilde{\nu}\tilde{H}^-_d - \overline{\tilde{e}}_R\mathbf{Y_e}\nu\tilde{H}^-_d + \overline{e}_R\mathbf{Y_e}e_L H^0_d + \overline{e}_R\mathbf{Y_e}\tilde{E}_L\tilde{H}^0_d + \overline{\tilde{e}}_R\mathbf{Y_e}e_L\tilde{H}^0_d \\
&+\mu\left(\tilde{H}^+_u\tilde{H}^-_d - \tilde{H}^0_u\tilde{H}^0_d\right)\Big\} + h.c.
\end{aligned} \tag{V.1.12}$$

MSSM scalar potential

Recall that the information about scalar interaction is contained in the scalar potential (that composed of mainly auxiliary fields F and D), which generally reads

$$V = \sum_i\left\{\frac{\partial W}{\partial\phi_i}\frac{\partial\overline{W}}{\partial\overline{\phi}_i} + \sum_{a,n}\frac{g_n^2}{2}\left[\overline{\phi}_i(T^a_{Rn})_{ij}\phi_j + \xi^a\right]^2\right\}. \tag{V.1.13}$$

[1] This assumption is somewhat artificial in the theoretical point of views since there is no such internal inconsistency if R-parity breaking terms are introduced into the Lagrangian, but does makes sense from phenomenological perspective (e.g the constraints by proton decay [42] and that the fact that LSP is a good DM candidates).

Similar to the way we find the superpotential, the scalar potential can be read-off from W^{MSSM} in Eq. (V.1.8) using its definition. The last three terms comes from the D-terms potential (those relate to the sum over a and n), and the rest is contributed by the F-terms

$$
\begin{aligned}
\mathcal{L}_V^{\text{MSSM}} = -\Big\{ & \left(\overline{H}_u^0 \overline{\tilde{u}}_L - \overline{H}_u^+ \overline{\tilde{d}}_L\right) \mathbf{Y_u}^\dagger \mathbf{Y_u} \left(\tilde{u}_L H_u^0 - \tilde{d}_L H_u^+\right) \\
& + \left(\overline{H}_d^- \overline{\tilde{u}}_L - \overline{H}_d^0 \overline{\tilde{d}}_L\right) \mathbf{Y_d}^\dagger \mathbf{Y_d} \left(\tilde{u}_L H_d^- - \tilde{d}_L H_d^0\right) \\
& + \left(\overline{H}_d^- \overline{\tilde{\nu}} - \overline{H}_d^0 \overline{\tilde{d}}_L\right) \mathbf{Y_d}^\dagger \mathbf{Y_d} \left(\tilde{u}_L H_d^- - \tilde{d}_L H_d^0\right) \\
& + \left(-H_u^0 \overline{\tilde{u}}_R \mathbf{Y_u} + H_d^- \overline{\tilde{d}}_R \mathbf{Y_d}\right) \left(-\mathbf{Y_u}^\dagger \tilde{u}_R \overline{H}_u^0 + \mathbf{Y_d}^\dagger \tilde{d}_R \overline{H}_d^-\right) \\
& + \left(H_u^+ \overline{\tilde{u}}_R \mathbf{Y_u} - H_d^0 \overline{\tilde{d}}_R \mathbf{Y_d}\right) \left(\mathbf{Y_u}^\dagger \tilde{u}_R \overline{H}_u^+ - \mathbf{Y_d}^\dagger \tilde{d}_R \overline{H}_d^0\right) \\
& + H_d^- \overline{\tilde{e}}_R \mathbf{Y_e} \mathbf{Y_e}^\dagger \tilde{e}_R \overline{H}_d^- \\
& + H_d^0 \overline{\tilde{e}}_R \mathbf{Y_e} \mathbf{Y_e}^\dagger \tilde{e}_R \overline{H}_d^0 \\
& + \left(-\overline{\tilde{u}}_R \mathbf{Y_u} \tilde{u}_L + \mu H_d^0\right) \left(-\overline{\tilde{u}}_L \mathbf{Y_u}^\dagger \tilde{u}_R + \mu \overline{H}_d^0\right) \\
& + \left(\overline{\tilde{u}}_R \mathbf{Y_u} \tilde{d}_L - \mu H_d^-\right) \left(\overline{\tilde{d}}_L \mathbf{Y_u}^\dagger \tilde{u}_R - \mu \overline{H}_d^-\right) \\
& + \left(\overline{\tilde{d}}_R \mathbf{Y_d} \tilde{u}_L + \overline{\tilde{e}}_R \mathbf{Y_e} \tilde{\nu} - \mu H_u^+\right) \left(\overline{\tilde{u}}_L \mathbf{Y_d}^\dagger \tilde{d}_R + \overline{\tilde{\nu}} \mathbf{Y_e}^\dagger \tilde{e}_R - \mu \overline{H}_u^+\right) \\
& + \left(-\overline{\tilde{d}}_R \mathbf{Y_d} \tilde{d}_L - \overline{\tilde{e}}_R \mathbf{Y_e} \tilde{e}_L + \mu H_u^0\right) \left(-\overline{\tilde{d}}_L \mathbf{Y_d}^\dagger \tilde{d}_R - \overline{\tilde{e}}_L \mathbf{Y_e}^\dagger \tilde{e}_R + \mu \overline{H}_u^0\right) \\
& + \frac{g_s^2}{2} \sum_a \left(\overline{\tilde{Q}} T^a \tilde{Q} + \overline{\tilde{u}}_R T^a \tilde{u}_R + \overline{\tilde{d}}_R T^a \tilde{d}_R\right)^2 \\
& + \frac{g^2}{2} \sum_b \left(\overline{\tilde{Q}} I^b \tilde{Q} + \overline{\tilde{L}} I^b \tilde{L} + \overline{H}_u I^b H_u + \overline{H}_d I^b H_d\right)^2 \\
& + \frac{g'^2}{2} \Bigg(\overline{\tilde{Q}} \frac{Y(Q)}{2} \tilde{Q} + \overline{\tilde{u}}_R \frac{Y(u)}{2} \tilde{u}_R + \overline{\tilde{d}}_R \frac{Y(d)}{2} \tilde{d}_R \\
& \qquad + \overline{\tilde{L}} \frac{Y(L)}{2} \tilde{L} + \overline{\tilde{e}}_R \frac{Y(e)}{2} \tilde{e} + \overline{H}_d \frac{Y(H_d)}{2} H_d + \overline{H}_u \frac{Y(H_u)}{2} H_u + \xi\Bigg)^2 \Big\}.
\end{aligned} \tag{V.1.14}
$$

MSSM soft-breaking Lagrangian

Finally, we proceed to construct the supersymmetry breaking soft term. The reason is that supersymmetry implies that a particle and its superpartner should have the same mass, which has been experimentally ruled out. Hence supersymmetry, if exists, must be broken. We want our model to maintain the cancellation of quadratically divergent terms in the radiative corrections of all scalar masses, at all orders in perturbation theory, thus the breaking term should be "soft". This requirement leads to the fact that only terms with highest mass-dimensional order three are considered. Below we follow the path drawn in Ref. [43]. In order to control symmetry breaking terms, we couple the so-called *chiral scalar spurion field* $\hat{A}$ to the constructed Lagrangian so that the total action remains supersymmetry invariant

$$\hat{A}(y) \equiv A + \sqrt{2}\theta a + \theta\theta F_A. \tag{V.1.15}$$

By identifying $A = 0$, the powers of $\hat{A}$ is cut-off and the most general real and chiral expression containing $\hat{A}$ and $\overline{\hat{A}}$ are

$$\mathscr{P}_{\text{real}} = c_{\text{real},1}\left(\hat{A} + \overline{\hat{A}}\right) + c_{real,2}\hat{A}\overline{\hat{A}}, \quad \mathscr{P}_{\text{chiral}} = c_{\text{chiral}}\hat{A}. \tag{V.1.16}$$

Inserting the above new real superfield to terms under full-space integrals in Kähler potential, and new chiral superfields into terms under half-space integrals (those belong to Super Yang-Mills Lagrangian and superpotential) to obtain the soft-breaking Lagrangian

$$\begin{aligned}\mathcal{L}^{\text{MSSM}}_{soft} = &\int \mathrm{d}^4\theta \left\{ \mathscr{P}_Q \overline{\hat{Q}} \exp\left[2g_s\hat{G} + 2g\hat{W} + g'Y(Q)\hat{B}\right]\hat{Q} + \mathscr{P}_u \overline{\hat{u}}^c \exp\left[2g_s\hat{G} + g'Y(u)\hat{B}\right]\hat{u}^c \right.\\ &+ \mathscr{P}_d \overline{\hat{d}}^c \exp\left[2g_s\hat{G} + g'Y(d)\hat{B}\right]\hat{d}^c + \mathscr{P}_L \overline{\hat{L}} \exp\left[2g\hat{W} + g'Y(L)\hat{B}\right]\hat{L}\\ &\left. + \mathscr{P}_e \overline{\hat{e}}^c \exp\left[g'Y(Q)\hat{B}\right]\hat{e}^c + \mathscr{P}_{H_i} \overline{\hat{H}}_i \exp\left[2g\hat{W} + g'Y(H_i)\hat{B}\right]\hat{H}_i \right\}\\ &+\left(\int \mathrm{d}^2\theta \left\{ -\frac{c_G}{2}\hat{A}\,\mathrm{Tr}\left[\mathcal{W}^\alpha(\hat{G})\mathcal{W}_\alpha(\hat{G})\right] - \frac{c_W}{2}\hat{A}\,\mathrm{Tr}\left[\mathcal{W}^\alpha(\hat{\mathcal{W}})W_\alpha(\hat{\mathcal{W}})\right] - \frac{c_B}{4}\hat{A}\mathcal{W}^\alpha(\hat{B})\mathcal{W}_\alpha(\hat{B}) \right.\right.\\ &\left.\left. - c_{uQH}\hat{A}\hat{u}^c\mathbf{Y_u}\hat{Q}\cdot\hat{H}_u + c_{dQH}\hat{A}\hat{d}^c\mathbf{Y_d}\hat{Q}\cdot\hat{H}_d + c_{eLH}\hat{A}\hat{e}^c\mathbf{Y_e}\hat{L}\cdot\hat{H}_d - c_{HH}\mu\hat{A}\hat{H}_u\cdot\hat{H}_d \right\} + h.c \right)\end{aligned} \tag{V.1.17}$$

with the general chiral superfield $\mathscr{P}_\phi$ built up on the chiral superfield ϕ as

$$\mathscr{P}_\phi = c_{\phi 1}\left(\hat{A} + \overline{\hat{A}}\right) + c_{\phi 2}\overline{\hat{A}}\hat{A}, \qquad \phi = Q, u, d, L, e, H_u, H_d. \tag{V.1.18}$$

The supersymmetry is broken by vacuum expectation value of spurion field $\hat{A}$, i.e $\hat{A} \to \theta\theta v_A$. After supersymmetry broken, the soft breaking Lagrangian is a sum of mass terms for scalar fields and gauginos and up to trilinear interactions between the scalar components of either purely chiral or purely antichiral multiplets. Explicitly it reads

$$\begin{aligned}\mathcal{L}^{\text{MSSM}}_{soft} = &-\frac{1}{2}\left(M_{\tilde{G}}\tilde{G}^a\tilde{G}^a + M_{\tilde{W}}\tilde{W}^b\tilde{W}^b + M_{\tilde{B}}\tilde{B}\tilde{B} + h.c\right)\\ &- m^2_{H_u}\overline{H}_u H_u - m^2_{H_d}\overline{H}_d H_d - (bH_u\cdot H_d + h.c)\\ &- \overline{\tilde{Q}}\mathbf{M}^2_{\tilde{Q}}\tilde{Q} - \overline{\tilde{L}}\mathbf{M}^2_{\tilde{L}}\tilde{L} - \overline{\tilde{u}^c}\mathbf{M}_{\tilde{u}}\tilde{u}^c - \overline{\tilde{d}}^c\mathbf{M}_{\tilde{d}}\tilde{d}^c - \overline{\tilde{e}}^c\mathbf{M}_{\tilde{e}}\tilde{e}^c\\ &- \left(\overline{\tilde{u}}^c\mathbf{Y_u}\mathbf{A_u}\tilde{Q}\cdot H_u - \overline{\tilde{d}}^c\mathbf{Y_d}\mathbf{A_u}\tilde{Q}\cdot H_d - \overline{\tilde{e}}^c\mathbf{Y_u}\mathbf{A_u}\tilde{Q}\cdot H_d + h.c\right).\end{aligned} \tag{V.1.19}$$

The full Lagrangian for Minimal Supersymmetric Standard Model is

$$\boxed{\mathcal{L}^{\text{MSSM}} = \mathcal{L}^{\text{MSSM}}_{\text{OS Kähler}} + \mathcal{L}^{\text{MSSM}}_{OS\ SYM} + \mathcal{L}^{\text{MSSM}}_{OS\ W} + \mathcal{L}^{\text{MSSM}}_{V} + \mathcal{L}^{\text{MSSM}}_{soft}}, \tag{V.1.20}$$

where each component Lagrangian has been derived in Eq. (V.1.3), (V.1.4), (V.1.12), (V.1.14) and (V.1.19).

V.1.3 Tree level mass spectrum in the MSSM

The complex MSSM (cMSSM) Higgs potential

We first consider the Higgs potential in MSSM, which can be read directly from Eq. (V.1.20)

$$\begin{aligned}V_H =&\mu^2|H_u|^2 + \mu^2|H_d|^2\\ &+ \frac{g^2}{2}\sum_b\left(\overline{H}_u I^b H_u + \overline{H}_d I^b H_d\right)^2 + \frac{g'^2}{2}\left[\overline{H}_u\frac{Y(H_u)}{2}H_u + \overline{H}_d\frac{Y(H_d)}{2}H_d + \xi\right]^2\\ &+ m^2_{H_u}|H_u|^2 + m^2_{H_d}|H_d|^2 + (bH_u\cdot H_d + h.c)\\ =&\left(\mu^2 + m^2_{H_u} + g'^2\frac{\xi}{2}\right)|H_u|^2 + \left(\mu^2 + m^2_{H_d} + g'^2\frac{\xi}{2}\right)|H_d|^2 + (bH_u\cdot H_d + h.c)\\ &+ \frac{g^2}{8}\sum_b\left(\overline{H}_u\sigma^b H_u + \overline{H}_d\sigma^b H_d\right)^2 + \frac{g'^2}{8}\left(|H_u|^2 - |H_u|^2\right)^2.\end{aligned} \tag{V.1.21}$$

where the first line is the F-term of superpotential, the second line is the contribution of D-term in scalar potential, and the final line comes from the soft-breaking Lagrangian. The term related to g can be simplified by expanding as

$$\begin{aligned}
\sum_b \left(\overline{H}_u\sigma^b H_u + \overline{H}_d\sigma^b H_d\right) =& \left(\overline{H}^0_u H^+_u + \overline{H}^+_u H^0_u + \overline{H}^-_d H^0_d + \overline{H}^0_d H^-_d\right)^2 \\
&+ \left(i\overline{H}^0_u H^+_u - i\overline{H}^+_u H^0_u + i\overline{H}^-_d H^0_d - i\overline{H}^0_d H^-_d\right)^2 \\
&+ \left(|H^+_u|^2 - |H^0_u|^2 + |H^0_d|^2 - |H^-_d|^2\right)^2 \\
=& 4|H^+_u|^2|H^0_u|^2 + 4|H^0_d|^2|H^-_d|^2 + 4\overline{H}^0_u H^+_u \overline{H}^0_d H^-_d + 4\overline{H}^+_u H^0_u \overline{H}^-_d H^0_d \\
&+ \left(|H^+_u|^2 + |H^0_u|^2 - |H^0_d|^2 - |H^-_d|^2\right)^2 \\
&- 4|H^+_u|^2|H^0_u|^2 + 4|H^0_u|^2|H^-_d|^2 + 4|H^+_u|^2|H^0_d|^2 - 4|H^0_d|^2|H^-_d|^2 \\
=& 4|H^0_u|^2|H^-_d|^2 + 4|H^+_u|^2|H^0_d|^2 + 4\overline{H}^0_u H^+_u \overline{H}^0_d H^-_d + 4\overline{H}^+_u H^0_u \overline{H}^-_d H^0_d \\
&+ \left(|H^+_u|^2 + |H^0_u|^2 - |H^0_d|^2 - |H^-_d|^2\right)^2 \\
=& 4|\overline{H}^+_u H^0_d + \overline{H}^0_u H^-_d|^2 + \left(|H^+_u|^2 + |H^0_u|^2 - |H^0_d|^2 - |H^-_d|^2\right)^2 . \qquad \text{(V.1.22)}
\end{aligned}$$

$$\text{(V.1.23)}$$

After plugging back the g^2 term into the Higgs potential and rearranging the order, we obtain

$$\begin{aligned}
V_H =& M^2_{H_u}|H_u|^2 + M^2_{H_d}|H_d|^2 + (bH_u\cdot H_d + h.c) + \frac{g^2+g'^2}{8}\left(|H_u|^2 - |H_d|^2\right)^2 + \frac{g^2}{2}\left(\overline{H}_u H_d\right)\left(\overline{H}_d H_u\right) \\
=& \left(\mu^2 + m^2_{H_u} + g'^2\frac{\xi}{2}\right)\left(|H^+_u|^2 + |H^0_u|^2\right) + \left(\mu^2 + m^2_{H_d} + g'^2\frac{\xi}{2}\right)\left(|H^0_d|^2 + |H^-_d|^2\right) \\
&+ \left[b\left(H^+_u H^-_d - H^0_u H^0_d\right) + h.c\right] \\
&+ \frac{g^2+g'^2}{8}\left(|H^+_u|^2 + |H^0_u|^2 - |H^0_d|^2 - |H^-_d|^2\right)^2 + \frac{g^2}{2}\left|\overline{H}^+_u H^0_d + \overline{H}^0_u H^-_d\right|^2, \qquad \text{(V.1.24)}
\end{aligned}$$

Here we introduce new parameters $M^2_{H_u}$ and $M^2_{H_d}$, defined as

$$M^2_{H_u} \equiv \mu^2 + m^2_{H_u} + g'^2\frac{\xi}{2}, \qquad M^2_{H_d} \equiv \mu^2 + m^2_{H_d} + g'^2\frac{\xi}{2}, \qquad \text{(V.1.25)}$$

Clearly ξ does not give any phenomenological impact since it can be absorbed into $m^2_{H_u}$ and $m^2_{H_d}$. From now on we will set this ξ to zero.

In order to break the $SU(2)_L \times U(1)_Y$, the Higgs potential is required to be bounded below. Note that this potential is gauge invariant, and thus the component fields have some degrees freedom to be gauge transformed. We want the abelian symmetry $U(1)_Q$ is preserved, thus the vev of charged component H^+_u and H^-_d must vanish. We now want to decompose the Higgs doublet components as variation around their VEVs as following

$$H_d = \begin{pmatrix} H^0_d \\ H^-_d \end{pmatrix} = \begin{pmatrix} \left(v_d + \phi^0_d + i\chi^0_d\right)/\sqrt{2} \\ \phi^-_d \end{pmatrix}, \qquad \text{(V.1.26)}$$

$$H_u = \begin{pmatrix} H^+_u \\ H^0_u \end{pmatrix} = e^{i\varphi_u}\begin{pmatrix} \phi^+_u \\ \left(v_u + \phi^0_u + i\chi^0_u\right)/\sqrt{2} \end{pmatrix}, \qquad \text{(V.1.27)}$$

where the vacuum expectation values of Higgs field can be chosen to be real. There is a possible complex phase between two Higgs sector. Plugging these equations into Eq. (V.1.24) gives us a general expression

$$\begin{aligned}
V_H =& M^2_{H_u}\left[|\phi^+_u|^2 + \frac{\left(v_u + \phi^0_u\right)^2}{2} + \frac{\left(\chi^0_u\right)^2}{2}\right] + M^2_{H_d}\left[|\phi^+_d|^2 + \frac{\left(v_d + \phi^0_d\right)^2}{2} + \frac{\left(\chi^0_d\right)^2}{2}\right] \\
&+ \left\{be^{i\varphi_u}\left[\phi^+_u\phi^-_d - \frac{1}{2}\left(v_u + \phi^0_u + i\chi^0_u\right)\left(v_d + \phi^0_d + i\chi^0_d\right)\right] + h.c\right\}
\end{aligned}$$

$$+\frac{g^2+g'^2}{8}\left[|\phi_u^+|^2+\frac{\left(v_u+\phi_u^0\right)^2}{2}+\frac{\left(\chi_u^0\right)^2}{2}-|\phi_d^-|^2-\frac{\left(v_d+\phi_d^0\right)^2}{2}-\frac{\left(\chi_d^0\right)^2}{2}\right]^2$$

$$+\frac{g^2}{2}\left|\phi_u^+\frac{\left(v_d+\phi_d^0+i\chi_d^0\right)}{\sqrt{2}}+\frac{\left(v_u+\phi_u^0-i\chi_u^0\right)}{2}\phi_d^-\right|^2. \tag{V.1.28}$$

We want to rearrange V_H in powers of the fields, i.e in the following form

$$V_H=\cdots-T_{\phi_d^0}\phi_d^0-T_{\phi_u^0}\phi_u^0-T_{\chi_d^0}\chi_d^0-T_{\chi_u^0}\chi_u^0$$

$$+\frac{1}{2}\begin{pmatrix}\phi_d^0 & \phi_u^0 & \chi_d^0 & \chi_u^0\end{pmatrix}\mathbf{M}_{\phi\phi\chi\chi}\begin{pmatrix}\phi_d^0\\ \phi_u^0\\ \chi_d^0\\ \chi_u^0\end{pmatrix}+\begin{pmatrix}\phi_d^- & \phi_u^-\end{pmatrix}\mathbf{M}_{\phi^\pm\phi^\pm}\begin{pmatrix}\phi_d^+\\ \phi_u^+\end{pmatrix}, \tag{V.1.29}$$

where the dots denote the trilinear and quartic terms of V_H that is not needed in the following calculations. It is obvious that the phase φ_u only appears alongside b, thus b can be chosen real. Expanding the Eq. (V.1.28) yields the tadpoles coefficients

$$T_{\phi_d^0}=-\left[M_{H_d}^2v_d-b\cos\varphi_u\ v_u-\frac{g^2+g'^2}{8}\left(v_u^2-v_d^2\right)v_d\right], \tag{V.1.30a}$$

$$T_{\phi_u^0}=-\left[M_{H_u}^2v_u-b\cos\varphi_u\ v_d-\frac{g^2+g'^2}{8}\left(v_d^2-v_u^2\right)v_u\right], \tag{V.1.30b}$$

$$T_{\chi_d^0}=-b\sin\varphi_u\ v_u, \tag{V.1.30c}$$

$$T_{\chi_u^0}=-b\sin\varphi_u\ v_d. \tag{V.1.30d}$$

Similarly, the mass matrices of components of Higgs fields read

$$\mathbf{M}_{\phi\phi\chi\chi}=\begin{pmatrix}\mathbf{M}_\phi & \mathbf{M}_{\phi\chi}\\ \mathbf{M}_{\phi\chi}^\dagger & \mathbf{M}_\chi\end{pmatrix}, \tag{V.1.31a}$$

$$\mathbf{M}_\phi=\begin{pmatrix}M_{H_d}^2+\frac{g^2+g'^2}{8}\left(3v_d^2-v_u^2\right) & -b\cos\varphi_u-\frac{g^2+g'^2}{4}v_uv_d\\ -b\cos\varphi_u-\frac{g^2+g'^2}{4}v_uv_d & M_{H_u}^2+\frac{g^2+g'^2}{8}\left(3v_u^2-v_d^2\right)\end{pmatrix}, \tag{V.1.31b}$$

$$\mathbf{M}_{\phi\chi}=\begin{pmatrix}0 & b\sin\varphi_u\\ b\sin\varphi_u & 0\end{pmatrix}, \tag{V.1.31c}$$

$$\mathbf{M}_\chi=\begin{pmatrix}M_{H_d}^2+\frac{g^2+g'^2}{8}\left(v_d^2-v_u^2\right) & b\cos\varphi_u\\ b\cos\varphi_u & M_{H_u}^2+\frac{g^2+g'^2}{8}\left(v_u^2-v_d^2\right)\end{pmatrix}, \tag{V.1.31d}$$

$$\mathbf{M}_{\phi^\pm,\phi^\pm}=\begin{pmatrix}M_{H_d}^2+\frac{g^2}{8}\left(v_u^2+v_d^2\right)+\frac{g'^2}{8}\left(v_d^2-v_u^2\right) & be^{i\varphi_u}+\frac{g^2}{4}v_uv_d\\ be^{-i\varphi_u}+\frac{g^2}{4}v_uv_d & M_{H_u}^2+\frac{g^2}{8}\left(v_d^2+v_u^2\right)+\frac{g'^2}{8}\left(v_u^2-v_d^2\right)\end{pmatrix}. \tag{V.1.31e}$$

The nonvanishing phase factor $\sin\varphi_u$ would lead to a mixing between CP-even fields ϕ_u^0,ϕ_d^0 and CP-odd fields χ_u^0,χ_d^0. At tree-level, we require v_u and v_d are indeed stationary points of the Higgs potential, the following equations must be satisfied

$$\left.\frac{\partial V_H}{\partial\phi_d^0}\right|_{\phi_d^0,\phi_u^0,\chi_d^0,\chi_u^0=0}=\left.\frac{\partial V_H}{\partial\phi_u^0}\right|_{\phi_d^0,\phi_u^0,\chi_d^0,\chi_u^0=0}=\left.\frac{\partial V_H}{\partial\chi_d^0}\right|_{\phi_d^0,\phi_u^0,\chi_d^0,\chi_u^0=0}=\left.\frac{\partial V_H}{\partial\chi_u^0}\right|_{\phi_d^0,\phi_u^0,\chi_d^0,\chi_u^0=0}=0,$$

$$\Rightarrow T_{\phi_d^0}=T_{\phi_u^0}=T_{\chi_d^0}=T_{\chi_u^0}=0,$$

$$\Rightarrow \begin{cases} M_{H_d}^2 v_d - b\cos\varphi_u \; v_u - \dfrac{g^2+g'^2}{8}\left(v_u^2 - v_d^2\right) v_d = 0 \\ M_{H_u}^2 v_u - b\cos\varphi_u \; v_d - \dfrac{g^2+g'^2}{8}\left(v_d^2 - v_u^2\right) v_u = 0 \, . \\ b\sin\varphi_u \; v_u = 0 \\ b\sin\varphi_u \; v_d = 0 \end{cases} \tag{V.1.32}$$

There are some significant results we obtain from the above set of equations for stationary value

- If we set the parameters of soft SUSY breaking terms to zero, i.e $m_{H_u} = m_{H_d} = b = 0$, then Higgs doublets can only achieve a non-degenerate minimum at $v_u = v_d = 0$. Such vanishing VEVs would not allow the electroweak breaking due to Higgs mechanism, proving the integral role of soft-SUSY breaking terms.

- If the parameters b, m_{H_u}, m_{H_d} are non-zero, the above set of equations implies the vanishing of phase factor φ_u at tree level. Thus CP conservation in the Higgs sector still holds in MSSM at the lowest level of perturbation.

- Using the first two equations, we can solve for the mass terms $M_{H_d}^2$ and $M_{H_u}^2$ in terms of parameter b:

$$M_{H_d}^2 = b\,\frac{v_u}{v_d} + \frac{g^2+g'^2}{8}\left(v_u^2 - v_d^2\right), \tag{V.1.33}$$

$$M_{H_u}^2 = b\,\frac{v_d}{v_u} + \frac{g^2+g'^2}{8}\left(v_d^2 - v_u^2\right). \tag{V.1.34}$$

  The above condition for stationary of Higgs potential together with the definition of M_{H_u} and M_{H_d} in (V.1.25) insists that the SUSY parameter μ is of order of electroweak breaking scale, constrained by the RHS of Eq. (V.1.33) and (V.1.34). This requires the extreme fine-tuning of μ, which is known as the μ problem. A possible solution is by generating the μ parameter dynamically through EWSB of a singlet field is proposed in the next section of this chapter.

We now introduce the new parameters $\tan\beta = v_u/v_d$, $M_Z^2 = (g^2+g'^2)(v_u^2+v_d^2)/4$ and $M_W^2 = g^2(v_u^2+v_d^2)/4$. The mass matrices and the parameters $M_{H_d}^2$ and $M_{H_u}^2$ using Eq. (V.1.33) and (V.1.34) are

$$\begin{aligned} M_\phi &= \begin{pmatrix} b\tan\beta + \dfrac{g^2+g'^2}{4}v_d^2 & -b - \dfrac{g^2+g'^2}{4}v_u v_d \\ -b - \dfrac{g^2+g'^2}{4}v_u v_d & b\cot\beta + \dfrac{g^2+g'^2}{4}v_u^2 \end{pmatrix} \\ &= \begin{pmatrix} b\tan\beta + M_Z^2\cos^2\beta & -b - M_Z^2\sin\beta\cos\beta \\ -b - M_Z^2\sin\beta\;\cos\beta & b\cos\beta + M_Z^2\sin^2\beta \end{pmatrix}, \end{aligned} \tag{V.1.35a}$$

$$M_{\phi\chi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{V.1.35b}$$

$$M_\chi = \begin{pmatrix} b\tan\beta & b \\ b & b\cot\beta \end{pmatrix}, \tag{V.1.35c}$$

$$\begin{aligned} M_{\phi^\pm\phi^\pm} &= \begin{pmatrix} b\tan\beta + \dfrac{g^2}{4}v_u^2 & b + \dfrac{g^2}{4}v_u v_d \\ b + \dfrac{g^2}{4}v_u v_d & b\cot\beta + \dfrac{g^2}{4}v_d^2 \end{pmatrix} \\ &= \begin{pmatrix} b\tan\beta + M_W^2\sin^2\beta & b + M_W^2\sin\beta\;\cos\beta \\ b + M_W^2\sin\beta\;\cos\beta & b\cot\beta + M_W^2\cos^2\beta \end{pmatrix}. \end{aligned} \tag{V.1.35d}$$

Diagonalization of Higgs mass matrices in lowest order

We apply unitary transformations on both neutral and charged Higgs sectors to bring those into diagonalized form

$$\begin{pmatrix} h \\ H \\ A \\ G \end{pmatrix} = \begin{pmatrix} \sin\alpha & \cos\alpha & 0 & 0 \\ \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & 0 & \sin\beta & \cos\beta \\ 0 & 0 & \cos\beta & -\sin\beta \end{pmatrix} \begin{pmatrix} \phi_d^0 \\ \phi_u^0 \\ \chi_d^0 \\ \chi_u^0 \end{pmatrix}, \tag{V.1.36}$$

$$\begin{pmatrix} H^{\pm} & G^{\pm} \end{pmatrix} = \begin{pmatrix} \sin\beta & \cos\beta \\ \cos\beta & -\sin\beta \end{pmatrix} \begin{pmatrix} \phi_d^{\pm} \\ \phi_u^{\pm} \end{pmatrix}, \tag{V.1.37}$$

and obtain the following mass eigenvalues

$$m_G^2 = m_{G^{\pm}}^2 = 0, \tag{V.1.38}$$

$$m_A^2 = \frac{b}{\sin\beta\ \cos\beta}, \tag{V.1.39}$$

$$m_h^2 = \frac{1}{2}\left[m_A^2 + m_Z^2 - \sqrt{(m_A^2 - m_Z^2) + 4m_A^2 M_Z^2 \sin^2 2\beta}\right], \tag{V.1.40}$$

$$m_H^2 = \frac{1}{2}\left[m_A^2 + m_Z^2 + \sqrt{(m_A^2 - m_Z^2) + 4m_A^2 M_Z^2 \sin^2 2\beta}\right], \tag{V.1.41}$$

$$m_{H^{\pm}}^2 = m_A^2 + M_W^2. \tag{V.1.42}$$

We can construct a ratio between the first two elements of the normalized eigenvectors $(\sin\alpha, \cos\alpha, 0, 0)$ (the one that corresponds to eigenvalue m_h^2) to obtain the tangent of mixing angle α. The result is

$$\begin{aligned} \tan\alpha &= \frac{\left(m_A^2 + M_Z^2\right)^2 \sin 2\beta}{(M_Z^2 - m_A^2)\cos 2\beta + \sqrt{m_A^4 + M_Z^4 - 2m_A^2 M_Z^2 \cos 4\beta}} \\ &= \frac{\left(m_A^2 + M_Z^2\right)^2 \sin\beta\ \cos\beta}{M_Z^2 \cos^2\beta + m_A^2 \sin^2\beta + m_h^2}. \end{aligned} \tag{V.1.43}$$

Constraints on parameters in Higgs sector

Here we represent two constraints on the value of b. As claimed above, we are always safely to choose b, $v_u = \sqrt{2}\langle H_u^0\rangle$ and $v_d = \sqrt{2}\langle H_d^0\rangle$ to be positive. The first constraint on the parameters in Higgs potential comes from the fact that on the trend $v_u \to v_d$ (thus kills the final term involving $g^2 + g'^2$), the potential is bounded below only if

$$\begin{aligned} &M_{H_u}^2 v_u^2 + M_{H_d}^2 v_d^2 - 2b v_u v_d \geq 0 \\ &\Rightarrow 2b \leq M_{H_u}^2 + M_{H_d}^2 = 2|\mu|^2 + m_{H_u}^2 + m_{H_d}^2 + g'^2 \varphi_u. \end{aligned} \tag{V.1.44}$$

Now consider the Higgs potential as a function of VEVs, and we will see shortly that investigating this potential around its minimum gives another constrain on the MSSM parameters. The potential now reads

$$V_H(v_u, v_d) = M_{H_u}^2 v_u^2 + M_{H_d}^2 v_d^2 - 2b v_u v_d + \frac{g^2 + g'^2}{8}(v_u^2 - v_d^2)^2. \tag{V.1.45}$$

At the minimum, the VEVs necessary satisfy the following equations

$$\frac{\partial V_H}{v_u} = 2M_{H_u}^2 v_u - 2b v_d + \frac{g^2 + g'^2}{2} v_u (v_u^2 - v_d^2) = 0, \tag{V.1.46}$$

$$\frac{\partial V_H}{v_d} = 2M_{H_d}^2 v_d - 2b v_u - \frac{g^2 + g'^2}{2} v_d (v_u^2 - v_d^2) = 0. \tag{V.1.47}$$

Let us introduce some new variables: $v^2 \equiv v_u^2 + v_d^2$, $(g^2 + g'^2)v^2 = 4M_Z^2$, $\tan\beta = v_u/v_d$. On one side,

$$\begin{aligned} &0 = v_d \frac{\partial V_H}{\partial v_u} + v_u \frac{\partial V_H}{\partial v_d} = 2(M_{H_u}^2 + M_{H_d}^2) v_u v_d - 2b(v_u^2 + v_d^2) \\ &\Rightarrow \frac{b}{M_{H_u}^2 + M_{H_d}^2} = \frac{v_u v_d}{v_u^2 + v_d^2} = \sin\beta\cos\beta \quad \Rightarrow \quad \tan^2\beta = \frac{b^2(1 + \tan^2\beta)^2}{(M_{H_u}^2 + M_{H_d}^2)^2}. \end{aligned} \tag{V.1.48}$$

On the other hand

$$0 = v_u \frac{\partial V_H}{\partial v_u} - v_d \frac{\partial V_H}{\partial v_d} = 2M_{H_u}^2 v_u^2 - 2M_{H_d}^2 v_d^2 + \frac{g^2 + g'^2}{2}(v_u^2 + v_d^2)(v_u^2 - v_d^2)$$

$$\Rightarrow 0 = v_u^2\left(M_{H_u}^2 + M_Z^2\right) - v_d^2\left(M_{H_d}^2 + M_Z^2\right) \quad \Rightarrow \quad \tan^2\beta = \frac{M_{H_d}^2 + M_Z^2}{M_{H_u}^2 + M_Z^2}. \tag{V.1.49}$$

Use Eq. (V.1.48) and (V.1.49) to obtain an expression without β variable:

$$\begin{aligned}
&\frac{M_{H_d}^2 + M_Z^2}{M_{H_u}^2 + M_Z^2} = \frac{b^2}{\left(M_{H_u}^2 + M_{H_d}^2\right)}\left(1 + \frac{M_{H_d}^2 + M_Z^2}{M_{H_u}^2 + M_Z^2}\right)^2 \\
&\Rightarrow b^2\left(M_{H_u}^2 + M_{H_d}^2 + 2M_Z^2\right)^2 = \left(M_{H_u}^2 + M_Z^2\right)\left(M_{H_d}^2 + M_Z^2\right)\left(M_{H_u}^2 + M_{H_d}^2\right)^2.
\end{aligned} \tag{V.1.50}$$

Now let us consider the following expression, and plugging in the Eq. (V.1.50) to replace the b variable by an expression containing only masses

$$\begin{aligned}
&\left(b^2 - M_u^2 M_d^2\right)\left(M_{H_u}^2 + M_{H_d}^2 + 2M_Z^2\right)^2 \\
&= \left(M_{H_u}^2 + M_Z^2\right)\left(M_{H_d}^2 + M_Z^2\right)\left(M_{H_u}^2 + M_{H_d}^2\right)^2 - M_u^2 M_d^2\left(M_{H_u}^2 + M_{H_d}^2 + 2M_Z^2\right)^2.
\end{aligned} \tag{V.1.51}$$

After some algebra to expand the RHS and collect the mutual factors, we end up with the following expression:

$$\begin{aligned}
&\left(b^2 - M_{H_u}^2 M_{H_d}^2\right)\left(M_{H_u}^2 + M_{H_d}^2 + 2M_Z^2\right)^2 \\
&= M_Z^2\left(M_{H_u} - M_{H_d}\right)^2\left(M_{H_u} + M_{H_d}\right)^2\left(M_{H_u}^2 + M_{H_d}^2 + M_Z^2\right).
\end{aligned} \tag{V.1.52}$$

The RHS is positive definite, so does the LHS. Thus we achieve the second constraint on variable b, in the form of inequality

$$b^2 \geq M_{H_u}^2 M_{H_d}^2 = \left(\mu^2 + m_{H_u}^2\right)\left(\mu^2 + m_{H_d}^2\right), \tag{V.1.53}$$

where the equality holds if and only if $M_{H_u} = M_{H_d} \Rightarrow m_{H_u} = m_{H_d}$. In summary, the parameters in MSSM Lagrangian must satisfy the two constraints in Eq. (V.1.44) and (V.1.53)

Gauge bosons masses

To obtain the mass of gauge boson, we consider the kinetic parts of Higgs' Kähler potential, which read

$$\mathcal{L} = \overline{D_\mu^{H_u} H_u} D_\mu^{H_u} H_u + \overline{D_\mu^{H_d} H_d} D_\mu^{H_d} H_d. \tag{V.1.54}$$

Since we need the mass terms only, we temporarily ignore the Higgs' perturbation around vacuum. The covariant derivatives acting on Higgs fields give

$$\begin{aligned}
D_\mu^{H_u} H_u &\to \left(\partial_\mu - ig\frac{\sigma^b}{2}W_\mu^b - ig'\frac{Y(H_u)}{2}B_\mu\right)\begin{pmatrix} 0 \\ v_u/\sqrt{2}\end{pmatrix} \\
&= -\frac{ig}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} 0 \\ \frac{v_u}{\sqrt{2}}\end{pmatrix} - \frac{i}{2}g'B_\mu\begin{pmatrix} 0 \\ \frac{v_u}{\sqrt{2}}\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{ig}{2}\left(W_\mu^1 - iW_\mu^2\right)v_u \\ \frac{ig}{2}W_\mu^3 v_u - \frac{i}{2}g'B_\mu v_u \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{ig}{2}v_u\left(W_\mu^1 - iW_\mu^2\right) \\ \frac{i}{2}v_u\left(gW_\mu^3 - g'B_\mu\right)\end{pmatrix},
\end{aligned} \tag{V.1.55}$$

$$\begin{aligned}
D_\mu^{H_d} H_d &\to \left(\partial_\mu - ig\frac{\sigma^b}{2}W_\mu^b - ig'\frac{Y(H_d)}{2}B_\mu\right)\begin{pmatrix} v_d/\sqrt{2} \\ 0\end{pmatrix} \\
&= -\frac{ig}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix}\begin{pmatrix} \frac{v_d}{\sqrt{2}} \\ 0\end{pmatrix} + \frac{i}{2}g'B_\mu\begin{pmatrix} \frac{v_d}{\sqrt{2}} \\ 0\end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{i}{2}v_d\left(gW_\mu^3 - g'B_\mu\right) \\ -\frac{ig}{2}v_d\left(W_\mu^1 + iW_\mu^2\right)\end{pmatrix}.
\end{aligned} \tag{V.1.56}$$

Expanding the Higgs kinetic part in terms of component gauge fields

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} -\frac{ig}{2}v_u\left(W^1_\mu - iW^2_\mu\right) \\ \frac{i}{2}v_u\left(gW^3_\mu - g'B_\mu\right)\end{pmatrix}^\dagger \begin{pmatrix} -\frac{ig}{2}v_u\left(W^1_\mu - iW^2_\mu\right) \\ \frac{i}{2}v_u\left(gW^3_\mu - g'B_\mu\right)\end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\frac{i}{2}v_d\left(gW^3_\mu - g'B_\mu\right) \\ -\frac{ig}{2}v_d\left(W^1_\mu + iW^2_\mu\right)\end{pmatrix}^\dagger \begin{pmatrix} -\frac{i}{2}v_d\left(gW^3_\mu - g'B_\mu\right) \\ -\frac{ig}{2}v_d\left(W^1_\mu + iW^2_\mu\right)\end{pmatrix}$$

$$= \frac{v_u^2 g^2}{8}|W^1_\mu - iW^2_\mu|^2 + \frac{v_u^2}{8}|gW^3_\mu - g'B_\mu|^2 + (v_u \leftrightarrow v_d)$$

$$= \frac{(v_u^2+v_d^2)g^2}{8}|W^1_\mu - iW^2_\mu|^2 + \frac{v_u^2+v_d^2}{8}|gW^3_\mu - g'B_\mu|^2. \tag{V.1.57}$$

Redefine the W-field as $W^\pm_\mu \equiv \dfrac{W^1_\mu \mp iW^2_\mu}{\sqrt{2}}$, and apply the Weinberg rotation

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} s_W & c_W \\ c_W & -s_W \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}, \text{ where the Weinberg angle: } \begin{cases} c_W &= \dfrac{g}{\sqrt{g^2+g'^2}} \\ s_W &= \dfrac{g'}{\sqrt{g^2+g'^2}} \end{cases}, \tag{V.1.58}$$

we finally obtain the mass term of W and B bosons

$$\mathcal{L} = \frac{g^2(v_u^2+v_d^2)}{8}(W^-_\mu)^\dagger W^{-\mu} + \frac{g^2(v_u^2+v_d^2)}{8}(W^+_\mu)^\dagger W^{+\mu} + \frac{(g^2+g'^2)(v_u^2+v_d^2)}{8}Z_\mu, \tag{V.1.59}$$

which gives the expected results, and explaining the name of the parameters we used above

$$M_{W_+} = M_{W_-} = M_W = \frac{g\sqrt{v_u^2+v_d^2}}{2}, \tag{V.1.60}$$

$$M_Z = \frac{\sqrt{g^2+g'^2}\sqrt{v_u^2+v_d^2}}{2}, \tag{V.1.61}$$

$$M_\gamma = 0. \tag{V.1.62}$$

Neutralinos & charginos masses

The four mass eigenstates *neutralinos* arise from the mixing between the neutral higgsinos $\tilde{H}^0_u$, $\tilde{H}^0_d$ and the neutral gauginos $\tilde{B}, \tilde{W}^3$. Similarly the charged higgsinos $\tilde{H}^+_u$ and $\tilde{H}^-_d$ and the pair of winos $\tilde{W}^\pm$ combine to form two mass eigenstates with electric charge ± 1 called *charginos*. These mass eigenstates are conventionally labelled in ascending order such that $m_{\tilde{\chi}^0_1} < m_{\tilde{\chi}^0_2} < m_{\tilde{\chi}^0_3} < m_{\tilde{\chi}^0_4}$ and $m_{\tilde{\chi}^\pm_1} < m_{\tilde{\chi}^\pm_2}$.

The Lagrangian for neutralino masses in the gauge-eigenstates $\tilde{\psi}^0 \equiv \left(\tilde{B}, \tilde{W}^3, \tilde{H}^0_d, \tilde{H}^0_u\right)$ is written in the following quadratic form

$$\mathcal{L}_{\tilde{\chi}^0} = -\frac{1}{2}\left(\tilde{\psi}^0\right)^T \mathbf{M}_{\tilde{\chi}^0}\tilde{\psi}^0 + h.c, \tag{V.1.63}$$

with the symmetric mass matrix

$$\mathbf{M}_{\tilde{\chi}^0} = \begin{pmatrix} M_1 & 0 & -M_Z s_W c_\beta & e^{-i\varphi_u}M_Z s_W s_\beta \\ 0 & M_2 & M_Z c_W c_\beta & -e^{-i\varphi_u}M_Z c_W s_\beta \\ -c_\beta s_W M_Z & c_\beta c_W M_Z & 0 & -\mu \\ s_\beta s_W M_Z & -e^{-i\varphi_u}s_\beta c_W M_Z & -\mu & 0 \end{pmatrix}. \tag{V.1.64}$$

The entries M_1 and M_2 comes from the soft SUSY breaking term (V.1.19) whereas the entries $-\mu$ originated from the supersymmetric higgsino mass terms in the scalar potential (V.1.14). Finally, the couplings of Higgs-higgsino-gaugino in the Kähler potential lead to the entries containing g and g'. This matrix can be diagonalized by apply a unitary transformation $\mathbf{N}$

$$\mathbf{N}^* M_{\tilde{\chi}^0}\mathbf{N}^\dagger = \begin{pmatrix} m_{\tilde{\chi}^0_1} & 0 & 0 & 0 \\ 0 & m_{\tilde{\chi}^0_2} & 0 & 0 \\ 0 & 0 & m_{\tilde{\chi}^0_3} & 0 \\ 0 & 0 & 0 & m_{\tilde{\chi}^0_4} \end{pmatrix}, \tag{V.1.65}$$

and the neutralinos are given by $\tilde{\chi}_i^0 = \mathbf{N}_{ij}\tilde{\psi}_j^0$.

The superpartners of the charged Higgs and gauge bosons form a charged basis, which is expressed in terms of the Weyl spinors as

$$\psi_R^- \equiv \begin{pmatrix} \tilde{W}^- \\ H_d^- \end{pmatrix}, \qquad \psi_L^+ \equiv \begin{pmatrix} \tilde{W}^+ \\ H_u^+ \end{pmatrix}. \tag{V.1.66}$$

The chargino Lagrangian in this basis is written as

$$\mathcal{L}_{\tilde{\chi}^\pm} = \left(\psi_R^-\right)^T M_{\tilde{\chi}^\pm}\psi_L^+ + h.c, \tag{V.1.67}$$

where the corresponding mass matrix

$$M_{\tilde{\chi}^\pm} = \begin{pmatrix} M_2 & \sqrt{2}s_\beta M_W e^{-i\varphi_u} \\ \sqrt{2}c_\beta M_W & \mu \end{pmatrix}, \tag{V.1.68}$$

with the M_2 entry sterming from the soft breaking (V.1.19), the μ entry from scalar potential (V.1.14) and the other two is from the Kähler potential (V.1.12). The chargino mass eigenvalues can be obtained with the help of two unitary 2×2 matrices U and V according to

$$\tilde{\chi}_L^+ = V\psi_L^+, \qquad \tilde{\chi}_R^- = U\psi_R^-. \tag{V.1.69}$$

Generally the rotation matrix U and V is different since the mass matrix is not in symmetric form. After diagonalization we obtain the masses of charginos

$$\mathbf{U}^* M_{\tilde{\chi}^\pm}\mathbf{V}^\dagger = \begin{pmatrix} m_{\tilde{\chi}_1^\pm} & 0 \\ 0 & m_{\tilde{\chi}_2^\pm} \end{pmatrix}, \tag{V.1.70}$$

where

$$m^2_{\tilde{\chi}_1^\pm}, m^2_{\tilde{\chi}_2^\pm} = \frac{1}{2}\left[|M_2|^2 + |\mu|^2 + 2M_W^2 \mp \sqrt{\left(|M_2|^2 + |\mu|^2 + 2M_W^2\right)^2 - 4\left|\mu M_2 - e^{-i\phi_u}M_W^2 s_{2\beta}\right|^2}\right]. \tag{V.1.71}$$

The gluino

The gluino is the superpartner of gluons. This sparticle is a color octet fermion and thus cannot mix with any other particles in the MSSM like neutralinos or charginos. Furthermore, the gluino does not couple with the Higgs field. Hence, the mass term of gluino is determined solely by the soft breaking parameter:

$$m_{\tilde{G}} = |M_3|\,. \tag{V.1.72}$$

This unique property also holds in the NMSSM where we make a modification by adding one singlet scalar superfield $\hat{S}$.

Quarks & leptons

The masses of quarks and leptons come from the EWSB of the trilinear Yukawa couplings with scalar Higgs field. All of these information are contained in the MSSM superpotential (V.1.12). The only differece is we introduce two Higgs doublet in the MSSM: H_u couples with up-type quarks and H_d couples with down-type quarks and leptons. Hence, the quarks and leptons masses and mixing angles in the family space not only determined by the Yukawa coupling matrix $\mathbf{Y}_f$ but also the parameter $\tan\beta = v_u/v_d$. Let us call y_ℓ, y_{q_d} and y_{q_u} are respectively the eigenvalues of the Yukawa matrix $\mathbf{Y}_e$, $\mathbf{Y}_d$ and $\mathbf{Y}_u$, where $\ell = e, \mu, \tau$, $q_d = d, s, b$ and $q_u = u, c, t$. The masses of the charged leptons, up-type quarks and down-type quarks have the similar form as follows

$$m_\ell = \frac{y_\ell v_d}{\sqrt{2}} = \frac{y_\ell v c_\beta}{\sqrt{2}}, \quad m_d = \frac{y_d v_d}{\sqrt{2}} = \frac{y_d v c_\beta}{\sqrt{2}}, \quad m_u = \frac{y_u v_u}{\sqrt{2}} = \frac{y_u v s_\beta}{\sqrt{2}} \tag{V.1.73}$$

While the experimental measurements suggest that mass eigenstates of charged leptons are identical with the flavour eigenstates, this is not the case for quarks. Similar to SM, the mixing among quarks in the family space is non-vanishing. Diagonalizing the Yukawa matrices $\mathbf{Y}_u, \mathbf{Y}_d$ thus requires four unitary matrices $\mathbf{V}^{L,R}_{u,d}$ such that

$$\mathbf{M}_u \equiv \mathrm{diag}(m_u, m_c, m_t) = \frac{v_u}{\sqrt{2}} \mathbf{V}^L_u \mathbf{Y}_u \mathbf{V}^R_u, \tag{V.1.74}$$

$$\mathbf{M}_d \equiv \mathrm{diag}(m_d, m_s, m_b) = \frac{v_d}{\sqrt{2}} \mathbf{V}^L_d \mathbf{Y}_d \mathbf{V}^R_d. \tag{V.1.75}$$

Squarks & sleptons

The mass matrices of sfermions (including squarks and sleptons) can be written collectively in the basis $(\tilde{f}_L, \tilde{f}_R)$ as

$$\mathbf{M}_{\tilde{f}} = \begin{pmatrix} M_Z^2 c_{2\beta} \left(I_3^f - Q^f s_W^2 \right) \mathbb{1}_{3\times 3} + \mathbf{M}_{\tilde{f}_L} + \mathbf{m}_f^* \mathbf{m}_f^T & \mathbf{m}_f^* \mathbf{X}_f^* \\ \mathbf{X}_f^T \mathbf{m}_f^T & \mathbf{m}_f^T \mathbf{m}_f^* + \mathbf{M}^2_{\tilde{f}_R} + M_Z^2 c_{2\beta} Q^f s_W^2 \mathbb{1}_{3\times 3} \end{pmatrix}, \tag{V.1.76}$$

where the index $f = e, u, d$; I_3^f and Q^f are the isospin and electric charge of the fermion. $\mathbf{M}^2_{\tilde{f}_L}$ is either $\mathbf{M}^2_{\tilde{Q}}$ for squarks or $\mathbf{M}^2_{\tilde{L}}$ for sleptons. $\mathbf{m}_f$ is the mass matrix for considered fermion in the generation space, which is $\mathbf{Y}_e v_d/\sqrt{2}$ for leptons, $\mathbf{Y}_d v_d/\sqrt{2}$ for down-type quarks or $e^{-i\varphi_u} \mathbf{Y}_u v_u/\sqrt{2}$ for up-type quarks. The 3×3 matrix $\mathbf{X}_f$ is defined via

$$\mathbf{X}_f \equiv \mathbf{A}_f - e^{-i\varphi_u} \mu^* \left(\cot\beta\right)^{2I_3^f} \mathbb{1}_{3\times 3}. \tag{V.1.77}$$

When there is no mixing term between different generations of sfermions and fermions, the matrices $\mathbf{M}_{\tilde{f}_{L,R}}$, $\mathbf{m}_f$, $\mathbf{X}_f$ are all diagonal matrices. In such case, the general sfermion mass matrix (V.1.76) is decomposed as a direct sum of 2×2 mass matrix in each generation, i.e $\mathbf{M}_{\tilde{f}} = \mathbf{M}_{\tilde{f}_1} \oplus \mathbf{M}_{\tilde{f}_2} \oplus \mathbf{M}_{\tilde{f}_3}$, and the diagonalization can be performed analytically by a rotation

$$\begin{pmatrix} \tilde{f}_{i,1} \\ \tilde{f}_{i,2} \end{pmatrix} = \mathbf{U}_{\tilde{f}} \begin{pmatrix} \tilde{f}_{iL} \\ \tilde{f}_{iR} \end{pmatrix} = \begin{pmatrix} -\sin\theta_{\tilde{f}_i} & \cos\theta_{\tilde{f}_i} \\ \cos\theta_{\tilde{f}_i} & \sin\theta_{\tilde{f}_i} \end{pmatrix} \begin{pmatrix} \tilde{f}_{iL} \\ \tilde{f}_{iR} \end{pmatrix}, \tag{V.1.78}$$

with the rotation angle

$$\begin{aligned} \cot\theta_{\tilde{f}_i} = -\frac{1}{2m_{f_i} X^*_{f_i}} \Bigg\{ & M^2_{\tilde{f}_{iL}} - M^2_{\tilde{f}_{iR}} + M_Z^2 c_{2\beta} \left(I_3^f - 2Q^f s_W^2 \right) \\ & + \sqrt{\left[M^2_{\tilde{f}_{iL}} - M^2_{\tilde{f}_{iR}} + M_Z^2 c_{2\beta} \left(I_3^f - 2Q^f s_W^2 \right) \right]^2 + 4m^2_{f_i} \left| X_{f_i} \right|} \Bigg\}, \end{aligned} \tag{V.1.79}$$

where $X_{f_i} = A_{f_i} - e^{-i\phi_u} \mu^* (\cot\beta)^{2I_3^f}$.

The mass eigenvalues of each 2×2 block matrix after diagonalization are

$$\begin{aligned} m^2_{\tilde{f}_i 1,2} = \frac{1}{2} \Bigg\{ & 2m^2_{f_i} + M^2_{\tilde{f}_{iL}} + M^2_{\tilde{f}_{iR}} + I_3^f M_Z^2 c_{2\beta} \\ & \mp \sqrt{\left[M^2_{\tilde{f}_{iL}} - M^2_{\tilde{f}_{iR}} + M_Z^2 c_{2\beta} \left(I_3^f - 2Q^f s_W^2 \right) \right]^2 + 4m^2_{f_i} \left| X_{f_i} \right|} \Bigg\}. \end{aligned} \tag{V.1.80}$$

As far as we do not concern about adding the right-chiral neutrinos into the MSSM (with similar argument in the SM: the right-chiral neutrinos have no gauge interaction with other particles and thus not being considered as a functional part in the model), the right-chiral sneutrino do not exist. The sneutrino masses is thus 1-dimensional, written in the basis $\tilde{\nu}_{i,L} = \tilde{\nu}_i$ as (assuming the decoupling between different flavours)

$$m^2_{\tilde{\nu}} = M^2_{\tilde{L}} + \frac{1}{2} M_Z^2 c_{2\beta}, \tag{V.1.81}$$

with the first contribution comes from the soft SUSY breaking term, and the latter is originated from the scalar potential.

V.2 Next-to-minimal Supersymmetric Standard Model

With the simplest extension of the SM containing the supersymmetry, we face the μ problem mentioned above. Recall that in the MSSM framework, the μ parameter has no priory reason for constraint the value of μ in the desired electroweak scale, i.e typically $\mu \sim M_{rmSUSY}$. On the other hand, various phenomenological analysis and experimental measurements suggests that $|\mu|$ should not exceed significantly the mass of Z-boson. The μ problem relates to the fact that a parameter at SUSY scale involves in the electroweak symmetry breaking, which require an extreme fine-tuning of MSSM parameter μ. This is the main motivation for one to go beyond the MSSM, and the section aims to represent such extension by adding a gauge singlet chiral superfield $\hat{S}$ into the particle contents. In the model so-called *Next-to-minimal Supersymmetric Standard Model* (NMSSM), the μ parameter is generated dynamically by the VEV of newly added singlet S and thus resolves the issue of MSSM.

V.2.1 NMSSM Particle Content

The general NMSSM Lagrangian is obtained from the MSSM Lagrangian by adding a gauge singlet chiral superfield $\hat{S}$ into the particle contents. We thus introduce to the NMSSM two more Higgs field (one scalar and one pseudoscalar) compared to the MSSM.

Super Field	**Spin 0**	**Spin 1/2**	$\mathbf{SU(3)_C \times SU(2)_L \times U(1)_Y}$
$\hat{S}$	S	$\tilde{S}$	$(\mathbf{1}, \mathbf{1}, \mathbf{0})$

Table V.3: The newly introduced complex scalar singlet in NMSSM.

NMSSM thus becomes the simplest extension of the MSSM with the μ term is generated by EWSB process, with only one more gauge singlet being considered. The rest of the particle contents is totally similar to that of MSSM given in the Table V.2 and Table V.1.

V.2.2 NMSSM Lagrangian

Before we start, we want to note that most of the NMSSM Lagrangian can be derived the same way as of the MSSM. We do not repeat those derivation, and represent in this section only the differences arising from the additional Higgs singlet. Explicit changes are the kinetic term for $\hat{S}$ in Kähler potential, the contribution of this singlet in the superpotential and finally the soft-SUSY breaking term. Let us consider this extension of MSSM which posesses a superpotential composed of a Yukawa term and a Higgs term

$$W^{\text{NMSSM}} = W^{\text{NMSSM}}_{Higgs} + W_{Yukawa}. \tag{V.2.1}$$

Apparently, the Yukawa couplings remains intact while a slight modification on the Higgs superpotential due to the newly added superfield $\hat{S}$ can be achieved as follows [1]

$$W_{Yukawa} = \hat{H}_d \cdot \hat{L}\mathbf{Y}_e \hat{e}^c + \hat{H}_d \cdot \hat{Q}\mathbf{Y}_d \hat{d}^c - \hat{H}_u \cdot \hat{Q}\mathbf{Y}_u \hat{u}^c, \tag{V.2.2}$$

$$W^{\text{NMSSM}}_{Higgs} = \left(\mu + \lambda\hat{S}\right)\hat{H}_u \cdot \hat{H}_d + \xi_F \hat{S} + \frac{1}{2}\mu' \hat{S}^2 + \frac{\kappa}{3}\hat{S}^3, \tag{V.2.3}$$

where the terms $\sim \lambda, \kappa$ are dimensionless parameters, the terms $\sim \mu, \mu'$ are supersymmetric mass terms, and the dimension-two parameter ξ_F parametrizes a supersymmetric tadpole term. Similarly the corresponding soft SUSY breaking masses and couplings is

$$\begin{aligned}
\mathcal{L}^{\text{NMSSM}}_{soft} = &-\frac{1}{2}\left(M_{\tilde{G}}\tilde{G}^a\tilde{G}^a + M_{\tilde{W}}\tilde{W}^b\tilde{W}^b + M_{\tilde{B}}\tilde{B}\tilde{B} + h.c\right) \\
&- \overline{\tilde{Q}}\mathbf{M}^2_{\tilde{Q}}\tilde{Q} - \overline{\tilde{L}}\mathbf{M}^2_{\tilde{L}}\tilde{L} - \overline{\tilde{u}^c}\mathbf{M}_{\tilde{u}}\tilde{u}^c - \overline{\tilde{d}}^c\mathbf{M}_{\tilde{d}}\tilde{d}^c - \overline{\tilde{e}}^c\mathbf{M}_{\tilde{e}}\tilde{e}^c \\
&- \left(m^2_{H_u}\overline{H}_u H_u + m^2_{H_d}\overline{H}_d H_d + m^2_S|S|^2\right) \\
&- \left(\lambda A_\lambda H_u \cdot H_d S + \frac{\kappa}{3}A_\kappa S^3 + m^2_3 H_u \cdot H_d + \frac{1}{2}m'^2_S S^2 + \xi_S S\right) \\
&- \left(\overline{\tilde{u}}^c\mathbf{Y_u}\mathbf{A_u}\tilde{Q} \cdot H_u - \overline{\tilde{d}}^c\mathbf{Y_d}\mathbf{A_u}\tilde{Q} \cdot H_d - \overline{\tilde{e}}^c\mathbf{Y_u}\mathbf{A_u}\tilde{Q} \cdot H_d + h.c\right),
\end{aligned} \tag{V.2.4}$$

[1] Here we keep the R-parity conservation that is dicussed in the construction of MSSM superpotential.

where the first three lines are respectively mass terms of gauginos, Higgs and sfermions. The fourth line contains interaction between scalars, and the last line is composed of trilinear couplings between Higgs field and sfermions.

We notice that all new dimensionful parameters in NMSSM (namely $\mu, \mu', \xi_F, m_3^2, m_S'^2, \xi_S$) also participate in the electroweak symmetry breaking, thus being bounded by the electroweak scale; but naturally these parameters arise naturally from the supersymmetry, and thus on SUSY scale. The exclusion of scale-dependent parameters can be achieved by requiring the Lagrangian to possess a $\mathbb{Z}_3$ symmetry (meaning to rotate all components of all chiral superfields with a phase $e^{2\pi i/3}$), thus only terms that are product of three superfields survive. In short, we obtain the $\mathbb{Z}_3$-invariant NMSSM from the general one by simply setting the dimensionful parameters to zero

$$m_3^2 = m_S'^2 = \xi_S = \mu = \mu' = \xi_F = 0. \tag{V.2.5}$$

The Higgs sector of the $\mathbb{Z}_3$-invariant NMSSM is thus specified by seven parameters λ, κ, $m_{H_u}^2$, $m_{H_d}^2$, m_S^2, A_λ and A_κ. From now on the $\mathbb{Z}_3$ symmetry will always be employed to eliminate the undesired parameters, and NMSSM in this context stands for $\mathbb{Z}_3$-invariant NMSSM.

To summarize, we are working with the supersymmetric potential

$$\begin{aligned} W_{NMSSM} = &\left(\hat{H}_d^0 \hat{e}_L - \hat{H}_d^- \hat{\nu}\right) \mathbf{Y}_e \hat{e}^c + \left(\hat{H}_d^0 \hat{d}_L - \hat{H}_d^- \hat{u}_L\right) \mathbf{Y}_d \hat{d}^c - \left(\hat{H}_u^+ \hat{d}_L - \hat{H}_u^0 \hat{u}_L\right) \mathbf{Y}_u \hat{u}^c \\ &+ \lambda \hat{S} \left(\hat{H}_u^+ \hat{H}_d^- - \hat{H}_u^0 \hat{H}_d^0\right) + \frac{\kappa}{3} \hat{S}^3, \end{aligned} \tag{V.2.6}$$

and the soft breaking term (after setting $m_3^2, m_S'^2$ and ξ_S to zero)

$$\begin{aligned} \mathcal{L}_{soft}^{\text{NMSSM}} = &- \frac{1}{2} \left(M_{\tilde{G}} \tilde{G}^a \tilde{G}^a + M_{\tilde{W}} \tilde{W}^b \tilde{W}^b + M_{\tilde{B}} \tilde{B} \tilde{B} + h.c\right) \\ &- \left(m_{H_u}^2 \overline{H}_u H_u + m_{H_d}^2 \overline{H}_d H_d + m_S^2 |S|^2\right) - \left(\lambda A_\lambda H_u \cdot H_d S + \frac{\kappa}{3} A_\kappa S^3\right) \\ &- \overline{\tilde{Q}} \mathbf{M}_{\tilde{Q}}^2 \tilde{Q} - \overline{\tilde{L}} \mathbf{M}_{\tilde{L}}^2 \tilde{L} - \overline{\tilde{u}^c} \mathbf{M}_{\tilde{u}} \tilde{u}^c - \overline{\tilde{d}}^c \mathbf{M}_{\tilde{d}} \tilde{d}^c - \overline{\tilde{e}}^c \mathbf{M}_{\tilde{e}} \tilde{e}^c \\ &- \left(\overline{\tilde{u}}^c \mathbf{Y_u A_u} \tilde{Q} \cdot H_u - \overline{\tilde{d}}^c \mathbf{Y_d A_u} \tilde{Q} \cdot H_d - \overline{\tilde{e}}^c \mathbf{Y_u A_u} \tilde{Q} \cdot H_d + h.c\right). \end{aligned} \tag{V.2.7}$$

V.2.3 Tree level mass spectrum in the NMSSM

With new singlet superfield $\hat{S}$ being added, the mass spectrum of NMSSM is slightly modified compared to that of MSSM and require careful investigations. This subsection thus aims to clarify the differences between these two supersymmetric extension of SM, including the Higgs sector, the neutralino. For other sectors, we refer the readers to Section V.1.3.

The Neutral Higgs Sector

From the general SUSY Lagrangian, one can extract the Higgs potential (which contains H_u, H_d and newly introduced singlet S) as follow

$$\begin{aligned} V_H^{\text{NMSSM}} = &\left(|\lambda S|^2 + m_{H_d}^2\right) \left(|H_d^-|^2 + |H_d^0|^2\right) + \left(|\lambda S|^2 + m_{H_u}^2\right) \left(|H_u^+|^2 + |H_u^0|^2\right) + m_S^2 |S|^2 \\ &+ \frac{g^2}{2} \left|\overline{H}_u^+ H_d^0 + \overline{H}_u^0 H_d^-\right|^2 + \frac{g^2 + g'^2}{8} \left(|H_u^+|^2 + |H_u^0|^2 - |H_d^0|^2 - |H_d^-|^2\right)^2 \\ &+ \left|\lambda \left(H_u^+ H_d^- - H_u^0 H_d^0\right) + \kappa S^2\right|^2 + \left[-\lambda A_\lambda S \left(H_u^+ H_d^- - H_u^0 H_d^0\right) + \frac{\kappa}{3} A_\kappa S^3 + h.c\right]. \end{aligned} \tag{V.2.8}$$

After the symmetry $SU(2)_L \times U(1)_Y$ is broken, each of the scalar Higgs field can be expanded around its vacuum generally as

$$H_d = \begin{pmatrix} (v_d + h_d + i a_d)/\sqrt{2} \\ h_d^- \end{pmatrix}, \quad H_u = e^{i\varphi_u} \begin{pmatrix} h_u^+ \\ (v_u + h_u + i a_u) \end{pmatrix}, \quad S = \frac{e^{i\varphi_s}}{\sqrt{2}} \left(v_s + h_s + i a_s\right), \tag{V.2.9}$$

where φ_u and φ_s are possible relative phases of H_u and S with respect to H_d. Due to $U(1)$ symmetry, the VEVs v_d, v_u and v_s can be chosen to be positive. After plugging these expansion into Eq. (V.2.8), the Higgs potential can be written as a sum of terms containing one neutral field (tadpoles), two fields (quadratic terms) and higher order (Higgs interactions). More specifically, we want to express the Higgs potential in the following form:

$$V_H^{\text{NMSSM}} = const + t_{h_d} h_d + t_{h_u} h_u + t_{h_s} h_s + t_{a_d} a_d + t_{a_u} a_u + t_{a_s} a_s \\ + \frac{1}{2}\phi^{0T}\mathbf{M}_{\phi\phi}\phi^0 + \overline{\phi}^c\mathbf{M}_{h^+h^-}\phi^c + \text{ higher order of } \phi, \tag{V.2.10}$$

where the neutral Higgs vector $\phi^{0T} = (h_d, h_u, h_s, a_d, a_u, a_s)$, the charged Higgs vector $\overline{\phi}^c = (h_d^-, \overline{h}_u^+)$, our task now is to determine the tadpole coefficients $t_{h_d}, t_{h_u}, t_{h_s}, t_{a_d}, t_{a_u}, t_{a_s}$, the neutral Higgs mass matrix $\mathbf{M}_{\phi\phi}$ and the charged Higgs mass matrix $\mathbf{M}_{h^+h^-}$. The const part has no contribution to the symmetry breaking process, and not of our interests. The final term contains three and four scalar interactions. For detailed discussion about the tadpole coefficients and the mass martices, see e.g [44, section 2, 3].

The tadpole coefficients are obtained via the first derivative of the Higgs potential with respect to the corresponding fields

$$t_{\phi_i} = \left\langle \frac{\partial V}{\partial \phi_i} \right\rangle = \left.\frac{\partial V}{\partial \phi_i}\right|_{\phi=0}, \quad \phi_i = h_d, h_u, h_s, a_d, a_u, a_s, \tag{V.2.11}$$

from which we can derive the following results

$$t_{h_d} = \frac{2M_W s_W c_\beta}{e}\left[m_{H_d}^2 + \frac{M_Z^2 c_{2\beta}}{2} + |\lambda|^2\left(\frac{2M_W^2 s_W^2 s_\beta^2}{e^2} + \frac{v_s^2}{2}\right) - \frac{|\lambda| v_s t_\beta}{2}\left(\sqrt{2}A_\lambda c_{\varphi_x} + v_s \kappa c_{\varphi_y}\right)\right], \tag{V.2.12}$$

$$t_{h_u} = \frac{2M_W s_W s_\beta}{e}\left[m_{H_u}^2 - \frac{M_Z^2 c_{2\beta}}{2} + |\lambda|^2\left(\frac{2M_W^2 s_W^2 c_\beta^2}{e^2} + \frac{v_s^2}{2}\right) - \frac{|\lambda| v_s}{2t_\beta}\left(\sqrt{2}A_\lambda c_{\varphi_x} + v_s \kappa c_{\varphi_y}\right)\right], \tag{V.2.13}$$

$$t_{h_s} = m_S^2 v_s + \frac{2M_W^2 s_W^2}{e^2}\left[|\lambda|^2 v_s - |\lambda| s_{2\beta}\left(\frac{|A_\lambda|}{\sqrt{2}} c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right)\right] + |\kappa|^2 v_s^3 + \frac{1}{\sqrt{2}}|A_\kappa||\kappa| v_s^2 c_{\varphi_z}, \tag{V.2.14}$$

$$t_{a_d} = \frac{M_W s_W s_\beta}{e}|\lambda| v_s\left(\sqrt{2}|A_\lambda| s_{\varphi_x} - |\kappa| v_s s_{\varphi_y}\right), \tag{V.2.15}$$

$$t_{a_u} = \frac{t_{a_d}}{t_\beta}, \tag{V.2.16}$$

$$t_{a_s} = \frac{2M_W^2 s_W^2 s_{2\beta}}{e^2}|\lambda|\left(\frac{1}{\sqrt{2}}|A_\lambda| s_{\varphi_x} + |\kappa| v_s s_{\varphi_y}\right) - \frac{1}{\sqrt{2}}|A_\kappa||\kappa| v_s^2 s_{\varphi_z}, \tag{V.2.17}$$

where we introduce the short hand notation for the phase combinations

$$\varphi_x \equiv \varphi_{A_\lambda} + \varphi_\lambda + \varphi_s + \varphi_u, \tag{V.2.18}$$

$$\varphi_y \equiv \varphi_\kappa - \varphi_\lambda + 2\varphi_s - \varphi_u, \tag{V.2.19}$$

$$\varphi_z = \varphi_{A_\kappa} + \varphi_\kappa + 3\varphi_s. \tag{V.2.20}$$

In order that the Higgs potential acquires a minimum values at VEVs, it is necessary that all tadpole coefficients vanishes. Since t_{a_u} and t_{a_d} is in a linear relation, we obtains a set of 5 independent constraints of the SUSY parameters. The squared masses parameters $m_{H_d}^2, m_{H_u}^2$ and m_S^2 can be solved by setting (V.2.12), (V.2.13) and (V.2.14) to zero, and from (V.2.15) we find a relation between s_{φ_x} and s_{φ_y}; all of those constraint would be used to simplify the mass matrices of neutral Higgs later.

We next consider the terms of the Higgs potential which are bilinear in the neutral Higgs boson, from those we can construct the 6×6 mass matrix $\mathbf{M}_{\phi\phi}$, which can be written in terms of 3×3 block matrices $\mathbf{M}_{hh}, \mathbf{M}_{aa}$ and $\mathbf{M}_{ha}$ as

$$\mathbf{M}_{\phi\phi} = \begin{pmatrix} \mathbf{M}_{hh} & \mathbf{M}_{ha} \\ \mathbf{M}_{ha}^T & \mathbf{M}_{aa} \end{pmatrix}, \tag{V.2.21}$$

and all of the component matrices are symmetric. Each of the matrix elements can be computed by taking derivative with respect to the corresponding fields, i.e

$$\left(\mathbf{M}_{\phi\phi}\right)_{\phi_i\phi_j} = \left.\frac{\partial^2 V_H^{\text{NMSSM}}}{\partial\phi_i\partial\phi_j}\right|_{\phi=0}. \tag{V.2.22}$$

The specific results of the entries of $\mathcal{M}_{hh}$ read

$$\mathbf{M}_{h_d h_d} = M_Z^2 c_\beta^2 + \frac{1}{2}|\lambda| v_s t_\beta \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right), \tag{V.2.23}$$

$$\mathbf{M}_{h_d h_u} = -\frac{1}{2} M_Z^2 s_{2\beta} - \frac{1}{2}|\lambda| v_s \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa rvert v_s c_{\varphi_y}\right) + 2|\lambda|^2 \frac{M_W^2 s_W^2}{e^2} s_{2\beta}, \tag{V.2.24}$$

$$\mathbf{M}_{h_u h_u} = M_Z^2 s_\beta^2 + \frac{1}{2}|\lambda| \frac{v_s}{t_\beta} \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right), \tag{V.2.25}$$

$$\mathbf{M}_{h_d h_s} = 2|\lambda|^2 \frac{M_W s_W}{e} c_\beta v_s - |\lambda| \frac{M_W s_W}{e} s_\beta \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + 2|\kappa| v_s c_{\varphi_y}\right), \tag{V.2.26}$$

$$\mathbf{M}_{h_u h_s} = 2|\lambda|^2 \frac{M_W s_W}{e} s_\beta v_s - |\lambda| \frac{M_W s_W}{e} c_\beta \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + 2|\kappa| v_s c_{\varphi_y}\right), \tag{V.2.27}$$

$$\mathbf{M}_{h_s h_s} = 2|\kappa|^2 v_s^2 + \frac{v_s}{\sqrt{2}}|\kappa||A_\kappa| c_{\varphi_z} + \sqrt{2}|\lambda||A_\lambda| \frac{M_W^2 s_W^2}{e^2 v_s} s_{2\beta} c_{\varphi_x}. \tag{V.2.28}$$

Similarly, the entries of $\mathbf{M}_{aa}$ which describes the mixing between the CP-odd components of the Higgs doublet and singlet fields are

$$\mathbf{M}_{a_d a_d} = \frac{1}{2}|\lambda| \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right) v_s t_\beta, \tag{V.2.29}$$

$$\mathbf{M}_{a_d a_u} = \frac{M_{a_d a_d}}{t_\beta} = \frac{1}{2}|\lambda| \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right) v_s, \tag{V.2.30}$$

$$\mathbf{M}_{a_u a_u} = \frac{M_{a_d a_d}}{t_\beta^2} = \frac{1}{2}|\lambda| \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right) \frac{v_s}{t_\beta}, \tag{V.2.31}$$

$$\mathbf{M}_{a_d a_s} = |\lambda| \frac{M_W s_W}{e} s_\beta \left(\sqrt{2}|A_\lambda| c_{\varphi_x} - 2|\kappa| v_s c_{\varphi_y}\right), \tag{V.2.32}$$

$$\mathbf{M}_{a_u a_s} = |\lambda| \frac{M_W s_W}{e} c_\beta \left(\sqrt{2}|A_\lambda| c_{\varphi_x} - 2|\kappa| v_s c_{\varphi_y}\right), \tag{V.2.33}$$

$$\mathbf{M}_{a_s a_s} = |\lambda| \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + 4|\kappa| v_s c_{\varphi_y}\right) \frac{M_W^2 s_W^2}{e^2 v_s} s_{2\beta} - 3|A_\kappa||\kappa| \frac{v_s}{\sqrt{2}} c_{\varphi_z}. \tag{V.2.34}$$

Finally, the mixing between the CP-even and CP-odd components of the Higgs doublet and singlet is described by the matrix $\mathbf{M}_{ha}$, whose elements (after plugging in condition of vanishing tadpoles) are

$$\mathbf{M}_{ha} = \begin{pmatrix} 0 & 0 & 3v_s s_\beta \\ 0 & 0 & 3v_s c_b eta \\ -v_s s_\beta & -v_s s_\beta & -4s_{2\beta}\dfrac{M_W s_W}{e} \end{pmatrix} \frac{M_W s_W}{e} |\kappa||\lambda| s_{\varphi_y}. \tag{V.2.35}$$

The Charged Higgs Sector

In the same manner, the bilinear terms of charged Higgs bosons can be rewritten in the matrix form

$$\begin{pmatrix} h_d^+ & h_u^+ \end{pmatrix} \mathbf{M}_{h^+h^-} \begin{pmatrix} h_d^- \\ h_u^- \end{pmatrix}. \tag{V.2.36}$$

By calculating the second derivatives of V_H^{NMSSM}, we obtain the mass matrix of the charged Higgs bosons as

$$\mathbf{M}_{h^+h^-} = \frac{1}{2}\begin{pmatrix} t_\beta & 1 \\ 1 & 1/t_\beta \end{pmatrix} \left[M_W^2 s_{2\beta} + |\lambda| v_s \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right) - 2|\lambda|^2 \frac{M_W^2 s_W^2}{e^2} s_{2\beta} \right]. \tag{V.2.37}$$

The mass eigenvalues in this case are much easier to calculate, since the $\det(\mathbf{M}_{h^+h^-}) = 0 \Rightarrow$ there is a massless charged Goldstone boson. The other mass eigenvalue is the trace of $\mathbf{M}_{h^+h^-}$:

$$M_{H^\pm}^2 = \text{Tr}\left(\mathbf{M}_{h^+h^-}\right) = M_W^2 + \frac{|\lambda| v_s}{s_{2\beta}} \left(\sqrt{2}|A_\lambda| c_{\varphi_x} + |\kappa| v_s c_{\varphi_y}\right) - 2|\lambda|^2 \frac{M_W^2 s_W^2}{e^2}. \tag{V.2.38}$$

The Neutralino

With one extra Higgs singlet, we now have five neutral sfermions which in the Weyl spinor basis reads $\tilde{\psi}^0 = \left(\tilde{B}, \tilde{W}^3, \tilde{H}_d^0, \tilde{H}_u^0, \tilde{S}\right)$, and the corresponding mass matrix after electroweak symmeetry breaking is

$$\mathbf{M}_{\tilde{\chi}^0} = \begin{pmatrix} M_1 & 0 & -M_Z s_W c_\beta & e^{-i\varphi_u} M_Z s_W s_\beta & 0 \\ 0 & M_2 & M_Z c_W c_\beta & -e^{-i\varphi_u} M_Z c_W s_\beta & 0 \\ -M_Z s_W c_\beta & M_Z c_W c_\beta & 0 & -\mu_{\text{eff}} & -\mu_\lambda s_\beta \\ e^{-i\varphi_u} s_\beta s_W M_Z & -e^{-i\varphi_u} s_\beta c_W M_Z & -\mu_{\text{eff}} & 0 & -\mu_\lambda c_\beta \\ 0 & 0 & -\mu_\lambda s_\beta & -\mu_\lambda c_\beta & \mu_\kappa \end{pmatrix} \tag{V.2.39}$$

and the Lagrange containing neutralino mass terms is written in this basis as $\mathcal{L}_{\tilde{\chi}^0} = (\tilde{\chi}^0)^T \mathbf{M}_{\tilde{\chi}^0} \tilde{\chi}^0$. Comparing this mass matrix with that of MSSM in (V.1.64), the differences are the extended fifth column and row due to the extension of the Higgs sector with the mass term of the higgsino component $\tilde{S}$

$$\mu_\kappa \equiv \sqrt{2}\kappa v_s, \tag{V.2.40}$$

the mixing between the Higgs doublet and the singlet parameterized by

$$\mu_\lambda = \frac{\lambda v}{\sqrt{2}}, \tag{V.2.41}$$

and the μ entries are replaced by dynamically generated term

$$\mu_{\text{eff}} = \frac{e^{i\varphi_s} \lambda v_s}{\sqrt{2}}. \tag{V.2.42}$$

With an appropriate unitary transformation $\mathbf{N}$, we obtain the mass eigenvalues of five neutralinos

$$\left(\tilde{\chi}_1^0, \tilde{\chi}_2^0, \tilde{\chi}_3^0, \tilde{\chi}_4^0, \tilde{\chi}_5^0\right)^T = \mathbf{N}\tilde{\psi}^0 \quad \Rightarrow \quad \text{diag}\left(m_{\tilde{\chi}_1^0}, m_{\tilde{\chi}_2^0}, m_{\tilde{\chi}_3^0}, m_{\tilde{\chi}_4^0}, m_{\tilde{\chi}_5^0}\right) = \mathbf{N}^* \mathbf{M}_{\tilde{\chi}^0} \mathbf{N}, \tag{V.2.43}$$

with the mass of neutralinos is in ascending order as usual. As mentioned, the neutralinos are colorless and electrically neutral, thus interact with other particles weakly. With the R-parity being considered, the lightest neutralino $\tilde{\chi}_1^0$ is the suitable candidate for the dark matter and plays the main role in our calculations in the next several chapters.

VI
The Lightest Neutralino as a Dark Matter Candidate

Outline

VI.1 Helicity Amplitude Method

The code for generating data on this thesis has been built from scratch with the input `Fortran` subroutines for amplitude of all relevant processes are generated using `FeynCALC`, and the input parameters of NMSSM in the SLHA format (Ref. [45]).

The present section aims to represent the helicity amplitude method that is used in calculating the cross section of neutralino pair-annihilation processes, together with relevant techniques to simplify the spinor products before implementing into the code. At the very first attempt, we choose to work in the Weyl representation (see Appendix A.1 for Dirac matrices in this representation) where the Weyl spinors corresponding to four-momentum $p = (E, \mathbf{p})$ are expressed as

$$\chi_+(p) = \frac{1}{\sqrt{2|\mathbf{p}|\left(|\mathbf{p}| + p_z\right)}} \begin{pmatrix} |\mathbf{p}| + p_z \\ p_x + ip_y \end{pmatrix}, \qquad \chi_-(p) = \frac{1}{\sqrt{2|\mathbf{p}|\left(|\mathbf{p}|| + p_z\right)}} \begin{pmatrix} -p_x + ip_y \\ |\mathbf{p}| + p_z \end{pmatrix}. \tag{VI.1.1}$$

Since the above definition is ambiguous in case $p_z = -|\mathbf{p}|$, we fix the notation by taking the limit where $p_y = 0$ and $p_x \to 0^+$, i.e

$$\chi_+(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_-(p) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{VI.1.2}$$

The Dirac spinors are constructed from the Weyl spinors as follows

$$u_\lambda(p) = \begin{pmatrix} \omega_{-\lambda}\chi_\lambda \\ \omega_\lambda\chi_\lambda \end{pmatrix}, \tag{VI.1.3}$$

$$v_\lambda(p) = \begin{pmatrix} \lambda\omega_{-\lambda}\chi_{-\lambda} \\ -\lambda\omega_\lambda\chi_{-\lambda} \end{pmatrix}, \tag{VI.1.4}$$

where $\lambda = \pm 1$ and $\omega_\pm = \sqrt{E \pm \mathbf{p}}$. There are however a relation between the two types of eigenspinors, which can help us to reduce the computational effort. Clearly from the definition of $u_\lambda(p)$ and $v_\lambda(p)$ that these two eigenspinors is closely related via

$$\begin{cases} v_{-\lambda}(p) = -\gamma^5\lambda u_\lambda(p) \\ u_{-\lambda}(p) = \gamma^5\lambda v_\lambda(p) \end{cases} \Rightarrow \begin{cases} v_\lambda(p) = \gamma^5\lambda u_{-\lambda}(p) \\ u_\lambda(p) = -\gamma^5\lambda v_{-\lambda}(p) \end{cases}. \tag{VI.1.5}$$

Taking bar conjugation of Eq. (VI.1.5) yields

$$\begin{cases} \overline{v}_\lambda(p) = -\lambda\overline{u}_{-\lambda}(p)\gamma^5 \\ \overline{u}_\lambda(p) = \lambda\overline{v}_{-\lambda}(p)\gamma^5 \end{cases}. \tag{VI.1.6}$$

Thus, knowing only $u_\lambda(p)$ or $v_\lambda(p)$ is enough for calculating the other, reducing the calculation cost. Thus the general spinor chain of the forms $\overline{u}\Gamma v$, $\overline{v}\Gamma u$, $\overline{u}\Gamma u$ and $\overline{v}\Gamma v$ can be cast in the form of $\overline{u}\Gamma' u$ and $\overline{v}\Gamma' v$, where Γ is a products of Dirac matrices, Γ' is the new products generating from Γ and applying identities (VI.1.5) and (VI.1.6).

There are however other methods to construct the scattering amplitude. Below we will give one that is useful in our calculations at tree-level, where the spinors representing particle or anti-particle states are built from more basic quantities, which are independent of the (anti)particle momentum. For convenience, let us call those entities "basic spinors". A full derivation and more generalized of the considered method can be found in Ref. [46].

VI.1.1 Basic Spinors

Let us begin to calculating the spinor products in detail. [1] We first want to build the spinors based on a specific basic spinors which are independent of the given momentum. Defining a light-like vectors k_0, and a space-like vector k_1 that satisfy the following properties

$$k_0 \cdot k_0 = 0, \quad k_1 \cdot k_1 = -1, \quad k_0 \cdot k_1 = 0. \tag{VI.1.7}$$

The traces of the slash notation of these two momenta has the following properties, which will be used repeatedly in the dervation of other formulas below

$$\begin{cases} \quad \mathrm{Tr}\left(\not{k}_0\not{k}_1\right) = 4k_0 \cdot k_1 = 0, \\ \quad \mathrm{Tr}\left(\gamma^5\not{k}_0\not{k}_1\right) = 0 \end{cases} \Rightarrow \begin{cases} \quad \mathrm{Tr}\left(P_L\not{k}_0\not{k}_1\right) = 4k_0 \cdot k_1 = 0, \\ \quad \mathrm{Tr}\left(P_R\not{k}_0\not{k}_1\right) = 0 \end{cases} \tag{VI.1.8}$$

where in the second identity, we used a property of the fifth gamma matrix: $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0$.

For simplicity, we will sometimes change the notation of P_L, P_R to P_- and P_+ respectively. The basic spinors are chosen to be

$$\begin{cases} w_-(k_0)\overline{w}_-(k_0) = P_-\not{k}_0, \\ w_+(k_0) = \not{k}_1 w_-(k_0) \end{cases} \tag{VI.1.9}$$

The positive helicity basic spinor $w_+(k_0)$ is constructed such that $w_+(k_0)\overline{w}_+(k_0) = P_+\not{k}_0$, and thus the completeness relation is satisfied automatically:

$$\sum_{\lambda=\pm1} w_\lambda(k_0)\overline{w}_\lambda(k_0) = \sum_{\lambda=\pm1} P_\lambda\not{k}_0 = \not{k}_0.$$

A general spinor of massive state can now be represented on the above basis

$$u_\lambda(p) = \frac{\not{p} + m}{\sqrt{2p \cdot k_0}} w_{-\lambda}(k_0) \tag{VI.1.10}$$

[1] We will follows the derivation in the lecture notes in the following link: `https://www.ippp.dur.ac.uk/~krauss/Lectures/QuarksLeptons`

$$v_\lambda(p) = \frac{\not{p} - m}{\sqrt{2p\cdot k_0}} w_\lambda(k_0) \tag{VI.1.11}$$

Let us prove that such construction of spinors satisfies automatically the completeness relation for u spinors

$$\begin{aligned}
\sum_{\lambda=\pm} u_\lambda(p)\overline{u}_\lambda(p) &= \sum_{\lambda=\pm}\left[\frac{\not{p}+m}{\sqrt{2p\cdot k_0}} w_{-\lambda}(k_0)\overline{w}_{-\lambda}(k_0)\frac{\not{p}+m}{\sqrt{2p\cdot k_0}}\right] \\
&= \frac{\left(\not{p}+m\right)\not{k}_0\left(\not{p}+m\right)}{2p\cdot k_0} = \frac{\left(\not{p}\not{k}_0\not{p}+m^2\not{k}_0\right)+m\left(\not{p}\not{k}_0+\not{k}_0\not{p}\right)}{2p\cdot k_0} \\
&= \frac{2(p\cdot k_0)\not{p}+\left(m^2-p^2\right)\not{k}_0+2(p\cdot k_0)m}{2p\cdot k_0} = \not{p}+m.
\end{aligned} \tag{VI.1.12}$$

Similarly, by a change of sign of the mass one obtains the completeness relation for v spinors.

As mentioned at the beginning of this section, it is general enough to consider the spinor chains starting with $\overline{u}$ and ending with u. In what follows, we denote the spinor u as a general spinor, and whenever it is a spinor representing a particle or antiparticle based on the sign of m, i.e

$$u_\lambda^{(\eta)}(p) = \frac{\not{p}+\eta m}{\sqrt{2p\cdot k_0}} w_{-\lambda}(k_0) = \begin{cases} \text{spinor } u \text{ if } \eta = + \longrightarrow \text{particle state} \\ \text{spinor } v \text{ if } \eta = - \longrightarrow \text{anti-particle state} \end{cases} \tag{VI.1.13}$$

In short, these formulas are the basics of constructing an explicit expression for the spinor products we represent below:

Basis construction of the spinors

The basic spinors:

$$\begin{cases}
w_+(k_0) = \not{k}_1 w_-(k_0) & \text{(VI.1.14a)} \\
w_-(k_0)\overline{w}_-(k_0) = P_-\not{k}_0 \qquad (k_0\cdot k_0 = 0,\ k_1\cdot k_1 = -1,\ k_0\cdot k_1 = 0)\,. & \text{(VI.1.14b)} \\
w_+(k_0)\overline{w}_+(k_0) = P_+\not{k}_0 & \text{(VI.1.14c)}
\end{cases}$$

Useful identities involving basic spinors

$$w_\lambda(k_0)\overline{w}_\lambda(k_0) = P_\lambda\not{k}_0, \tag{VI.1.15a}$$

$$w_\lambda(k_0)\overline{w}_{-\lambda}(k_0) = \lambda P_\lambda\not{k}_1\not{k}_0, \tag{VI.1.15b}$$

$$\overline{w}_{\lambda_1}(k_0)w_{\lambda_2}(k_0) = \frac{\delta_{\lambda_1,-\lambda_2}}{2}(k_0\cdot k_1). \tag{VI.1.15c}$$

A general eigenspinors can be expressed in the given basis as

$$u_\lambda^{(\eta)}(p) = \frac{\not{p}+\eta m}{\sqrt{2p\cdot k_0}} w_{-\lambda}(k_0), \tag{VI.1.16}$$

with the equation of motion is rewritten as

$$\left(\not{p}-\eta m\right)u_\lambda^{(\eta)}(p) = 0 \qquad (\eta = \pm). \tag{VI.1.17}$$

The completeness relation for a general eigenspinors read

$$\sum_{\lambda=\pm} u_\lambda^{(\eta)}(p)\overline{u}_\lambda^{(\eta)}(p) = \not{p}+\eta m, \tag{VI.1.18}$$

from which we derive the following useful identities

$$\not{p} = \frac{1}{2}\sum_{\eta=\pm}\sum_{\lambda=\pm} u_\lambda^{(\eta)}(p)\overline{u}_\lambda^{(\eta)}(p), \tag{VI.1.19a}$$

$$\mathbb{1} = \frac{1}{2}\sum_{\eta=\pm}\sum_{\lambda=\pm} \eta u_\lambda^{(\eta)}(p)\overline{u}_\lambda^{(\eta)}(p). \tag{VI.1.19b}$$

The first and most simple spinor products is direct product between two spinors

$$\begin{aligned} S(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2) &\equiv \overline{u}_{\lambda_1}^{(\eta_1)}(p_1)u_{\lambda_2}^{(\eta_2)}(p_2) && \text{(VI.1.20)}\\ &= \overline{w}_{-\lambda_1}(k_0)\frac{\left(\not{p}_1+\eta_1 m_1\right)}{\sqrt{2k_0\cdot p_1}}\frac{\left(\not{p}_2+\eta_2 m_2\right)}{\sqrt{2k_0\cdot p_2}}w_{-\lambda_2}(k_0)\\ &= \mathrm{Tr}\left[w_{-\lambda_2}(k_0)\overline{w}_{-\lambda_1}(k_0)\frac{\left(\not{p}_1+\eta_1 m_1\right)}{\sqrt{2k_0\cdot p_1}}\frac{\left(\not{p}_2+\eta_2 m_2\right)}{\sqrt{2k_0\cdot p_2}}\right]. && \text{(VI.1.21)}\end{aligned}$$

Let us consider the above scalar in two seperate cases. When two spinors have the same helicities, the given scalar becomes

$$\begin{aligned} S(p_1,\lambda,\eta_1;p_2,\lambda,\eta_2) &= \frac{1}{2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}}\,\mathrm{Tr}\left[P_{-\lambda}\not{k}_0\left(\not{p}_1+\eta_1 m_1\right)\left(\not{p}_2+\eta_2 m_2\right)\right]\\ &= \frac{1}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}}\,\mathrm{Tr}\left[P_{-\lambda}\not{k}_0\left(\eta_2 m_2\not{p}_1+\eta_1 m_1\not{p}_2\right)\right]\\ &= \frac{\eta_2 m_2(k_0\cdot p_1)+\eta_1 m_1\,(k_0\cdot p_2)}{\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}} = \eta_1 m_1\sqrt{\frac{k_0\cdot p_2}{k_0\cdot p_1}}+\eta_2 m_2\sqrt{\frac{k_0\cdot p_1}{k_0\cdot p_2}}. && \text{(VI.1.22)}\end{aligned}$$

Similarly when the two component spinors have different helicities

$$\begin{aligned} S(p_1,\lambda,\eta_1;p_2,-\lambda,\eta_2) &= \frac{\lambda}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}}\,\mathrm{Tr}\left[P_\lambda\not{k}_1\not{k}_0\left(\not{p}_1+\eta_1 m_1\right)\left(\not{p}_2+\eta_2 m_2\right)\right]\\ &= \frac{\lambda}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}}\,\mathrm{Tr}\left[P_\lambda\not{k}_1\not{k}_0\left(\not{p}_1\not{p}_2+\eta_1\eta_2 m_1 m_2\right)\right]\\ &= \frac{\lambda}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}}\,\mathrm{Tr}\left[P_\lambda\not{k}_1\not{k}_0\left(\not{p}_1\not{p}_2\right)\right]\\ &= \frac{\lambda}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}}\left[(k_0\cdot p_1)(k_1\cdot p_2)-i\lambda\varepsilon^{\mu\nu\rho\kappa}k_{1\mu}k_{0\nu}p_{1\rho}p_{2\kappa}\right]. && \text{(VI.1.23)}\end{aligned}$$

The above formula seems impractical at the first glance, but by choosing the basis vectors k_0 and k_1 it can be simplified greatly. For example:

$$k_0 = (1,1,0,0), \qquad k_1 = (0,0,1,0), \tag{VI.1.24}$$

results in

$$S(p_1,\lambda,\eta_1;p_2,-\lambda,\eta_2) = \lambda\,(p_1^y - i\lambda p_1^z)\sqrt{\frac{p_2^0-p_2^x}{p_1^0-p_1^x}} - \lambda\,(p_2^y - i\lambda p_2^z)\sqrt{\frac{p_1^0-p_1^x}{p_2^0-p_2^x}}. \tag{VI.1.25}$$

We generalize the above spinor products by inserting a linear combination of projection operators between $\overline{u}$ and u:

$$\begin{aligned} &Y(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2;c_L,c_R)\\ &\equiv \overline{u}_{\lambda_1}^{(\eta_1)}(p_1)\left[c_L P_L + c_R P_R\right]u_{\lambda_2}^{(\eta_2)}(p_2,\lambda_2) && \text{(VI.1.26)}\\ &= \frac{\overline{w}_{-\lambda_1}(k_0)\left(\not{p}_1+\eta_1 m_1\right)(c_L P_L + c_R P_R)\left(\not{p}_2+\eta_2 m_2\right)w_{-\lambda_2}(k_0)}{2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}}\end{aligned}$$

$$= \frac{\text{Tr}\left\{w_{-\lambda_2}(k_0)\overline{w}_{-\lambda_1}(k_0)\left[P_L\left(c_R\not{p}_1 + c_L\eta_1 m_1\right) + P_R\left(c_L\not{p}_1 + c_R\eta_1 m_1\right)\right]\left(\not{p}_2 + \eta_2 m_2\right)\right\}}{2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}}. \tag{VI.1.27}$$

Applying the property $w_\lambda(k_0)\overline{w}_\lambda(k_0) = P_\lambda \not{k}_0 = \not{k}_0 P_{-\lambda}$ in the case both helicity are identical gives

$$Y(p_1,+,\eta_1;p_2,+,\eta_2;c_L,c_R) = \frac{\text{Tr}\left\{\not{k}_0 P_R\left[P_L\left(c_R\not{p}_1 + c_L\eta_1 m_1\right) + P_R\left(c_L\not{p}_1 + c_R\eta_1 m_1\right)\right]\left(\not{p}_2 + \eta_2 m_2\right)\right\}}{2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}}$$

$$= c_L\eta_2 m_2\frac{\sqrt{p_1\cdot k_0}}{\sqrt{p_2\cdot k_0}} + c_R\eta_1 m_1\frac{\sqrt{p_2\cdot k_0}}{\sqrt{p_1\cdot k_0}}. \tag{VI.1.28}$$

Similarly for identical negative helicity states

$$Y(p_1,-,\eta_1;p_2,-,\eta_2;c_L,c_R) = c_R\eta_2 m_2\frac{\sqrt{p_1\cdot k_0}}{\sqrt{p_2\cdot k_0}} + c_L\eta_1 m_1\frac{\sqrt{p_2\cdot k_0}}{\sqrt{p_1\cdot k_0}}. \tag{VI.1.29}$$

For different helicity states, the scalar Y becomes

- $Y(p_1,+,\eta_1;p_2,-,\eta_2;c_L,c_R)$

$$= \frac{\text{Tr}\left\{P_R\not{k}_1\not{k}_0\left[P_L\left(c_R\not{p}_1 + c_L\eta_1 m_1\right) + P_R\left(c_L\not{p}_1 + c_R\eta_1 m_1\right)\right]\left(\not{p}_2 + \eta_2 m_2\right)\right\}}{2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}}$$

$$= \frac{\text{Tr}\left[c_L P_R\not{k}_1\not{k}_0\not{p}_1\not{p}_2\right]}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}} = c_L S(p_1,+,\eta_1;p_2,-,\eta_2). \tag{VI.1.30}$$

- $Y(p_1,-,\eta_1;p_2,+,\eta_2;c_L,c_R)$

$$= \frac{\text{Tr}\left\{-P_L\not{k}_1\not{k}_0\left[P_L\left(c_R\not{p}_1 + c_L\eta_1 m_1\right) + P_R\left(c_L\not{p}_1 + c_R\eta_1 m_1\right)\right]\left(\not{p}_2 + \eta_2 m_2\right)\right\}}{2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}}$$

$$= -\frac{\text{Tr}\left[c_R P_L\not{k}_1\not{k}_0\not{p}_1\not{p}_2\right]}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}} = c_R S(p_1,-,\eta_1;p_2,+,\eta_2). \tag{VI.1.31}$$

Having the value of scalar Y, one can proceed to calculate more complicated expression such as

$$X(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2;Q;c_L,c_R) \equiv \overline{u}^{\eta_1}_{\lambda_1}(p_1)\not{Q}\left(c_L P_L + c_R P_R\right)u^{\eta_2}_{\lambda_2}(p_2) \tag{VI.1.32}$$

$$= \frac{1}{2}\sum_{\lambda=\pm}\sum_{\eta=\pm}\left[\overline{u}^{\eta_1}_{\lambda_1}(p_1)\left(P_L + P_R\right)u^{(\eta)}_{\lambda}(Q)\right]\left[\overline{u}^{(\eta)}_{\lambda}(Q)\left(c_L P_L + c_R P_R\right)u^{\eta_2}_{\lambda_2}(p_2)\right] \tag{VI.1.33}$$

$$= \frac{1}{2}\sum_{\lambda=\pm}\sum_{\eta=\pm}Y(p_1,\lambda_1,\eta_1;Q,\lambda,\eta;1,1)\times Y(Q,\lambda,\eta;p_2,\lambda_2,\eta_2;c_L,c_R), \tag{VI.1.34}$$

where we insert a derived form of completeness relation, Eq. (VI.1.19a), at the first equality. This method can be extended into the case $\overline{u}^{(\eta_1)}_{\lambda_1}(p_1)\Gamma u^{(\eta_2)}_{\lambda_2}(p_2)$, where Γ is a product of slash notation of four vectors, by a successive insertion of the identity matrix in the form

$$\mathbb{1}_{4\times 4} = P_L + P_R = \frac{1}{2}\sum_{\eta=\pm}\sum_{\lambda=\pm}\eta u^{(\eta)}_{\lambda}(p)\overline{u}^{(\eta)}_{\lambda}(p).$$

VI.1.2 Fierz Identities

Let us make a detour to the discussion of the Fierz identities. This set of identities is frequently seen in the particle physics context to rewrite a product of two Dirac bilinears in terms of a linear combination of other products of bilinears where the four spinors are in different orders. As mentioned in [47], the Fierz identities assure that the following conversion is always possible

$$\left(\psi_1 A\psi_2\right)\left(\psi_3 B\psi_4\right) = \sum\left(\psi_1 C\psi_4\right)\left(\psi_3 D\psi_2\right). \tag{VI.1.35}$$

Here we do not intend to represent the detailed derivation of the full set of Fierz identities as well as the broad range of their applications. We rather want to show a specific case in calculating the amplitude neutralino pair-annihilation where the Fierz identities show their practicality.

The basic idea is to choose a suitable complete set of 4×4 matrices, and express a general 4×4 complex matrix in terms of the chosen basis. Since we choose to work in the chiral representation of gamma matrices, it is pragmatic to work with the chiral basis, defined by the following set

$$\left\{\Gamma^A\right\} = \{P_R, P_L, P_R\gamma^\mu, P_L\gamma^\mu, 2\Sigma^{\mu\nu}\}, \qquad (\mu,\nu = 0,1,2,3) \tag{VI.1.36}$$

where $\mu < \nu$ to avoid redundancy, and $\Sigma^{mu\nu} \equiv i[\gamma^\mu, \gamma^\nu]/4$. The respective dual basis is given by

$$\{\Gamma_A\} = \{P_R, P_L, P_L\gamma_\mu, P_R\gamma_\mu, \Sigma_{\mu\nu}\}, \qquad (\mu,\nu = 0,1,2,3) \tag{VI.1.37}$$

The orthogonality between the two bases are given by

$$\mathrm{Tr}\left[\Gamma_A\Gamma^B\right] = 2\delta_A^B. \tag{VI.1.38}$$

This identity allows the expansion of an arbitrary complex 4×4 matrix X in terms of the basis $\left\{\Gamma^A\right\}$ as

$$X = X_A\Gamma^A, \qquad X_A = \frac{1}{2}\,\mathrm{Tr}\left[X\Gamma_A\right]. \tag{VI.1.39}$$

Rewritten the above expansion in element-wise manner gives

$$\begin{aligned} X_{ij} = \frac{1}{2}X_{k\ell}\Gamma_{A\ell k}\Gamma^A_{ij} &\Rightarrow X_{k\ell}\left[\delta_{ik}\delta_{j\ell} - \frac{1}{2}\Gamma_{A\ell k}\Gamma^A_{ij}\right] = 0 \\ &\Rightarrow (\mathbb{1})_{ik}(\mathbb{1})_{j\ell} = \frac{1}{2}\Gamma_{A\ell k}\Gamma^A_{ij}. \end{aligned} \tag{VI.1.40}$$

It is clear from this relation that one sees the reordering of the indices. For consistency with [47], we introduce a new notation for the matrix indices by parentheses ( ) and brackets [ ], such that each open or close parenthesis/bracket represents an independent index. Then Eq. (VI.1.40) is expressed in the form

$$(\ )[\ ] = \frac{1}{2}(\Gamma_A][\Gamma^A) \tag{VI.1.41}$$

$$= \frac{1}{2}\left\{(P_R][P_R) + (P_L][P_L) + (P_R\gamma^\mu][P_L\gamma_\mu) + (P_L\gamma^\mu][P_R\gamma_\mu) + 2(\Sigma^{\mu\nu}][\Sigma_{\mu\nu})\right\}. \tag{VI.1.42}$$

The first application we want to represent is using the derived Fierz identity to simplify the following chirally projected combination

$$\begin{aligned} (P_R\gamma^\mu)\left[P_L\gamma_\mu\right] &= \frac{1}{4}\,\mathrm{Tr}\left[P_R\gamma^\mu\Gamma_A P_L\gamma_\mu\Gamma_B\right](\Gamma^B][\Gamma^A) \\ &= \frac{1}{4}\,\mathrm{Tr}\left[P_R\gamma^\mu P_L P_L\gamma_\mu P_R\right](P_R][P_L) = \frac{1}{4}\,\mathrm{Tr}\left[P_R\gamma^\mu\gamma_\mu\right](P_R][P_L) \\ &= 2(P_R][P_L). \end{aligned} \tag{VI.1.43}$$

Note that at the second equality we have used $\gamma^\mu\gamma_\mu = 4 \times \mathbb{1}$ and $\gamma^\mu\Sigma_{\alpha\beta}\gamma_\mu = 0$. A swap between P_R and P_L is similar to interchanging the pair of parentheses and brackets, and we directly obtain

$$(P_L\gamma^\mu)\left[P_R\gamma_\mu\right] = 2(P_L][P_R). \tag{VI.1.44}$$

Another combination that can be manipulated by the Fierz identities is

$$\begin{aligned} (P_R\gamma^\mu)\left[P_R\gamma_\mu\right] &= \frac{1}{4}\,\mathrm{Tr}\left[P_R\gamma^\mu\Gamma_A P_R\gamma_\mu\Gamma_D\right](\Gamma^B][\Gamma^A) \\ &= \frac{1}{4}\,\mathrm{Tr}\left[P_R\gamma^\mu P_L\gamma_\nu P_R\gamma_\mu L\gamma^\rho\right](P_R\gamma^\rho][P_R\gamma^\nu) = \frac{1}{4}\,\mathrm{Tr}\left[P_R\gamma^\mu\gamma_\nu\gamma_\mu\gamma_\rho\right](P_R\gamma^\rho][P_R\gamma^\nu) \\ &= -\frac{1}{2}\,\mathrm{Tr}\left[P_R\gamma_\nu\gamma_\rho\right](P_R\gamma^\rho][P_R\gamma^\nu) = -(P_R\gamma^\mu][P_R\gamma_\mu). \end{aligned} \tag{VI.1.45}$$

Similarly, we have

$$(P_L\gamma^\mu)\left[P_L\gamma_\mu\right] = -(P_L\gamma^\mu][P_L\gamma_\mu). \tag{VI.1.46}$$

The above combination is called *form invariant*, since the structure of the products do not change after performing the Fierz transformation.

The formula (VI.1.44) and (VI.1.43) turns out to be useful in the amplitude of the process $\tilde{\chi}^0\tilde{\chi}^0 \to f\bar{f}$ in the s-channel with mediated Z bosons, where the spinor chain is of the form

$$\left[\overline{u}^{(\eta_1)}_{\lambda_1} P_\lambda u^{(\eta_2)}_{\lambda_2}\right]\left[\overline{u}^{(\eta_3)}_{\lambda_3} P_{\lambda'} u^{(\eta_4)}_{\lambda_4}\right]. \tag{VI.1.47}$$

Note: There are many more general spinor product structures that we do not mention here, since those do not shows up during the calculation of the annihilation amplitude of the neutralinos. One of the straightforward extension of (VI.1.47) would be

$$\begin{aligned}&Z\left(\{p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2;c_L^{12},c_R^{12}\};\{p_3,\lambda_3,\eta_3;p_4,\lambda_4,\eta_4;c_L^{34},c_R^{34}\}\right)\\ &\equiv\left[\overline{u}^{(\eta_1)}_{\lambda_1}(p_1)\gamma^\mu\left(c_L^{12}P_L+c_R^{12}P_R\right)u^{(\eta_2)}_{\lambda_2}(p_2)\right]\left[\overline{u}^{(\eta_3)}_{\lambda_3}(p_3)\gamma^\mu\left(c_L^{34}P_L+c_R^{34}P_R\right)u^{(\eta_4)}_{\lambda_4}(p_4)\right].\end{aligned} \tag{VI.1.48}$$

By using a set of so-called *Chisholm identity*, we can break all the spinor product with a contracted gamma matrix in the middle into a combination of dyadic products of spinors. See [46, **Eq. (10)**] for massless case and [46, **Eq. (38-42)**] for a massive generalization of the Chisholm identities. With this trick, we can decompose the spinor products (VI.1.48) into an expression of scalar Y defined in (VI.1.27).

In summary, below we list a collection of useful identities that can be implemented into the scattering amplitude source code to significantly boost the calculating speed. The following identities, of course, not being meant to be one of the most effective methods in calculating the amplitude numerically in our case, but should be the most basic and direct methods we should consider when building things from scratch.

List of useful identities in calculating amplitude at tree-level

- Conversion between spinors of particle and anti-particle states

$$\begin{cases}\overline{v}_\lambda(p)=-\lambda\overline{u}_{-\lambda}(p)\gamma^5\\ \overline{u}_\lambda(p)=\lambda\overline{v}_{-\lambda}(p)\gamma^5\end{cases} \tag{VI.1.49}$$

$$\begin{cases}v_\lambda(p)=\gamma^5\lambda u_{-\lambda}(p)\\ u_\lambda(p)=-\gamma^5\lambda v_{-\lambda}(p)\end{cases} \tag{VI.1.50}$$

- Direct product of a spinor and its bar conjugation

$$S(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2)\equiv\overline{u}^{(\eta_1)}_{\lambda_1}(p_1)u^{(\eta_2)}_{\lambda_2}(p_2). \tag{VI.1.51}$$

For states with identical helicities

$$S(p_1,\lambda,\eta_1;p_2,\lambda,\eta_2)=\eta_1 m_1\sqrt{\frac{k_0\cdot p_2}{k_0\cdot p_1}}+\eta_2 m_2\sqrt{\frac{k_0\cdot p_1}{k_0\cdot p_2}}. \tag{VI.1.52}$$

For states with different helicities

$$S(p_1,\lambda,\eta_1;p_2,-\lambda,\eta_2)=\frac{\lambda}{2\sqrt{(k_0\cdot p_1)(k_0\cdot p_2)}}\left[(k_0\cdot p_1)(k_1\cdot p_2)-i\lambda\varepsilon^{\mu\nu\rho\kappa}k_{1\mu}k_{0\nu}p_{1\rho}p_{2\kappa}\right] \tag{VI.1.53}$$

$$=\lambda\left(p_1^y-i\lambda p_1^z\right)\sqrt{\frac{p_2^0-p_2^x}{p_1^0-p_1^x}}-\lambda\left(p_2^y-i\lambda p_2^z\right)\sqrt{\frac{p_1^0-p_1^x}{p_2^0-p_2^x}}. \tag{VI.1.54}$$

where the second equality corresponds to the choice $k_0=(1,1,0,0)$ and $k_1=(0,0,1,0)$.

- Product of spinors containing a linear combination of projection operators

$$Y(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2;c_L,c_R)\equiv\overline{u}^{(\eta_1)}_{\lambda_1}(p_1)\left[c_LP_L+c_RP_R\right]u^{(\eta_2)}_{\lambda_2}(p_2,\lambda_2) \tag{VI.1.55}$$

For states with identical helicities

$$Y(p_1,+,\eta_1;p_2,+,\eta_2;c_L,c_R)=c_L\eta_2 m_2\frac{\sqrt{p_1\cdot k_0}}{\sqrt{p_2\cdot k_0}}+c_R\eta_1 m_1\frac{\sqrt{p_2\cdot k_0}}{\sqrt{p_1\cdot k_0}} \tag{VI.1.56}$$

$$Y(p_1,-,\eta_1;p_2,-,\eta_2;c_L,c_R) = c_R\eta_2 m_2\frac{\sqrt{p_1\cdot k_0}}{\sqrt{p_2\cdot k_0}} + c_L\eta_1 m_1\frac{\sqrt{p_2\cdot k_0}}{\sqrt{p_1\cdot k_0}}. \qquad \text{(VI.1.57)}$$

For states with different helicities

$$Y(p_1,+,\eta_1;p_2,-,\eta_2;c_L,c_R) = c_L S(p_1,+,\eta_1;p_2,-,\eta_2), \qquad \text{(VI.1.58)}$$

$$Y(p_1,-,\eta_1;p_2,+,\eta_2;c_L,c_R) = c_R S(p_1,-,\eta_1;p_2,+,\eta_2). \qquad \text{(VI.1.59)}$$

- Product of spinors containing product of slashed four-vectors. Here we consider the simplest case with only one slashed notation in the chain

$$X(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2;Q;c_L,c_R) \equiv \overline{u}^{\eta_1}_{\lambda_1}(p_1)\not{Q}\,(c_L P_L + c_R P_R)\,u^{\eta_2}_{\lambda_2}(p_2) \qquad \text{(VI.1.60)}$$

These type of spinor products can be expressed in terms of scalar Y as follows

$$\begin{aligned}
&X(p_1,\lambda_1,\eta_1;p_2,\lambda_2,\eta_2;Q;c_L,c_R)\\
&= \frac{1}{2}\sum_{\lambda=\pm}\sum_{\eta=\pm}\left[\overline{u}^{\eta_1}_{\lambda_1}(p_1)\,(P_L+P_R)\,u^{(\eta)}_{\lambda}(Q)\right]\left[\overline{u}^{(\eta)}_{\lambda}(Q)\,(c_L P_L + c_R P_R)\,u^{\eta_2}_{\lambda_2}(p_2)\right] && \text{(VI.1.61)}\\
&= \frac{1}{2}\sum_{\lambda=\pm}\sum_{\eta=\pm} Y(p_1,\lambda_1,\eta_1;Q,\lambda,\eta;1,1)\times Y(Q,\lambda,\eta;p_2,\lambda_2,\eta_2;c_L,c_R). && \text{(VI.1.62)}
\end{aligned}$$

- Fierz identities: Choosing the chiral basis $\left\{\Gamma^A\right\} = \{P_R, P_L, P_R\gamma^\mu, P_L\gamma^\mu, 2\Sigma^{\mu\nu}\}$, the orthogonality and completeness can be expressed as

$$\mathrm{Tr}\left[\Gamma_A\Gamma^B\right] = 2\delta^B_A, \qquad \text{(VI.1.63)}$$

$$(\ )[\ ] = \frac{1}{2}(\Gamma_A][\Gamma^A), \qquad \text{(VI.1.64)}$$

from which we derive

$$(P_R\gamma^\mu)\,[P_L\gamma_\mu] = 2(P_R][P_L), \qquad \text{(VI.1.65)}$$

$$(P_L\gamma^\mu)\,[P_R\gamma_\mu] = 2(P_L][P_R). \qquad \text{(VI.1.66)}$$

VI.2 Neutralino Pair Annihilations Amplitudes at Tree Level

In this section we represent the calculations of the annihilation amplitudes of all possible processes $\tilde{\chi}^0_1\tilde{\chi}^0_1 \to SM + SM'$, with $\tilde{\chi}^0_1$ being the lightest neutralino of all 5 neutralinos mass eigenstates. As mentioned before, we have not included the coannihilations into the calculations of the DM relics yet; this would be one of the main tasks in future work. The final states are SM-like particles including fermion-antifermion pairs, weak gauge boson pairs (WW and ZZ), one Higgs boson and one weak gauge boson and pair of Higgs boson (with 5 states of neutral Higgs bosons and 2 states of charged Higgs bosons in the NMSSM).

The relevant Feynman rules for NMSSM are derived from the full MSSM Lagragian (V.1.20), with modified superpotential (V.2.3) and new soft-breaking term (V.2.4); these rules has been represented in Ref. [48]. In practice, all of the vertices, mass matrices and tadpoles coefficients can be derived using `SARAH` (Ref. [49]) - a mathematica package for building and analyzing SUSY and non-SUSY models.

As mentioned before, we choose to represent the spinors and Dirac matrices in Weyl basis in order to exploit the list of identities derived in the above section for calculating amplitude at tree-level. It is thus a matter of simplicity to use the same way to denote the coupling vertices. Consider the couplings between particle A, B and C with the corresponding indices g_A, g_B and g_C for the degrees of freedom of each type of particle (which can be generation, color or its order in the mass eigenstates); such coupling is denoted as

$$C^{ABC}_{g_A,g_B,g_C}. \qquad \text{(VI.2.1)}$$

Furthermore, if there are two fermions out of three composite particles of the vertex (here assuming A and B), the interaction can in general treat the left and right chiral fermions differently. In such cases, the coupling constant is decomposed as

$$iC^{LABC}_{g_A,g_B,g_C}\,(\Psi_A P_L\Gamma\Psi_B) + iC^{RABC}_{g_A,g_B,g_C}\,(\Psi_A P_R\Gamma\Psi_B)\,, \qquad \text{(VI.2.2)}$$

with Ψ_A, Ψ_B are the spinors represent states of A and B respectively, Γ is a product of Dirac matrices. Additionally, we will denote the product between two spinors, a spinor and a 4×4 matrix and between Dirac matrices by a dot.

Finally, one should note that due to the fact that incoming particles are electrically neutral, the intermediate bosons in s-channel must also be neutral. In all of the following cases, the s-channel mediators is either the Z boson or one of the five neutral Higgs bosons in the NMSSM. To avoid singularities that happen when the intermediated particle on mass-shell, we use the Breit-Wigner propagators

$$\frac{1}{s - M_{h,g_h}^2} \longrightarrow \frac{1}{s - M_{h,g_h}^2 + iM_{h,g_h}\Gamma_{h,g_h}}, \tag{VI.2.3}$$

$$\frac{1}{s - M_Z^2} \longrightarrow \frac{1}{s - M_Z^2 + iM_Z\Gamma_Z}. \tag{VI.2.4}$$

With the input mass of the lightest neutralino $m_{\tilde{\chi}^0} \approx 190.749$ (GeV) for calculations, s-channel resonances can only occur with neutral Higgs mediators (for detailed set of the input parameters see Section VII.1). We summarize all annihilation processes with the involving channels and mediators in the following tables. These are also the processes being taken into account in the neutralino relic density in this thesis.

Process		Exchanged particles			Degrees of freedom
		s-channel	t-channel	u-channel	
Fermions final states	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to \ell^i\overline{\ell}^i$	$Z,\, h_m$	$\tilde{\ell}^m$	$\tilde{\ell}^m$	$i = e, \mu, \tau$
	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to \nu^i\overline{\nu}^i$	Z	$\tilde{\nu}^i$	$\tilde{\nu}^i$	$i = e, \mu, \tau$
	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to q_u^{i\alpha}\overline{q}_u^{j\beta}$	$Z,\, h_m$	$\tilde{q}_u^{k\gamma}$	$\tilde{q}_u^{k\gamma}$	$i,j,k = 1,2,3$ $\alpha,\beta,\gamma = R,G,B$ $q_u = u,c,t$
	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to q_d^{i\alpha}\overline{q}_d^{j\beta}$	$Z,\, h_m$	$\tilde{q}_d^{k\gamma}$	$\tilde{q}_d^{k\gamma}$	$i,j,k = 1,2,3$ $\alpha,\beta,\gamma = R,G,B$ $q_d = d,s,b$
Weak gauge bosons final states	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to W^+W^-$	$Z,\, h_m$	$\tilde{\chi}_n$	$\tilde{\chi}_n$	$m = 1,\cdots,5$ $n = 1,2$
	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to ZZ$	h_m	$\tilde{\chi}_m^0$	$\tilde{\chi}_m^0$	$m = 1,\cdots,5$
One Higgs & one weak gauge bosons	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to H_n^{\pm}W^{\mp}$	$Z,\, h_m$	$\tilde{\chi}_m$	$\tilde{\chi}_m$	$m,n = 1,2$
	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_nZ$	$Z,\, h_m$	$\tilde{\chi}_m^0$	$\tilde{\chi}_m^0$	$m,n = 1,\cdots,5$
Higgs bosons final states	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to H_n^+H_l^-$	$Z,\, h_m$	$\tilde{\chi}_m$	$\tilde{\chi}_m$	$n,l = 1,2$ $m = 1,\cdots,5$
	$\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_nh_l$	$Z,\, h_m$	$\tilde{\chi}_m^0$	$\tilde{\chi}_m^0$	$n,l,m = 1,\cdots,5$

Table VI.1: A complete set of relevant processes of tree-level neutralino annihilation into two body final states in the NMSSM. The collection of charged leptons is denoted as ℓ^i, and the corresponding neutrino ν^i. The up-type quarks and down-type quarks are denoted respectively $q_u^{i\alpha}$ and $q_d^{i\alpha}$ with i and α are the generation index and color index respectively. The NMSSM particle content contains five neutral Higgs bosons and two charged Higgs bosons, which are also considered in the annihilation products of neutralino pair.

VI.2.1 Annihilation into SM Fermion Pairs

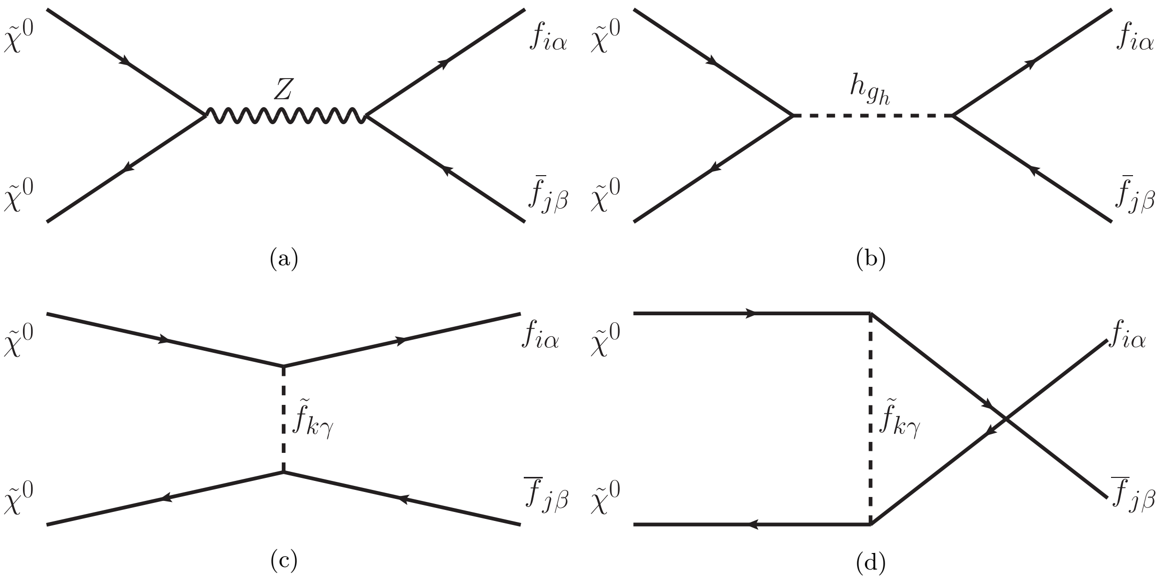


Figure VI.1: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to f\overline{f}$. Figure VI.1a is the s-channel with Z-boson mediated diagram; Figure VI.1b is the s-channel with Higgs mediated diagram; Figure VI.1c and Figure VI.1d are diagrams for t-channel and u-channel respectively, mediated by a sfermion. Here the indices $i, j, k = 1, 2, 3$ indicate the generation of the fermion, and α, β are color indices. Note that with pair of neutrino-antineutrino final states there will be no diagrams with exchanged Higgs.

The tree-level amplitude of annihilation of lightest neutralino pair into pair of fermion-antifermion includes four channels: two s-channel mediated by Z and Higgs bosons, one t-channel and one u-channel mediated by a sfermion. Note that the incoming particles are colorless forces the outgoing particles have opposite color charge, that explains the appearance of $\delta_{\alpha\beta}$ in the amplitude. The result is

$$
\begin{aligned}
&\mathcal{M}_{\tilde{\chi}_1^0\tilde{\chi}_1^0\to f_{i\alpha}\overline{f}_{j\beta}} \\
&= \frac{\delta_{i,j}\delta_{\alpha,\beta}}{s - M_Z^2 + iM_Z\Gamma_Z}\Big\{ - C^{\mathrm{RffZ}} C^{L\tilde{\chi}^0\tilde{\chi}^0}_{1,1} \bar{v}(p_2,h_2).\gamma^\mu.P_L.u(p_1,h_1)\,\bar{u}(p_3,h_3).\gamma_\mu.P_R.v(p_4,h_4) \\
&\qquad - C^{\mathrm{RffZ}} C^{R\tilde{\chi}^0\tilde{\chi}^0}_{1,1} \bar{u}(p_3,h_3).\gamma^\mu.P_R.v(p_4,h_4)\,\bar{v}(p_2,h_2).\gamma_\mu.P_R.u(p_1,h_1) \\
&\qquad - C^{\mathrm{LffZ}} C^{R\tilde{\chi}^0\tilde{\chi}^0}_{1,1} \bar{u}(p_3,h_3).\gamma^\mu.P_L.v(p_4,h_4)\,\bar{v}(p_2,h_2).\gamma_\mu.P_R.u(p_1,h_1) \\
&\qquad + C^{\mathrm{LffZ}} C^{L\tilde{\chi}^0\tilde{\chi}^0}_{1,1} \left(-\bar{u}(p_3,h_3).\gamma^\mu.P_L.v(p_4,h_4)\right)\bar{v}(p_2,h_2).\gamma_\mu.P_L.u(p_1,h_1)\Big\} \\
&+ \sum_{g_h=1}^{5} \frac{\delta_{i,j}\delta_{\alpha,\beta}}{s - M_{h,g_h}^2 + iM_{h,g_h}\Gamma_{h,g_h}}\Big\{ - C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,g_h} C^{\mathrm{Lhff}}_{i,j,g_h} \bar{u}(p_3,h_3).P_L.v(p_4,h_4)\,\bar{v}(p_2,h_2).P_L.u(p_1,h_1) \\
&\qquad - C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,g_h} C^{\mathrm{Lhff}}_{i,j,g_h} \bar{u}(p_3,h_3).P_L.v(p_4,h_4)\,\bar{v}(p_2,h_2).P_R.u(p_1,h_1) \\
&\qquad - C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,g_h} C^{\mathrm{Rhff}}_{i,j,g_h} \bar{u}(p_3,h_3).P_R.v(p_4,h_4)\,\bar{v}(p_2,h_2).P_L.u(p_1,h_1) \\
&\qquad - C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,g_h} C^{\mathrm{Rhff}}_{i,j,g_h} \bar{u}(p_3,h_3).P_R.v(p_4,h_4)\,\bar{v}(p_2,h_2).P_R.u(p_1,h_1)\Big\} \\
&+ \sum_{k=1}^{3}\sum_{\gamma=R,G,B} \frac{\delta_{\alpha,\beta}}{t - M_{\tilde{f},g_h}^2}\Big\{ - \left(C^{R\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{L\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}(p_3,h_3).P_L.u(p_1,h_1)\,\bar{v}(p_2,h_2).P_L.v(p_4,h_4) \\
&\qquad - \left(C^{L\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{L\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}(p_3,h_3).P_R.u(p_1,h_1)\,\bar{v}(p_2,h_2).P_L.v(p_4,h_4) \\
&\qquad - \left(C^{R\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{R\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}(p_3,h_3).P_L.u(p_1,h_1)\,\bar{v}(p_2,h_2).P_R.v(p_4,h_4)
\end{aligned}
$$

$$- \left(C^{L\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{R\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}\left(p_3,h_3\right).P_R.u\left(p_1,h_1\right)\bar{v}\left(p_2,h_2\right).P_R.v\left(p_4,h_4\right)\Bigg\}$$

$$\begin{aligned}
&+\sum_{k=1}^{3}\sum_{\gamma=R,G,B}\frac{\delta_{\alpha,\beta}}{u-M^2_{\tilde{f},g_h}}\Bigg\{\left(C^{R\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{L\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}\left(p_3,h_3\right).P_L.u\left(p_2,h_2\right)\bar{v}\left(p_1,h_1\right).P_L.v\left(p_4,h_4\right)\\
&\qquad + \left(C^{L\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{L\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}\left(p_3,h_3\right).P_R.u\left(p_2,h_2\right)\bar{v}\left(p_1,h_1\right).P_L.v\left(p_4,h_4\right)\\
&\qquad + \left(C^{R\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{R\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}\left(p_3,h_3\right).P_L.u\left(p_2,h_2\right)\bar{v}\left(p_1,h_1\right).P_R.v\left(p_4,h_4\right)\\
&\qquad + \left(C^{L\tilde{\chi}^0 f\tilde{f}}_{1,i,g_h}\right)^* C^{R\tilde{\chi}^0 f\tilde{f}}_{1,j,g_h} \bar{u}\left(p_3,h_3\right).P_R.u\left(p_2,h_2\right)\bar{v}\left(p_1,h_1\right).P_R.v\left(p_4,h_4\right)\Bigg\},
\end{aligned} \tag{VI.2.5}$$

where f can be charged leptons, up-type or down-type quarks and neutrinos. This amplitude contains the simplest spinor structures of all considered annihilation processes:

$$\overline{u}P_{L,R}v, \quad \overline{v}\cdot P_{L,R}\cdot u, \quad \left(\overline{v}\cdot\gamma^\mu\cdot P_{\{L,R\}}u\right)\left(\overline{u}\cdot\gamma_\mu\cdot P_{\{L,R\}}v\right), \tag{VI.2.6}$$

which we can apply the identities (VI.1.55), (VI.1.66) and (VI.1.65) to reduce the amount of numerical calculations.

VI.2.2 Annihilation into Weak Gauge Boson Pairs

Process $\tilde{\chi}^0_1\tilde{\chi}^0_1 \to W^+W^-$

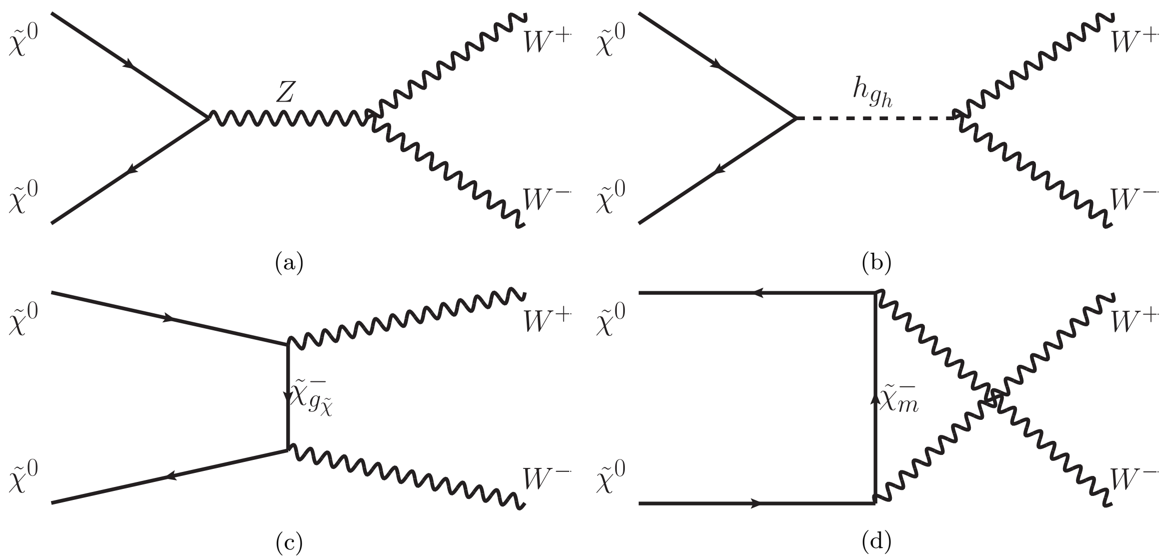


Figure VI.2: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to W^+W^-$. Figure VI.2a is the s-channel with Z-boson mediated diagram; Fig. VI.2b is the s-channel with Higgs mediated diagram; Fig. VI.2c and Fig. VI.2d are diagrams for t-channel and u-channel respectively, mediated by a chargino. Here the index $m = 1, 2$ indicates which chargino is being considered. This notation for indexing chargino will be used in other diagrams from now on.

The analytic tree-level amplitude of neutralino annihilation with a pair of Z boson as final states reads

$$\begin{aligned}
&\mathcal{M}_{\tilde{\chi}^0_1\tilde{\chi}^0_1\to W^+W^-}\\
&= \frac{C^{\mathrm{WWZ}}\bar{v}\left(p_2,h_2\right)}{s-M_Z^2+iM_Z\Gamma_Z}.\Bigg\{-C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left(p_3.\varepsilon^*_{p_3,\lambda_3}\right)\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_R + 2C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left(p_3.\varepsilon^*_{p_4,\lambda_4}\right)\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_R\\
&\qquad - 2C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left(p_4.\varepsilon^*_{p_3,\lambda_3}\right)\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_R + C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left(p_4.\varepsilon^*_{p_4,\lambda_4}\right)\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_R
\end{aligned}$$

$$
\begin{aligned}
&- C_{1,1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z}\left(\varepsilon^*_{p_3,\lambda_3}.\varepsilon^*_{p_4,\lambda_4}\right)(\gamma.p_3).P_R + C_{1,1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z}\left(\varepsilon^*_{p_3,\lambda_3}.\varepsilon^*_{p_4,\lambda_4}\right)(\gamma.p_4).P_R \\
&- C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}\left(p_3.\varepsilon^*_{p_3,\lambda_3}\right)\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_L + 2\left(p_3.\varepsilon^*_{p_4,\lambda_4}\right)C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_L \\
&- 2C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}\left(p_4.\varepsilon^*_{p_3,\lambda_3}\right)\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_L + C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}\left(p_4.\varepsilon^*_{p_4,\lambda_4}\right)\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_L \\
&- C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}\left(\varepsilon^*_{p_3,\lambda_3}.\varepsilon^*_{p_4,\lambda_4}\right)(\gamma.p_3).P_L + C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}\left(\varepsilon^*_{p_3,\lambda_3}.\varepsilon^*_{p_4,\lambda_4}\right)(\gamma.p_4).P_L\Big\}.u(p_1,h_1)
\end{aligned}
$$

$$
+\sum_{g_h=1}^{5}\frac{C_{g_h}^{\mathrm{hWW}}\left(\varepsilon^*_{p_3,\lambda_3}.\varepsilon^*_{p_4,\lambda_4}\right)}{s-M^2_{h,g_h}+i\Gamma_{g_h}M_{h,g_h}}\bar{v}(p_2,h_2).\Big\{C_{1,1,g_h}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L + C_{1,1,g_h}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R\Big\}.u(p_1,h_1)
$$

$$
\begin{aligned}
+\sum_{g_{\tilde{\chi}}=1}^{2}\frac{1}{t-m^2_{\tilde{\chi},g_{\tilde{\chi}}}}\bar{v}(p_2,h_2).\Big\{ &-\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).(\gamma.p_4).\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_R \\
&-\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).(\gamma.p_4).\left(\gamma.\varepsilon^*_{p_3}\right).P_L \\
&+\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).\left(\gamma.\varepsilon^*_{p_3}\right).P_R \\
&+\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon^*_{p_4}\right).\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_L \\
&-\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_L \\
&-\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_R \\
&+2\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\left(p_2.\varepsilon^*_{p_4,\lambda_4}\right)\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_L \\
&+2\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\left(p_2.\varepsilon^*_{p_4,\lambda_4}\right)\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).P_R\Big\}.u(p_1,h_1)
\end{aligned}
$$

$$
\begin{aligned}
+\sum_{g_{\tilde{\chi}}=1}^{2}\frac{1}{u-m^2_{\tilde{\chi},g_{\tilde{\chi}}}}\bar{v}(p_2,h_2).\Big\{ &-\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).(\gamma.p_3).\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_L \\
&-\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).(\gamma.p_3).\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_R \\
&+\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).\left(\gamma.\varepsilon^*_{p_4}\right).P_L \\
&+\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_R \\
&-\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_L \\
&-\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W} m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right).\left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right).P_R \\
&+2\left(C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0 W}\left(\gamma.\varepsilon^*_{p_4}\right).P_L\left(p_2.\varepsilon^*_{p_3,\lambda_3}\right) \\
&+2\left(C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\right)^* C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0 W}\left(\gamma.\varepsilon^*_{p_4}\right).P_R\left(p_2.\varepsilon^*_{p_3,\lambda_3}\right)\Big\}.u(p_1,h_1),
\end{aligned}
\tag{VI.2.7}
$$

where the polarization vector $\varepsilon^\mu_{p,\lambda}$ with $p=(E,p_x,p_y,p_z)$ is defined as

$$
\left\{
\begin{array}{l}
\varepsilon^\mu(p,1) = \left(|\mathbf{p}|\,p_T\right)^{-1}\left(0,p_xp_z,p_yp_z,-p_T^2\right) \\
\varepsilon^\mu(p,2) = \left(p_T\right)^{-1}\left(0,-p_y,-p_x,0\right) \\
\varepsilon^\mu(p,3) = \left(E/m\,|\mathbf{p}|\right)\left(|\mathbf{p}|^2/E,p_x,p_y,p_z\right)
\end{array}
\right.,
\tag{VI.2.8}
$$

and

$$
m = \sqrt{E^2-|\mathbf{p}|^2}, \tag{VI.2.9}
$$

$$
p_T^2 = p_x^2+p_y^2. \tag{VI.2.10}
$$

Note that the definitions of $\varepsilon^\mu(p,1)$ and $\varepsilon^\mu(p,2)$ are ambiguous in the limit $p_T \to 0$, we fix the notation by taking the limit where $p_y = 0$ and

$$\begin{cases} p_x \to 0^+ & (p_z > 0) \\ p_x \to 0^- & (p_z < 0) \end{cases}. \tag{VI.2.11}$$

The helicity eigenvectors for $\lambda = \pm, 0$ are defined as

$$\varepsilon^\mu_{p,\lambda=\pm} = \frac{1}{\sqrt{2}} \left(\mp\varepsilon^\mu(p,1) - i\varepsilon^\mu(p,2)\right), \tag{VI.2.12}$$

$$\varepsilon^\mu_{p,\lambda=0} = \varepsilon^\mu(p,3). \tag{VI.2.13}$$

This process contains the most complex spinor chains compared to other tree-level neutralino annihilation, namely

$$\bar{v} \cdot P_{\{L,R\}} \cdot (\gamma \cdot p_1)(\gamma \cdot p_2)(\gamma \cdot p_3) \cdot P_{\{L,R\}} \cdot u. \tag{VI.2.14}$$

In massless limit, the above structure is easier to handle, especially the polarization vector can be written in terms of the spinor of corresponding momentum and such expression can be break to simpler spinor chains which can be further simplify using the identities derived at the beginning of the chapter. We however employed the massive representation of spinors as well as polarization vectors; we have not figured out an efficient methods to decompose the spinor structure (VI.2.14) into sum of simpler spinor products yet, thus we import directly the 4×4 matrices $\gamma \cdot p$ into the code in the following form

$$\gamma \cdot p = \gamma^\mu p_\mu = \begin{pmatrix} 0 & 0 & p^0 - p^3 & -p^1 + ip^2 \\ 0 & 0 & -p^1 - ip^2 & p^0 + p^3 \\ p^0 + p^3 & p^1 - ip^2 & 0 & 0 \\ p^1 + ip^2 & p^0 - p^3 & 0 & 0 \end{pmatrix}, \tag{VI.2.15}$$

and continuously perform the block matrix multiplications.

Process $\tilde{\chi}^0_1\tilde{\chi}^0_1 \to ZZ$

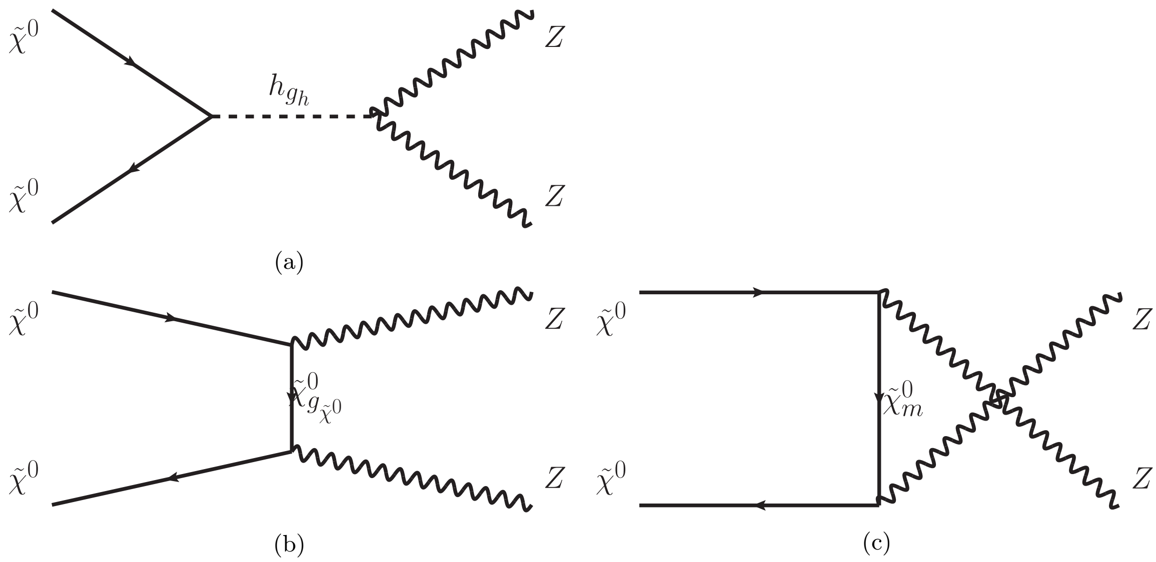


Figure VI.3: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to ZZ$. Figure VI.3a is the s-channel with Higgs mediated diagram; Fig. VI.3b and Fig. VI.3c are diagrams for t-channel and u-channel respectively, mediated by a neutralino. Here the index $m = 1, \ldots, 5$ indicates which neutralino is being considered. This notation for indexing neutralino will be used in other diagrams from now on.

The analytic tree-level amplitude of neutralino annihilation with a pair of Z boson as final states reads

$\mathcal{M}_{\tilde{\chi}^0_1\tilde{\chi}^0_1 \to ZZ}$

$$= \sum_{g_h=1}^{5} \frac{C_{g_h}^{\mathrm{hZZ}}}{s - M_{h,g_h}^2 + iM_{h,g_h}\Gamma_{g_h}} \bar{v}(p_2,h_2) . \left\{ P_L \varepsilon^*_{p_3,\lambda_3} . \varepsilon^*_{p_4,\lambda_4} C_{1,1,g_h}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0} + P_R \varepsilon^*_{p_3,\lambda_3} . \varepsilon^*_{p_4,\lambda_4} C_{1,1,g_h}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0} \right\} . u(p_1,h_1)$$

$$\begin{aligned}
\sum_{g_{\tilde{\chi}^0}=1}^{5} \bar{v}(p_2,h_2) . \Big\{ & - \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . (\gamma.p_4) . \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . P_R C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& - \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . (\gamma.p_4) . \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . P_L C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& + \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . P_R C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} m_{\tilde{\chi}^0,1} \\
& + \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . \left(\gamma.\varepsilon^*_{p_3}\right) . P_L C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} m_{\tilde{\chi}^0,1} \\
& - \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . P_R C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} m_{1,g_{\tilde{\chi}0}} \\
& - \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . P_L C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} m_{1,g_{\tilde{\chi}0}} \\
& + 2 \left(\gamma.\varepsilon^*_{p_3}\right) . P_R p_2 . \varepsilon^*_{p_4,\lambda_4} C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& + 2 \left(\gamma.\varepsilon^*_{p_3}\right) . P_L p_2 . \varepsilon^*_{p_4,\lambda_4} C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} \Big\} . u(p_1,h_1)
\end{aligned}$$

$$\begin{aligned}
+ \sum_{g_{\tilde{\chi}^0}=1}^{5} \frac{1}{u - m_{1,g_{\tilde{\chi}0}}^2} \bar{v}(p_2,h_2) . \Big\{ & - \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . (\gamma.p_3) . \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . P_R C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& + 2 \left(\gamma.\varepsilon^*_{p_4}\right) . P_R p_2 . \varepsilon^*_{p_3,\lambda_3} C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& - \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . (\gamma.p_3) . \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . P_L C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& + 2 \left(\gamma.\varepsilon^*_{p_4}\right) . P_L p_2 . \varepsilon^*_{p_3,\lambda_3} C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} \\
& + \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . P_R C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} m_{\tilde{\chi}^0,1} \\
& + \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . \left(\gamma.\varepsilon^*_{p_4}\right) . P_L C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} m_{\tilde{\chi}^0,1} \\
& - \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . P_R C_{1,g_{\tilde{\chi}0}}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} m_{1,g_{\tilde{\chi}0}} \\
& - \left(\gamma.\varepsilon^*_{p_3,\lambda_3}\right) . \left(\gamma.\varepsilon^*_{p_4,\lambda_4}\right) . P_L C_{1,g_{\tilde{\chi}0}}^{R\tilde{\chi}^0\tilde{\chi}^0 Z} C_{g_{\tilde{\chi}0},1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z} m_{1,g_{\tilde{\chi}0}} \Big\} . u(p_1,h_1)
\end{aligned} \tag{VI.2.16}$$

VI.2.3 Annihilation into One Higgs Boson & One Weak Gauge Boson

Process $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to H^{\pm}W^{\mp}$: pair of charged bosons final states.

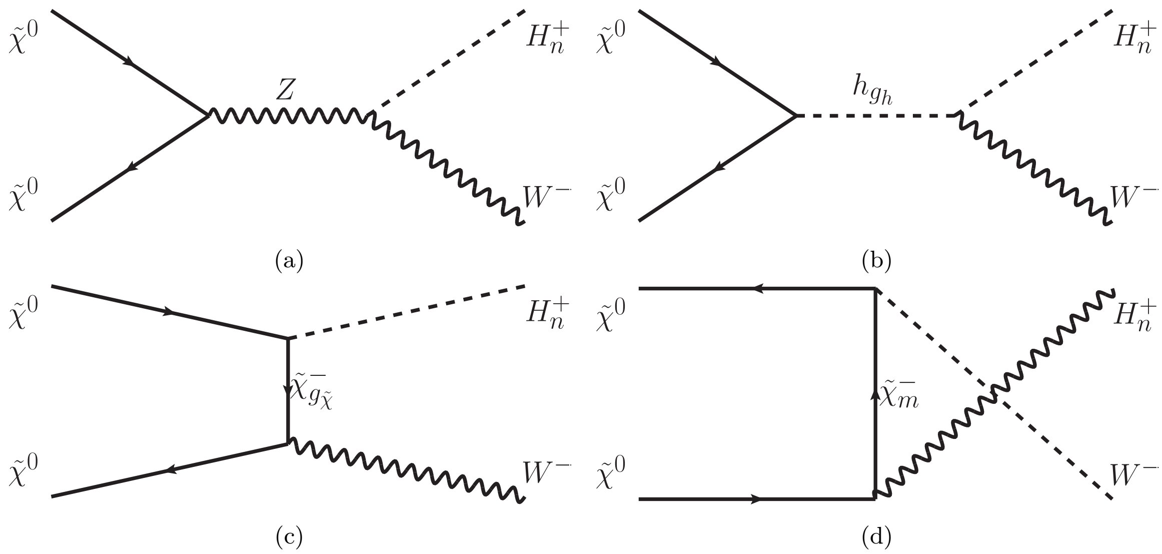


Figure VI.4: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to H^{\pm}W^{\mp}$. Figure VI.4a is the s-channel with Z-boson mediated diagram; Fig. VI.4b is the s-channel with Higgs mediated diagram; Fig. VI.4c and Fig. VI.4d are diagrams for t-channel and u-channel respectively, mediated by a chargino (by changing the electric charge, we obtain the diagram for the process $\tilde{\chi}^0\tilde{\chi}^0 \to H^+W^-$).

The analytic tree-level amplitude of neutralino annihilation with a charged Higgs boson and W boson as final states reads

$$
\begin{aligned}
&\mathcal{M}_{\tilde{\chi}_1^0\tilde{\chi}_1^0\to H_n W}\\
&= \frac{1}{s - M_Z^2 + iM_Z\Gamma_Z}\bar{v}(p_2,h_2).\Big\{C_n^{H^+W^-Z}C_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0Z}\left(\gamma.\varepsilon_{p_4}^*\right).P_L + C_n^{H^+W^-Z}C_{1,1}^{R\tilde{\chi}^0\tilde{\chi}^0Z}\left(\gamma.\varepsilon_{p_4}^*\right).P_R\Big\}.u(p_1,h_1)\\
&+ \sum_{g_h=1}^{5}\frac{1}{s - M_{h,g_h}^2 + iM_{h,g_h}\Gamma_{g_h}}\bar{v}(p_2,h_2).\Big\{\left(p_4.\varepsilon_{p_4}^*\right)C_{g_h,n}^{\mathrm{hH}^+W^-}C_{1,1,g_h}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L + \left(p_4.\varepsilon_{p_4}^*\right)C_{g_h,n}^{\mathrm{hH}^+W^-}C_{1,1,g_h}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R\\
&\qquad + 2\left(p_3.\varepsilon_{p_4}^*\right)C_{g_h,n}^{\mathrm{hH}^+W^-}C_{1,1,g_h}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L\\
&\qquad + 2\left(p_3.\varepsilon_{p_4}^*\right)C_{g_h,n}^{\mathrm{hH}^+W^-}C_{1,1,g_h}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R\Big\}.u(p_1,h_1)\\
&+ \sum_{g_{\tilde{\chi}}=1}^{2}\frac{1}{t - m_{\tilde{\chi},g_{\tilde{\chi}}}^2}\bar{v}(p_2,h_2).\Big\{-2iC_{n,g_{\tilde{\chi}},1}^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}P_Lp_2.\varepsilon_{p_4}^* - 2iC_{n,g_{\tilde{\chi}},1}^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0W}p_2.\varepsilon_{p_4}^*P_R\\
&\qquad + iC_{n,g_{\tilde{\chi}},1}^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}\left(\gamma.\varepsilon_{p_4}^*\right).\left(\gamma.p_4\right).P_L + iC_{n,g_{\tilde{\chi}},1}^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0W}\left(\gamma.\varepsilon_{p_4}^*\right).\left(\gamma.p_4\right).P_R\\
&\qquad - iC_{n,g_{\tilde{\chi}},1}^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon_{p_4}^*\right).P_L - iC^{\mathrm{RH}^+\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon_{p_4}^*\right).P_R\\
&\qquad + iC_{n,g_{\tilde{\chi}},1}^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon_{p_4}^*\right).P_R\\
&\qquad + iC_{n,g_{\tilde{\chi}},1}^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon_{p_4}^*\right).P_L\Big\}.u(p_1,h_1)\\
&+ \sum_{g_{\tilde{\chi}}=1}^{2}\frac{1}{u - m_{\tilde{\chi},g_{\tilde{\chi}}}^2}\bar{v}(p_2,h_2).\Big\{-iC_{n,g_{\tilde{\chi}},1}^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}\left(\gamma.p_3\right).\left(\gamma.\varepsilon_{p_4}^*\right).P_L\\
&\qquad - iC_{n,g_{\tilde{\chi}},1}^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0W}\left(\gamma.p_3\right).\left(\gamma.\varepsilon_{p_4}^*\right).P_R\\
&\qquad - iC_{n,g_{\tilde{\chi}},1}^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon_{p_4}^*\right).P_L\\
&\qquad - iC^{\mathrm{RH}^+\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{R\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi}^0,1}\left(\gamma.\varepsilon_{p_4}^*\right).P_R\\
&\qquad - iC_{n,g_{\tilde{\chi}},1}^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}C_{g_{\tilde{\chi}},1}^{L\tilde{\chi}\tilde{\chi}^0W}m_{\tilde{\chi},g_{\tilde{\chi}}}\left(\gamma.\varepsilon_{p_4}^*\right).P_L
\end{aligned}
$$

$$- iC^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{R\tilde{\chi}\tilde{\chi}^0 W}_{g_{\tilde{\chi}},1} m_{\tilde{\chi},g_{\tilde{\chi}}} \left(\gamma.\varepsilon^*_{p_4}\right).P_R \Big\}.u\left(p_1,h_1\right) \tag{VI.2.17}$$

Process $\tilde{\chi}^0_1\tilde{\chi}^0_1 \to h_n Z$: pair of neutral bosons final states.

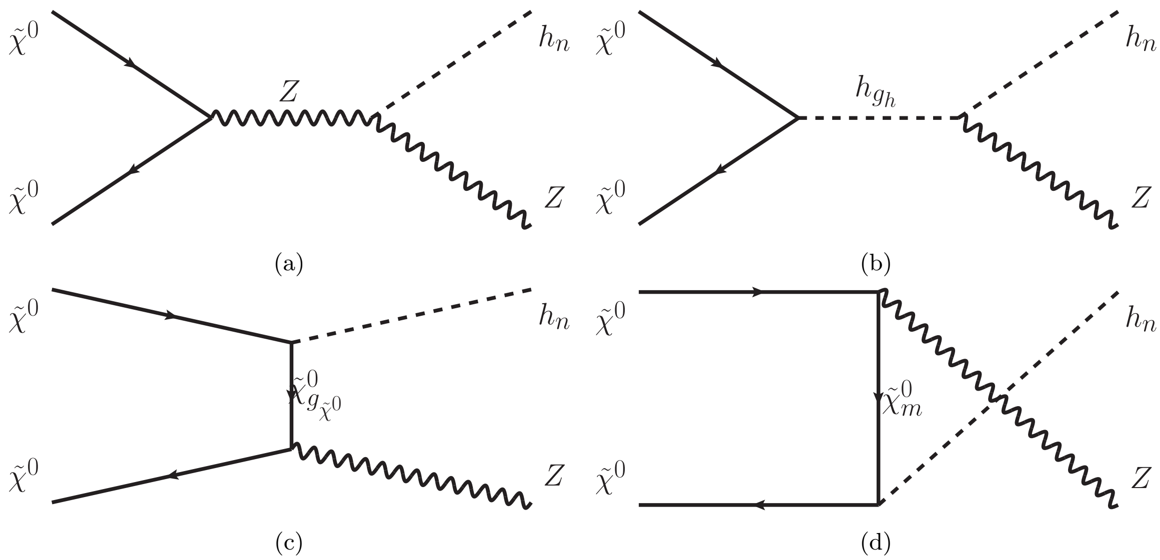


Figure VI.5: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to h_n Z$. Figure VI.5a is the s-channel with Z-boson mediated diagram; Fig. VI.5b is the s-channel with Higgs mediated diagram; Fig. VI.5c and Fig. VI.5d are diagrams for t-channel and u-channel respectively, mediated by a neutralino.

The analytic tree-level amplitude of neutralino annihilation with a neutral Higgs boson and Z boson as final states reads

$$\begin{aligned}
&\mathcal{M}_{\tilde{\chi}^0_1\tilde{\chi}^0_1\to h_n Z} \\
&= \frac{1}{s - M_Z^2 + iM_Z\Gamma_Z}\bar{v}\left(p_2,h_2\right).\Big\{ - iC^{\mathrm{hZZ}}_{n} C^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left(\gamma.\varepsilon^*_{p_4}\right).P_L - iC^{\mathrm{hZZ}}_{n} C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left(\gamma.\varepsilon^*_{p_4}\right).P_R\Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_h=1}^{5} \frac{1}{s - M^2_{h,n} + iM_{h,n}\Gamma_n}\bar{v}\left(p_2,h_2\right).\Big\{ - 2C^{\mathrm{hhZ}}_{n,n} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,n}\left(p_3.\varepsilon^*_{p_4}\right)P_L - 2C^{\mathrm{hhZ}}_{n,n} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,n}\left(p_3.\varepsilon^*_{p_4}\right)P_R \\
&\qquad - C^{\mathrm{hhZ}}_{n,n} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,n}\left(p_4.\varepsilon^*_{p_4}\right)P_L - C^{\mathrm{hhZ}}_{n,n} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,n}\left(p_4.\varepsilon^*_{p_4}\right)P_R\Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_{\tilde{\chi}^0}=1}^{5} \frac{1}{t - m^2_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}}\bar{v}\left(p_2,h_2\right).\Big\{ - 2iC^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(p_2.\varepsilon^*_{p_4}\right)P_L - 2iC^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(p_2.\varepsilon^*_{p_4}\right)P_R \\
&\qquad + iC^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(\gamma.\varepsilon^*_{p_4}\right).\left(\gamma.p_4\right).P_L + iC^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(\gamma.\varepsilon^*_{p_4}\right).\left(\gamma.p_4\right).P_R \\
&\qquad - im_{\tilde{\chi}^0,1} C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(\gamma.\varepsilon^*_{p_4}\right).P_L - im_{\tilde{\chi}^0,1} C^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(\gamma.\varepsilon^*_{p_4}\right).P_R \\
&\qquad + im_{\tilde{\chi}^0,g_{\tilde{\chi}^0}} C^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(\gamma.\varepsilon^*_{p_4}\right).P_L \\
&\qquad + im_{\tilde{\chi}^0,g_{\tilde{\chi}^0}} C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,g_{\tilde{\chi}^0}} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{g_{\tilde{\chi}^0},1,n}\left(\gamma.\varepsilon^*_{p_4}\right).P_R\Big\}.u\left(p_1,h_1\right)
\end{aligned} \tag{VI.2.18}$$

$$+ \sum_{g_{\tilde{\chi}^0}=1}^{5} \frac{1}{u - m^2_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}}\bar{v}\left(p_2,h_2\right).\Big\{ iC^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{g_{\tilde{\chi}^0},1} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,g_{\tilde{\chi}^0},n}\left(\gamma.p_3\right).\left(\gamma.\varepsilon^*_{p_4}\right).P_L + iC^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{g_{\tilde{\chi}^0},1} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,g_{\tilde{\chi}^0},n}\left(\gamma.p_3\right).\left(\gamma.\varepsilon^*_{p_4}\right).P_R$$

$$
\begin{aligned}
&+ im_{\tilde{\chi}^0,1} C^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{g_{\tilde{\chi}^0},1} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,g_{\tilde{\chi}^0},n} \left(\gamma.\varepsilon^*_{p_4}\right).P_L + im_{\tilde{\chi}^0,1} C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{g_{\tilde{\chi}^0},1} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,g_{\tilde{\chi}^0},n} \left(\gamma.\varepsilon^*_{p_4}\right).P_R \\
&+ im_{\tilde{\chi}^0,g_{\tilde{\chi}^0}} C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{g_{\tilde{\chi}^0},1} C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,g_{\tilde{\chi}^0},n} \left(\gamma.\varepsilon^*_{p_4}\right).P_R \\
&+ im_{\tilde{\chi}^0,g_{\tilde{\chi}^0}} C^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{g_{\tilde{\chi}^0},1} C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,g_{\tilde{\chi}^0},n} \left(\gamma.\varepsilon^*_{p_4}\right).P_L \Big\}.u\left(p_1,h_1\right)
\end{aligned} \tag{VI.2.19}
$$

VI.2.4 Annihilation into Higgs Boson Pairs

Process $\tilde{\chi}^0_1\tilde{\chi}^0_1 \to H^+_n H^-_\ell$: pair of charged Higgs bosons final states

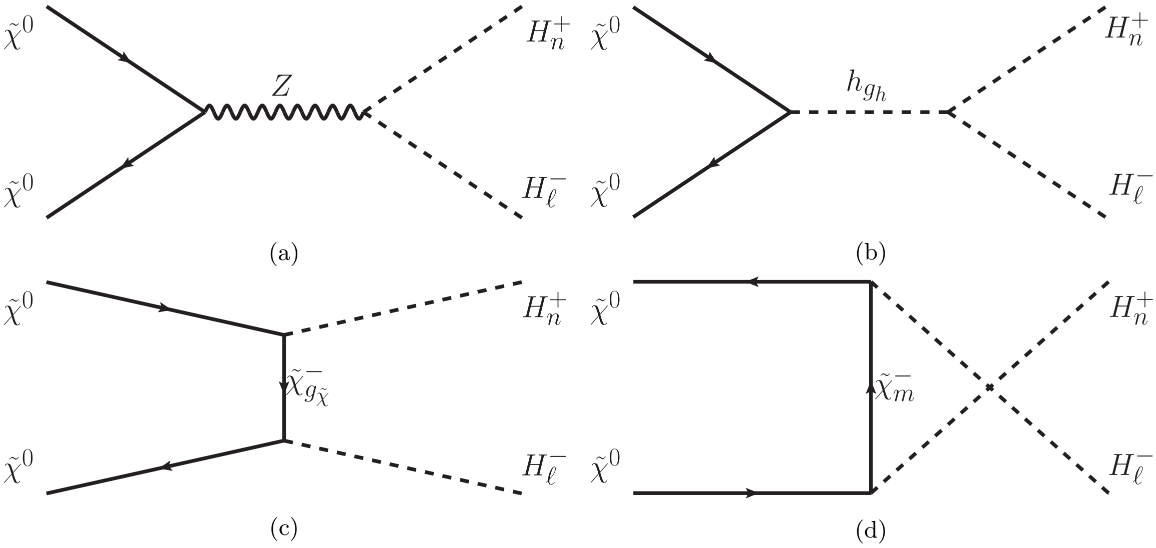


Figure VI.6: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to H^-_n H^+_\ell$. Figure VI.6a is the s-channel with Z-boson mediated diagram; Fig. VI.6b is the s-channel with Higgs mediated diagram; Fig. VI.6c and Fig. VI.6d are diagrams for t-channel and u-channel respectively, mediated by a chargino.

The analytic tree-level amplitude of neutralino annihilation with charged Higgs bosons as final states reads

$$
\begin{aligned}
&\mathcal{M}_{\tilde{\chi}^0_1\tilde{\chi}^0_1 \to H^+_n H^-_\ell} \\
&= \frac{C^{\mathrm{HHZ}}\delta_{n,\ell}}{s - M_Z^2 + iM_Z\Gamma_Z} \bar{v}\left(p_2,h_2\right).\Big\{ C^{L\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left[\gamma.(p_3-p_4)\right].P_L + C^{R\tilde{\chi}^0\tilde{\chi}^0 Z}_{1,1}\left[\gamma.(p_3-p_4)\right].P_R \Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_h=1}^{5} \frac{C^{\mathrm{hHH}}_{\ell,n,g_h}}{s - M^2_{h,g_h} + iM_{h,g_h}\Gamma_{g_h}} \bar{v}\left(p_2,h_2\right).\Big\{ P_L C^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,g_h} + P_R C^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}_{1,1,g_h} \Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_{\tilde{\chi}}=1}^{2} \frac{1}{t - m^2_{\tilde{\chi},g_{\tilde{\chi}}}} \bar{v}\left(p_2,h_2\right).\Big\{ C^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{LH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell}\left[(\gamma.p_4) + m_{\tilde{\chi}^0}\right].P_R + C^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{LH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell}\left[(\gamma.p_4) + m_{\tilde{\chi},g_{\tilde{\chi}}}\right] P_L \\
&\qquad + C^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{RH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell} m_{\tilde{\chi}^0,1} P_L + C^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{RH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell} m_{\tilde{\chi},g_{\tilde{\chi}}} P_R \Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_{\tilde{\chi}}=1}^{2} \frac{1}{u - m^2_{\tilde{\chi},g_{\tilde{\chi}}}} \bar{v}\left(p_2,h_2\right).\Big\{ C^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{LH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell}\left[(\gamma.p_3) + m_{\tilde{\chi}^0}\right].P_L + C^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{RH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell}\left[(\gamma.p_3) + m_{\tilde{\chi}^0}\right].P_R \\
&\qquad + C^{\mathrm{LH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{LH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell} m_{\tilde{\chi},g_{\tilde{\chi}}} P_L + C^{\mathrm{RH}\tilde{\chi}\tilde{\chi}^0}_{n,g_{\tilde{\chi}},1} C^{\mathrm{RH}\tilde{\chi}}_{g_{\tilde{\chi}},1,\ell} m_{\tilde{\chi},g_{\tilde{\chi}}} P_R \Big\}.u\left(p_1,h_1\right)
\end{aligned} \tag{VI.2.20}
$$

Process $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to \mathrm{h}_n\mathrm{h}_\ell$: pair of neutral Higgs bosons final states

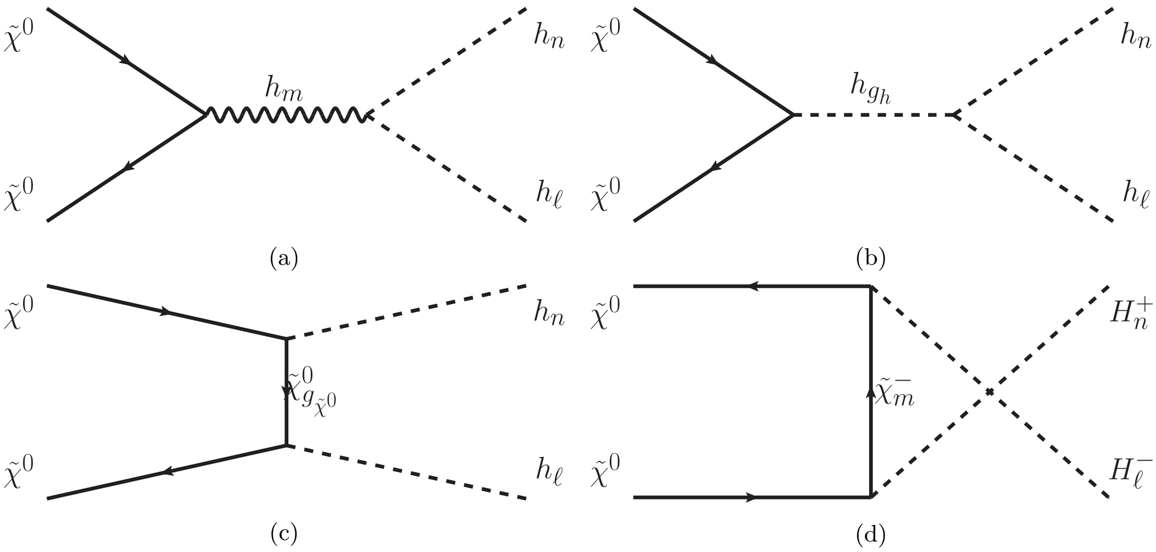


Figure VI.7: Diagrams of processes $\tilde{\chi}^0\tilde{\chi}^0 \to H_n^- H_\ell^+$. Figure VI.7a is the s-channel with Z-boson mediated diagram; Fig. VI.7b is the s-channel with Higgs mediated diagram; Fig. VI.7c and Fig. VI.7d are diagrams for t-channel and u-channel respectively, mediated by a chargino.

The analytic tree-level amplitude of neutralino annihilation with neutral Higgs as final states reads

$$
\begin{aligned}
&\mathcal{M}_{\tilde{\chi}_1^0\tilde{\chi}_1^0\to h_n h_\ell} \\
&= \frac{1}{s-M_Z^2+iM_Z\Gamma_Z}\bar{v}\left(p_2,h_2\right).\Big\{iC_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}C_{n,\ell}^{\mathrm{hhZ}}\left(\gamma.p_3\right).P_L - iC_{1,1}^{L\tilde{\chi}^0\tilde{\chi}^0 Z}C_{n,\ell}^{\mathrm{hhZ}}\left(\gamma.p_4\right).P_L \\
&\qquad + iC_{1,1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z}C_{n,\ell}^{\mathrm{hhZ}}\left(\gamma.p_3\right).P_R - iC_{1,1}^{R\tilde{\chi}^0\tilde{\chi}^0 Z}C_{n,\ell}^{\mathrm{hhZ}}\left(\gamma.p_4\right).P_R\Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_h=1}^{5}\frac{1}{s-M_{h,g_h}^2+iM_{h,g_h}\Gamma_{g_h}}\bar{v}\left(p_2,h_2\right).\Big\{C_{1,1,g_h}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{n,\ell,g_h}^{\mathrm{hhh}}P_L + C_{1,1,g_h}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{n,\ell,g_h}^{\mathrm{hhh}}P_R\Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_{\tilde{\chi}^0}=1}^{5}\frac{1}{t-m_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}^2}\bar{v}\left(p_2,h_2\right).\Big\{C_{1,g_{\tilde{\chi}^0},\ell}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,n}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}\left(\gamma.p_4\right).P_L + C_{1,g_{\tilde{\chi}^0},\ell}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,n}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}\left(\gamma.p_4\right).P_R \\
&\qquad + m_{\tilde{\chi}^0,1}C_{1,g_{\tilde{\chi}^0},\ell}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,n}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L + m_{\tilde{\chi}^0,1}C_{1,g_{\tilde{\chi}^0},\ell}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,n}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R \\
&\qquad + m_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}C_{1,g_{\tilde{\chi}^0},\ell}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,n}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L + m_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}C_{1,g_{\tilde{\chi}^0},\ell}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,n}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R\Big\}.u\left(p_1,h_1\right) \\
&+ \sum_{g_{\tilde{\chi}^0}=1}^{5}\frac{1}{u-m_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}^2}\bar{v}\left(p_2,h_2\right).\Big\{C_{1,g_{\tilde{\chi}^0},n}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,\ell}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}\left(\gamma.p_3\right).P_L + C_{1,g_{\tilde{\chi}^0},n}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,\ell}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}\left(\gamma.p_3\right).P_R \\
&\qquad + m_{\tilde{\chi}^0,1}C_{1,g_{\tilde{\chi}^0},n}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,\ell}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L + m_{\tilde{\chi}^0,1}C_{1,g_{\tilde{\chi}^0},n}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,\ell}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R \\
&\qquad + m_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}C_{1,g_{\tilde{\chi}^0},n}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,\ell}^{\mathrm{Lh}\tilde{\chi}^0\tilde{\chi}^0}P_L + m_{\tilde{\chi}^0,g_{\tilde{\chi}^0}}C_{1,g_{\tilde{\chi}^0},n}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}C_{g_{\tilde{\chi}^0},1,\ell}^{\mathrm{Rh}\tilde{\chi}^0\tilde{\chi}^0}P_R\Big\}.u\left(p_1,h_1\right)
\end{aligned}
\tag{VI.2.21}
$$

VI.3 Cross section of neutralino annihilations at tree-level

Solving for the neutralino relic density requires performing calculations related to the total cross section of all possible neutralino pair-annihilation processes. As mentioned in Section II.2.1, taking thermal average of cross section times Møller velocity using Eq. (II.2.25) helps us overcome the case of resonances (which is a problem if one tries to expand the cross section as powers of energy), where we just need to modify the propagators of the resonances as

$$\frac{1}{s-m_R^2} \to \frac{1}{s-m_R^2+im_R\Gamma_R}, \tag{VI.3.1}$$

with the mass and total decay width of the corresponding resonance m_R and Γ_R. Within all processes listed in the above section and the input parameters listed in Section VII.1, the s-channel diagrams are mediated by only three most massive neutral Higgs h_3, h_4 and h_5 which satisfies $M_{h_5} > M_{h_4} > M_{h_3} > 2m_{\tilde{\chi}_1^0}$. These resonances result in the peaks as we can observe from the following figures, which shows the dependence of the cross section $\times v_{\text{lab}}$ of each annihilation process. The NMSSM parameters, together with the total decay width are imported from the output SLHA file produced by the package **NMSSMCALC**.

For cross check purposes, as well as investigating the contribution of each processes to the total cross section explicitly, we represent below the plot of cross section of each neutralino pair-annihilation process. The 2 to 2 total cross sections is calculated in the helicity amplitude framework via the integral formula we derive in Appendix C, which reads

$$\begin{aligned}\sigma_\chi v_{\text{lab}} &= \frac{\beta_f(s)}{64\pi^2\left(s-2m_\chi^2\right)} \sum_{X,X'} \int \mathrm{d}\Omega\, \frac{1}{S_{X,X'}} \overline{|\mathcal{M}|^2}_{\tilde{\chi}\tilde{\chi}\to XX'}(s,\theta) \\ &= \frac{1}{32\pi} \frac{\beta_f(s)}{s-2m_\chi^2} \int_0^\pi \mathrm{d}\theta\, \sin\theta \times \sum_{X,X'} \frac{1}{S_{X,X'}} \overline{|\mathcal{M}|^2}_{\tilde{\chi}\tilde{\chi}\to XX'}(s,\theta),\end{aligned} \tag{VI.3.2}$$

where the sum over all possible final states if they are kinematically allowed, i.e the energy s is larger than the threshold energy

$$\sqrt{s}_{\text{threshold}} = \max\left(2m_\chi, m_X + m_X'\right), \tag{VI.3.3}$$

and $S_{X,X'}$ is the symmetric factor of final states. For most of our processes $S_{X,X'} = 1$ except

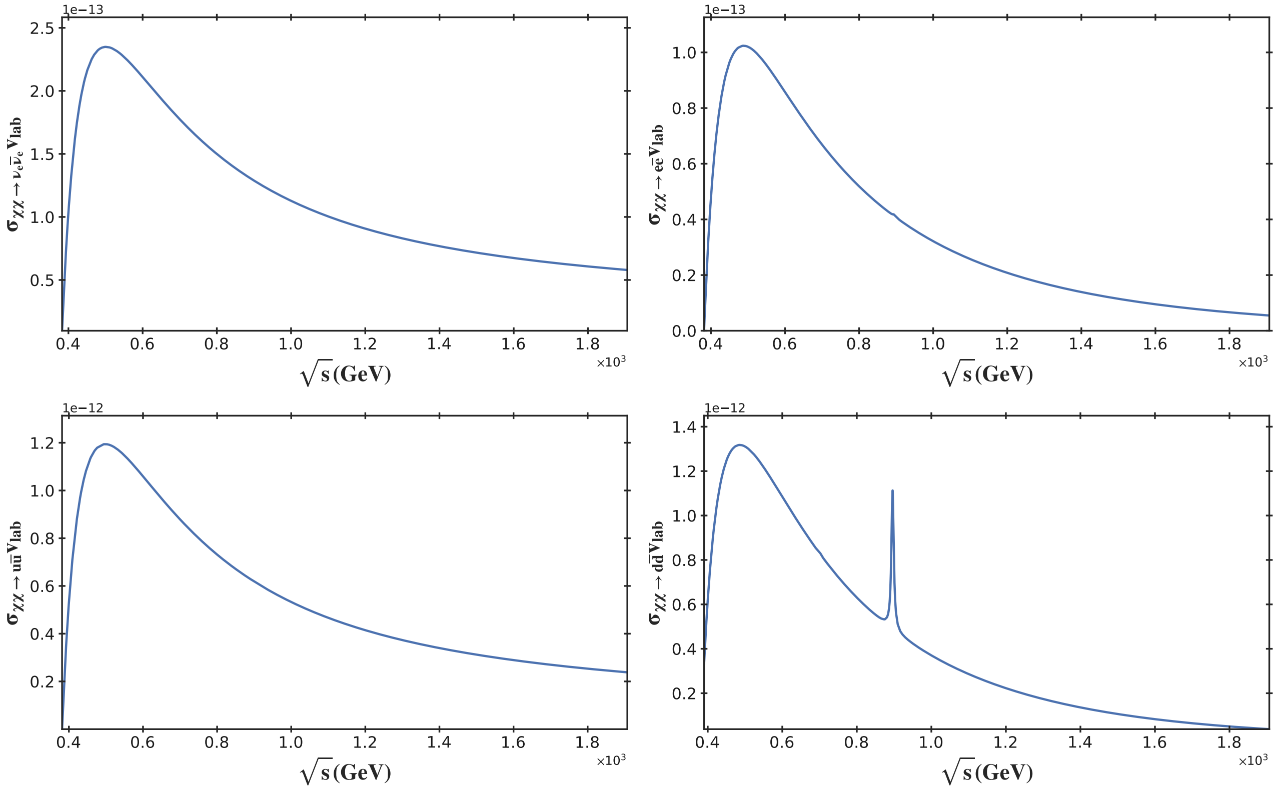
$$(X,X') = (Z,Z), (h_i,h_i) \quad \Rightarrow \quad S_{X,X'} = 2, \quad (i = 1, \cdots, 5). \tag{VI.3.4}$$

Thus, each process can be calculated independently and the total cross section is obtained by simply summing over all possible processes. We start our calculations for each process from the opening threshold of a channel and stop calculating when $\sigma_\chi v_{\text{lab}}$ is significantly small (but after consider all of the resonances). We choose to calculate the quantity σv_{lab} of all processes at four phase-space point 381.5 GeV, 600 GeV, 900 GeV and 2000 GeV. The numerical results are shown in the table below.

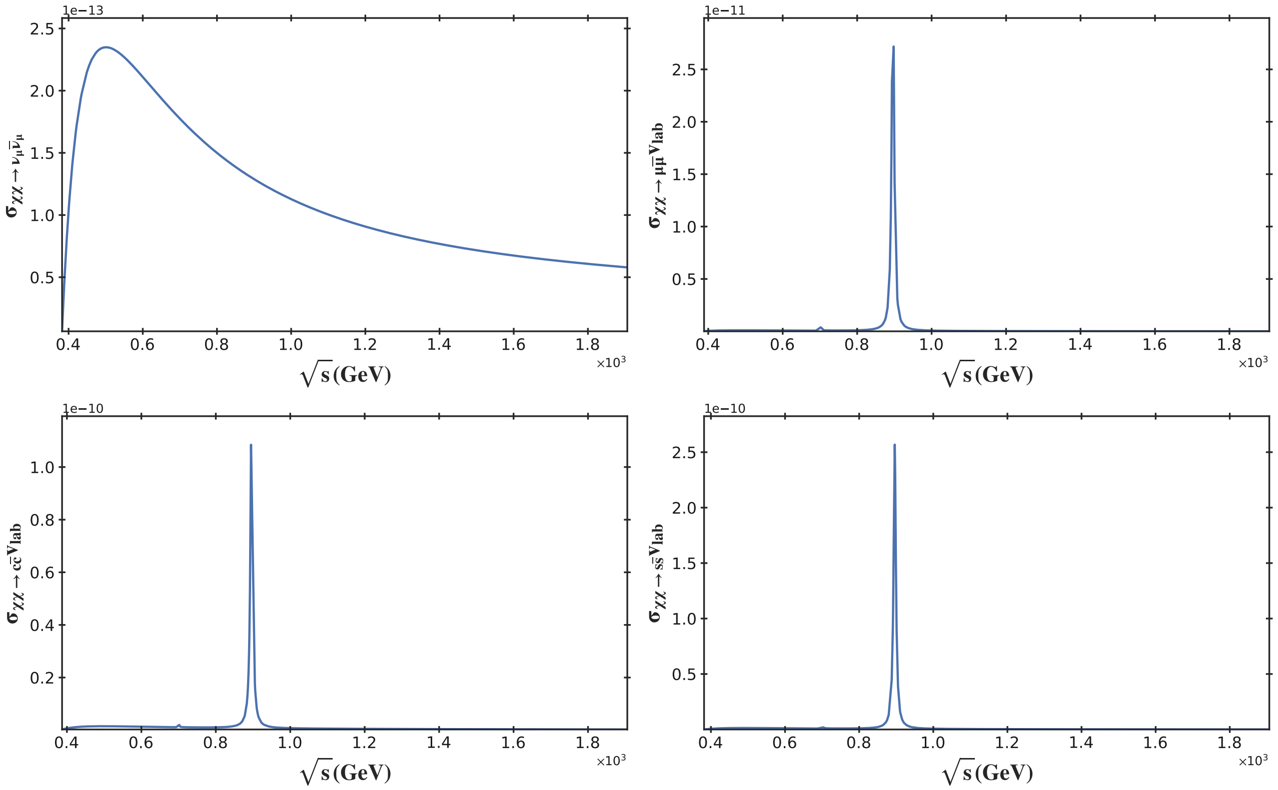
$\sigma_{\tilde{\chi}_1^0\tilde{\chi}_1^0\to XX'}v_{\mathrm{lab}}$ (GeV^{-2})	Center-of-mass energy $\sqrt{s}$			
Final states X, X'	$2m_{\tilde{\chi}_1^0} \approx 381.5$ (GeV)	600(GeV)	900 (GeV)	2000 (GeV)
$\nu_e^+\nu_e^-$	≈ 0	$2.11010451 \times 10^{-13}$	$1.28864336 \times 10^{-13}$	$5.57444397 \times 10^{-14}$
$\nu_\mu^+\nu_\mu^-$	≈ 0	$2.11010451 \times 10^{-13}$	$1.28864336 \times 10^{-13}$	$5.57444397 \times 10^{-14}$
$\nu_\tau^+\nu_\tau^-$	≈ 0	$3.18479493 \times 10^{-13}$	$2.53424563 \times 10^{-13}$	$1.44305000 \times 10^{-13}$
e^+e^-	≈ 0	$8.57761661 \times 10^{-14}$	$4.10067766 \times 10^{-14}$	$4.66799443 \times 10^{-15}$
$\mu^+\mu^-$	≈ 0	$8.89453994 \times 10^{-14}$	$1.63936669 \times 10^{-11}$	$5.17109493 \times 10^{-15}$
$\tau^+\tau^-$	≈ 0	$1.03548665 \times 10^{-12}$	$4.61973989 \times 10^{-9}$	$1.09119965 \times 10^{-13}$
$u\bar{u}$	≈ 0	$3.53116411 \times 10^{-13}$	$2.06265905 \times 10^{-13}$	$7.55337684 \times 10^{-14}$
$c\bar{c}$	≈ 0	$4.37561462 \times 10^{-13}$	$1.86005264 \times 10^{-11}$	$8.23452817 \times 10^{-14}$
$t\bar{t}$	$4.35772218 \times 10^{-10}$	$1.75469915 \times 10^{-9}$	$2.95655690 \times 10^{-7}$	$1.46937733 \times 10^{-9}$
$d\bar{d}$	≈ 0	$3.61638649 \times 10^{-13}$	$2.57283024 \times 10^{-13}$	$9.95175353 \times 10^{-15}$
$s\bar{s}$	≈ 0	$3.69306668 \times 10^{-13}$	$3.98208342 \times 10^{-11}$	$1.11730742 \times 10^{-14}$
$b\bar{b}$	≈ 0	$1.73985630 \times 10^{-11}$	$7.66315446 \times 10^{-8}$	$1.56310218 \times 10^{-12}$
W^+W^-	$1.16704447 \times 10^{-8}$	$7.64536636 \times 10^{-9}$	$3.97180627 \times 10^{-9}$	$1.17115287 \times 10^{-9}$
ZZ	$6.40521303 \times 10^{-9}$	$4.66750518 \times 10^{-9}$	$2.52648540 \times 10^{-9}$	$7.11489855 \times 10^{-10}$
H^+W^-	0	0	0	$1.75289523 - 010$
H^-W^+	0	0	0	$1.75289523 - 010$
h_1Z	≈ 0	$6.68526330 \times 10^{-12}$	$9.15645916 \times 10^{-8}$	$6.33933928 \times 10^{-12}$
h_2Z	$1.82969902 \times 10^{-10}$	$1.18890481 \times 10^{-10}$	$1.18720600 \times 10^{-10}$	$2.43375784 \times 10^{-11}$
h_3Z	0	0	$1.60002279 \times 10^{-10}$	$5.26495323 \times 10^{-13}$
h_4Z	0	0	0	$1.41821207 \times 10^{-11}$
h_5Z	0	0	0	$6.95994708 \times 10^{-11}$
H^+H^-	0	0	0	$1.34570129 \times 10^{-11}$
h_1h_1	≈ 0	$3.85242205 \times 10^{-11}$	$4.57024850 \times 10^{-10}$	$1.10585636 \times 10^{-11}$
h_1h_2	≈ 0	$3.09662963 \times 10^{-11}$	$1.04940541 \times 10^{-8}$	$6.94762061 \times 10^{-12}$
h_1h_3	0	0	$6.21700688 \times 10^{-11}$	$3.45494760 \times 10^{-11}$
h_1h_4	0	0	0	$7.63099749 \times 10^{-12}$
h_1h_5	0	0	0	$1.53987293 \times 10^{-12}$
h_2h_2	≈ 0	$7.43744611 \times 10^{-10}$	$6.64346737 \times 10^{-10}$	$7.90431061 \times 10^{-11}$
h_2h_3	0	0	$2.15822107 \times 10^{-9}$	$1.77204704 \times 10^{-11}$
h_2h_4	0	0	0	$1.93361338 \times 10^{-10}$
h_2h_5	0	0	0	$2.70702730 \times 10^{-11}$
h_3h_3	0	0	0	$7.11349342 \times 10^{-12}$
h_3h_4	0	0	0	$1.20191466 \times 10^{-12}$
h_3h_5	0	0	0	$4.19554348 \times 10^{-13}$
h_4h_4	0	0	0	$1.19067674 \times 10^{-11}$
h_4h_5	0	0	0	$9.56055545 \times 10^{-12}$
h_5h_5	0	0	0	$1.10658358 \times 10^{-11}$
Total	$1.90088510 \times 10^{-8}$	$1.40884387 \times 10^{-8}$	$3.83482579 \times 10^{-7}$	$3.71799249 \times 10^{-9}$

Table VI.2: Numerical value of σv_{lab} of each process at four energy point 381.5 GeV, 600 GeV, 900 GeV and 2000 GeV.

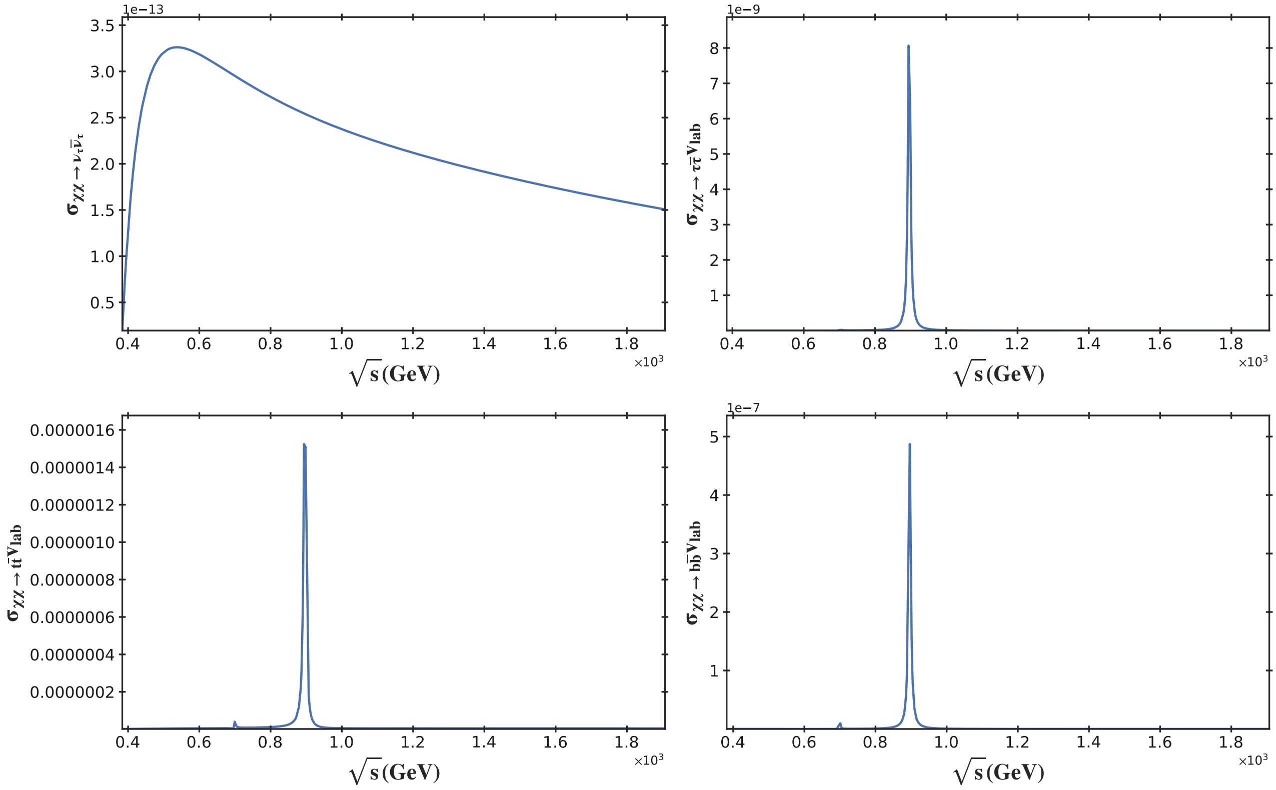
To show the energy dependence of σv_{lab}, we visualize the numerical results via the figures below. The calculation of each processes start at the threshold energy.



(a) Fermionic final states of 1st generation.



(b) Fermionic final states of 2nd generation.



(c) Fermionic final states of 3$^{\text{rd}}$ generation.

Figure VI.8: Cross section times laboratory velocity of neutralino pair annihilation with fermionic final states including charged leptons, neutrinos and two types of quarks in each generation.

The annihilation of neutralino pair into SM fermion pair reveals several interesting features that we want to mention. Firstly, in case of annihilation into a pair of light charged leptons in the first generation or a pair of neutrinos, all of the σv_{lab} have the same trend where they start from approximately 0, reach a maximum at around 500 GeV and gradually decline after that. For these processes, the s-channel mediated by Z boson gives the main contribution to the cross section; the effects of s-channel with Higgs exchanged is so small that can be safely neglected due to the negligible couplings between these fermions with Higgs (neutrinos have no couplings with Higgs at all). For the heavy fermionic final states, the couplings between SM fermions and Higgs bosons become much larger, and s-channel of Higgs exchanged becomes the most significant influence. This fact shows obviously through the sharp spikes near the resonances, especially the spikes appear in the range (800 GeV - 1000 GeV) which dominate the whole plot range.

Secondly, with the mass of lightest neutralino around 191 GeV (see Table VII.1), all of the fermionic final states processes (including top quark-anti top quark) have the threshold energy being identical to the total rest energy of $\tilde{\chi}_1^0$ pair. Although the fermionic final states are always open for the annihilations, at the limit of zero relative velocity (i.e at the threshold $\sqrt{s} \to 2m_{\tilde{\chi}_1^0}$) we observe that $\sigma v_{\text{lab}} \approx 0$ (see the first column of Table VI.2). This phenomenon has been considered in many context (e.g [50–52]). In short, the so-called s-wave [1] helicity suppression is originated from the fact that two incoming particles are Majorana fermions (whose charge conjugation yields itself). Since all of the quantum numbers of the incoming states are identical, the Fermi statistics forces their helicities to be opposite, hence the helicities of final states are also opposite. This results in a proportional coefficient of order $m_f/m_{\tilde{\chi}_1^0}$, and thus σv_{lab} adopt a helicy suppression factor of order $m_f^2/m_{\tilde{\chi}_1^0}^2$. A general treatment of generic dark matter interaction structures including the s-wave suppression can be found in [53].

[1] According to [52], calculations such as neutralino relic abundance, the flux of energetic neutrinos from neutralino annihilation in the sun and earth, and fluxes of anomalous cosmic rays produced by neutralino annihilation in the Galactic halo, it is generally sufficient to consider the expansion of annihilation cross section times relative velocity as:

$$\sigma_\chi v = a + bv^2 + \mathcal{O}(v^4), \tag{VI.3.5}$$

where a is the s-wave contribution at zero relative velocity and b contains contributions from both the s and p waves. The s-wave contribution accounts for the value $\sigma_\chi v$ when the incoming particles are at rest, which is useful for calculations of indirect-detection rates.

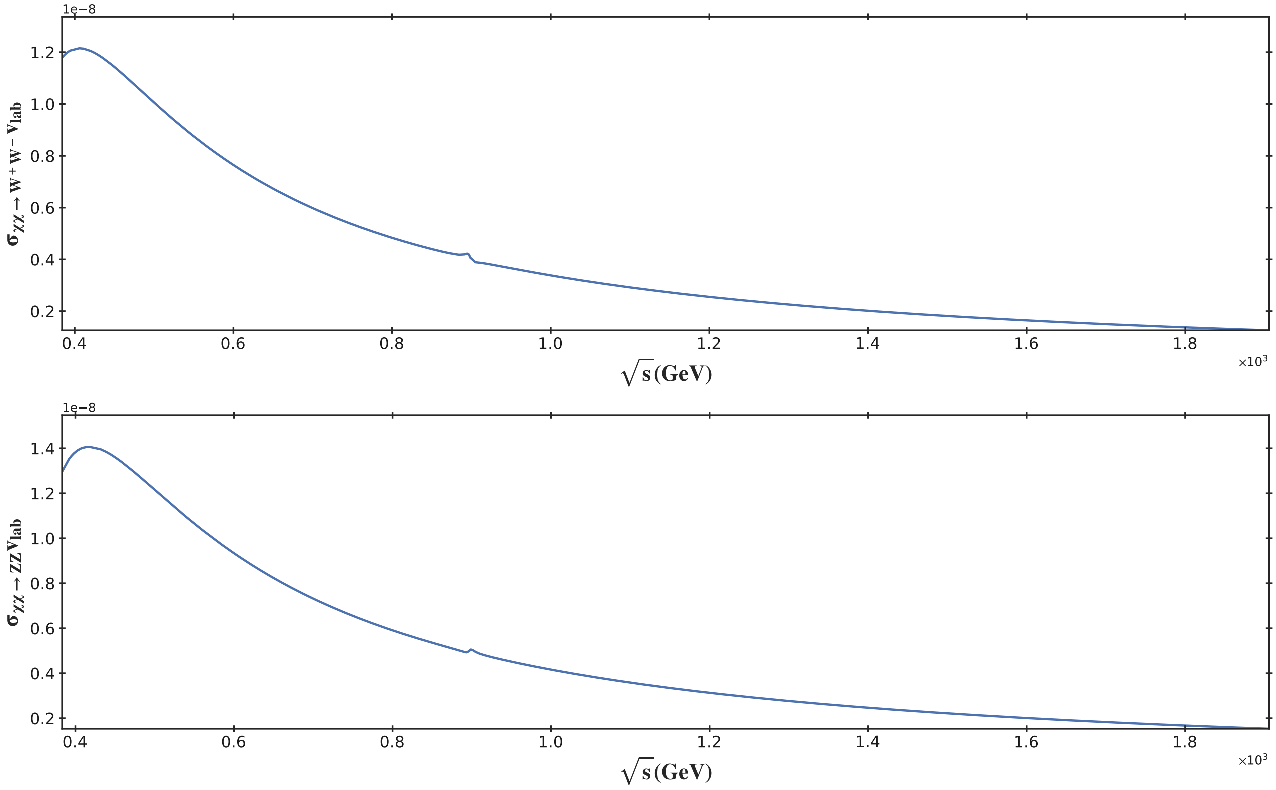


Figure VI.9: Cross section times laboratory velocity of neutralino pair annihilation with weak gauge boson final states. Top figure: $\sigma_\chi v_{\mathrm{lab}}$ of the process $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to W^+W^-$; bottom figure: $\sigma_\chi v_{\mathrm{lab}}$ of the process $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to ZZ$.

Similar to the fermionic final states, the production of W^+W^- and ZZ via neutralino pair annihilations are open for a large range of the collision energy due to large mass of $\tilde{\chi}_1^0$. These processes have a large contribution from velocity-independent part; while most of other tree-level amplitudes vanishes at the limit $v \to 0$, the annihilations into weak gauge boson pairs attain a value of order $\sim 10^{-8}$ GeV and deviate only several percent from the global maximum value. The weak gauge bosons also couple with Higgs boson in s-channel, which produces a visible small peaks at the resonances of h_4 and h_5. It is obviously to guess that (and indeed have already been numerically checked) s-channel with Higgs exchanged can be ignored in the calculations of total cross section.

With the value of $\sigma v_{\mathrm{lab}} \sim 10^{-8}$ GeV and slowly decreasing after reaching its global maximum, these two processes give the dominant contribution in almost all possible allowed energy. Exceptions are around the Higgs resonances, some other reactions couple much stronger with Higgs bosons and thus receive a huge enhancement from s-channel of exchanged Higgs. This enhancement is shown clearly in the plot of total σv_{lab}, see Fig. VI.12.

At low energy, the s-channel mediated by Z boson of these processes gives small contribution compared to t and u channels. For instance consider the cross section times relative velocity at threshold, the s-channel of Z only gives a result of ~ 2000 times smaller than when all channels are considered. At high energy limit, we observe that while t and u channels seperately have much bigger value than s channel with Z exchanged, the t and u channels partly cancels with each other to give a contribution of the same order with that of s channel.

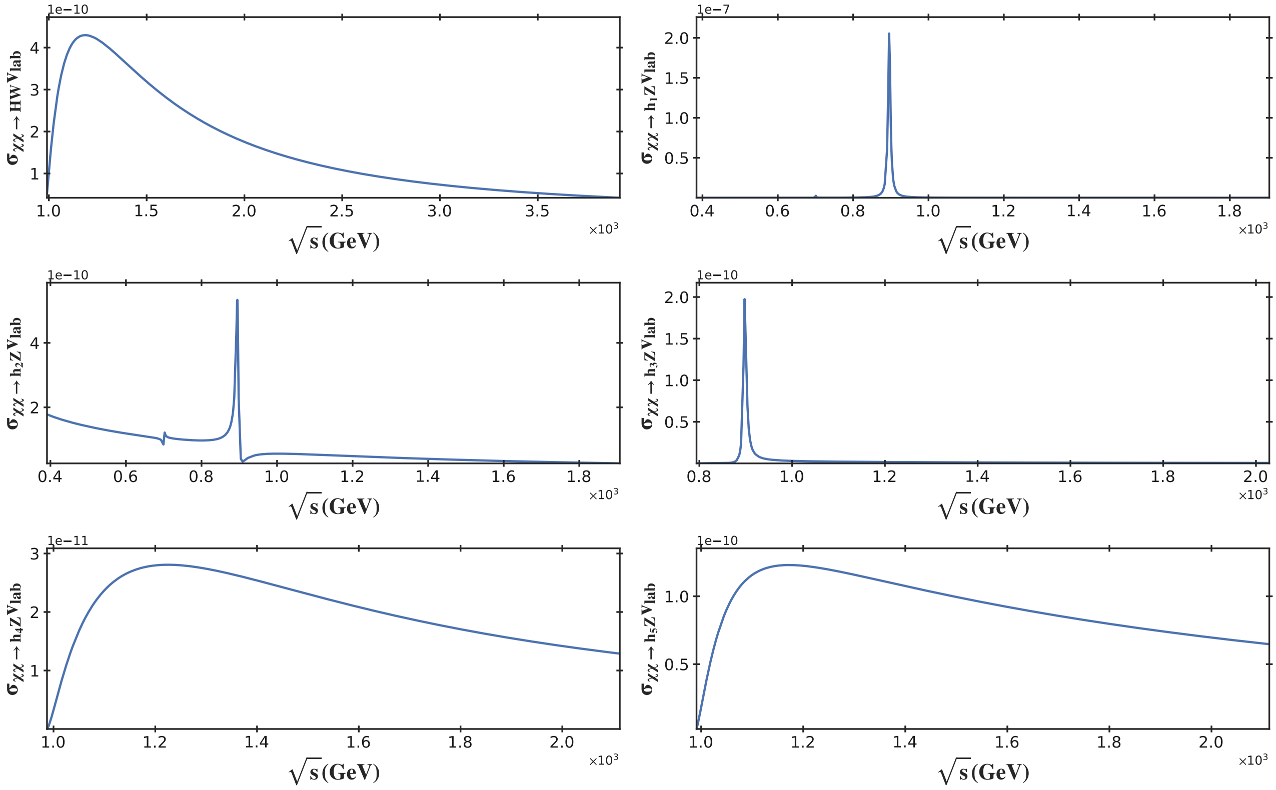


Figure VI.10: Cross section times laboratory velocity of neutralino pair annihilation with one weak gauge boson and one Higgs boson final states, including $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to HW$ and $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to hZ$.

Due to the distinctive mass spectrum of the neutral Higgs and charged Higgs bosons, the plot of each process with one Higgs boson and one weak gauge boson is quite different from each other. Firstly, only processes with final states contains h_1 and h_2 (two of the lightest neutral Higgs) are open at the limit $v \to 0$. At the threshold $\sqrt{s} = 2m_{\tilde{\chi}_1^0}$, the value $\sigma_{\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_1 Z} v_{\rm lab}$ is about $10^{-13}\,\mathrm{GeV}^{-2}$, while $\sigma_{\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_2 Z} v_{\rm lab} \sim 10^{-10}\,\mathrm{GeV}^{-2}$; other processes do not open at this point yet. Of all 6 cases shown in the figures, only those with sufficiently light final state (meaning $h_1 Z$, $h_2 Z$ and $h_3 Z$) have high peaks at resonances of h_4 and h_5; other cases contains too heavy final states to produce these resonances and thus have the "typical" shape: $\sigma v_{\rm lab}$ starts from 0 when these channels open, reaches a global maximum within the energy range 1100 GeV-1200 GeV, and slowly drops thereafter.

We also notice in the process $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_2 Z$, there is a destructive interference between s-channel mediated by Higgs bosons with t and u channels: all of these channel alone give $\sigma v_{\rm lab}$ a value of order 10^{-10}, but in total the contribution from these channels cancel out and produce the result of order 10^{-11}. This is the only situation within the 6 cases that have such a destructive behaviour with the unique local maximum at energy higher than all of Higgs resonances.

The typical value of $\sigma v_{\rm lab}$ in the final states of one Higgs and one weak gauge boson is about $\sim 10^{-10}, 10^{-11}$ GeV, much smaller when compared with the contribution of e.g weak gauge bosons final states where the typical value is in the range 10^{-8}. In addition, reactions that produce heavy Higgs bosons open only at high energy value and is not have negligible contribution to the total cross section at low energy limit. When taking the thermal average at temperature below the freeze-out T_f, the high energy tail would not give much contribution, and thus the neutralino annihilation to one Higgs boson and one weak gauge boson is not kinematically favourable. The only exception is the annihilation into $h_1 Z$ with the value of velocity-weighted cross section of order $10^{-7}\,\mathrm{GeV}^{-2}$ within the range of total energy around 800 GeV-1000 GeV. This enhancement from resonances of h_4 and h_5 make the cross section of $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_1 Z$ become the dominant process around $\sqrt{s} = 900$ GeV, explaining the blow up of total cross section as can be easily observed in Fig. VI.12.

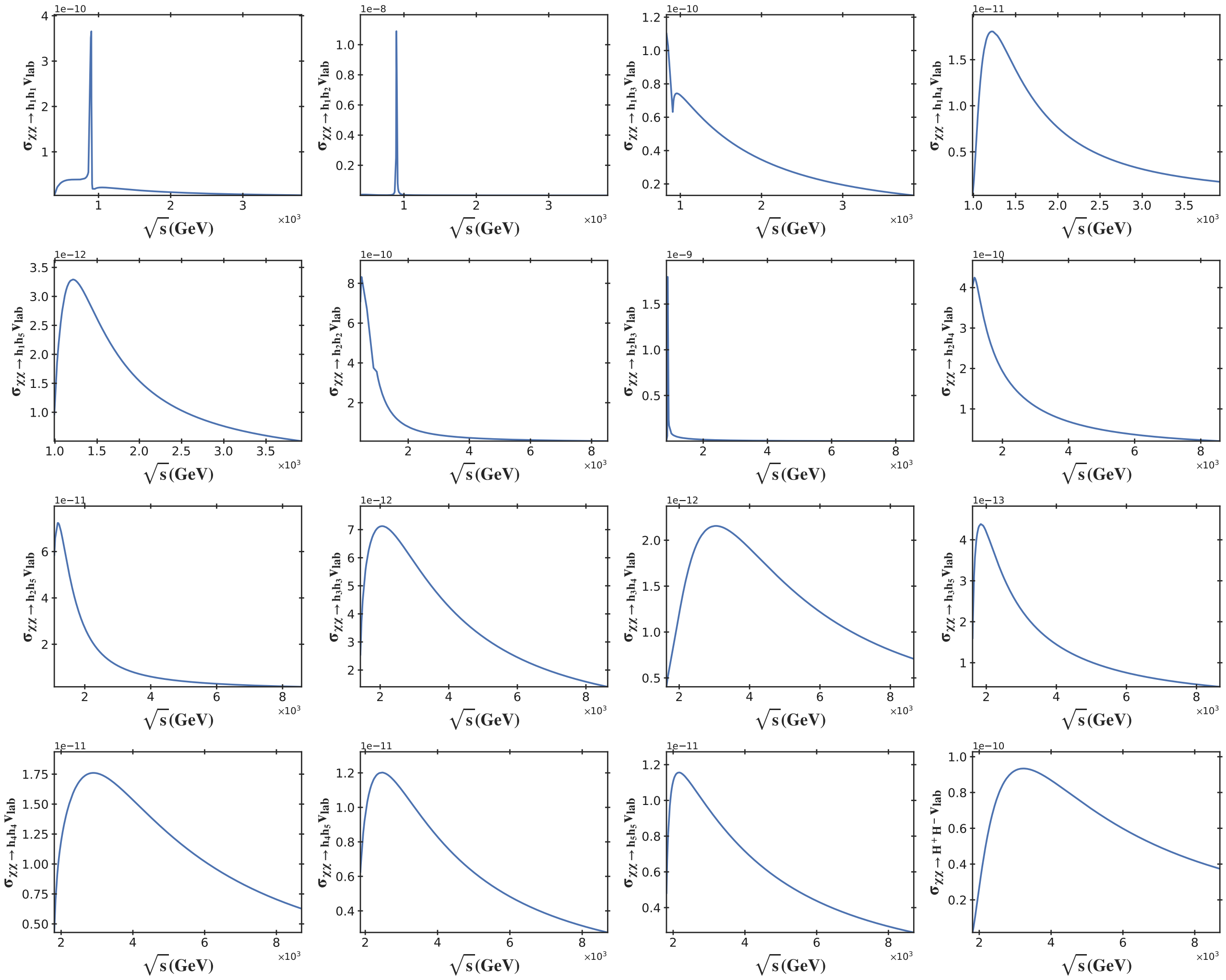


Figure VI.11: Cross section times laboratory velocity of neutralino pair annihilation with Higgs bosons final states, including $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to H^+H^-$ and $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to h_ih_j$, $i, j = 1, \ldots, 5$.

The final list of processes we consider are those with a pair of Higgs bosons in the final states. These reactions have quite similar behaviour as the case of neutralino annihilation to one Higgs boson and one weak gauge boson due to the mass spectrum of Higgs bosons results in the phenomenon that only reactions with production not containing the heavy Higgs (namely $H^\pm$, h_3, h_4 and h_5) have large impact on the thermal average $\langle\sigma_\chi v_{\text{lab}}\rangle$. Furthermore, the processes with heavy final Higgs bosons have smaller value of cross section compared to e.g $\tilde{\chi}_1^0\tilde{\chi}_1^0 \to W^+W^-$, even at large energy limit.

Another resemblance between the Higgs pair-production and one Higgs plus one weak gauge boson production from colliding two neutralino is that these processes can be catagorized into three main types based on the shape of line shown in figures, namely:

- Those with heavy final states (i.e total mass of final states larger than $M_{h,5}$) contain no Higgs resonance from s-channel and thus having the "typical" shape of a bump concentrated on the low-energy limit with the slowly decreasing trend at high energy tail.
- Those experience a spiky peak at the resonances and monotically drops when the difference between total energy - mass of resonance is larger than the width of the resonance.
- The remains are processes with a destructive behaviour after experience a resonance, usually a resonance of h_4, h_5 with energy around 900 GeV. These cases have not been fully analyzed and understood yet, which hopefully would be explainable after we consider the analytic expression of these processes as proposed in the outlook of Chapter VIII.

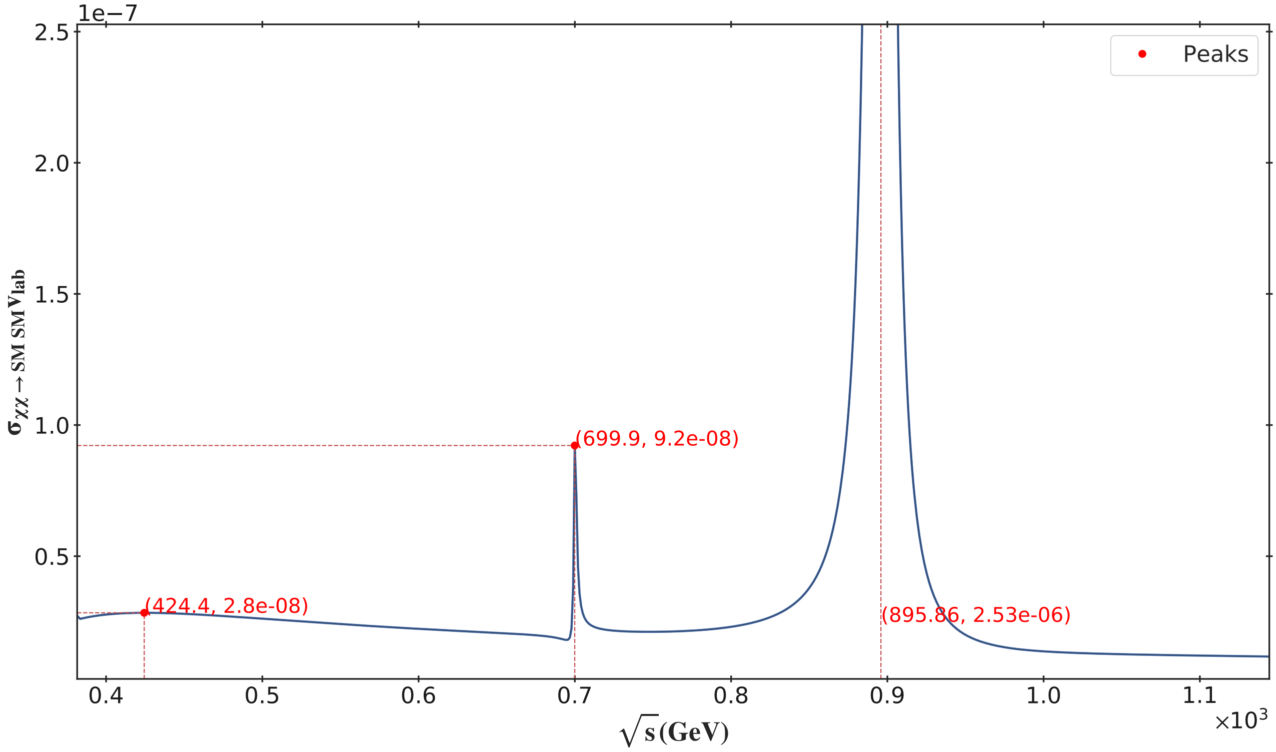


Figure VI.12: The total cross section times laboratory velocity of all neutralino pair annihilation into SM-like particles. The red points indicate the local maximum of σv_{lab}.

Fig. VI.12 shows three peaks of the total velocity-weighted annihilation rate, with the final peak actually being the combination of two resonances from h_4 and h_5. Since their masses are nearly degenerated, only a high peak is observed at energy around $890 - 900$ GeV. The second peak (from left to right) is due to the resonance of s-channel mediated by h_3, which is much shorter and narrower than the resonance of h_3 and h_4. The numerical results reveal that t-channel and u-channel are the dominant channels at sufficiently low energy in most processes (i.e near the threshold of opening a channel), results in the first bump at around 424 GeV. At higher scattering energy, the s-channel mediated by either Z or Higgs bosons becomes the dominate channel. This analysis shows that in calculating the integration over energy numerically (which we perform in calculating thermal average of $\sigma_\chi v_{\mathrm{lab}}$), carefulness should be paid for treating the resonances caused by mediators of s-channels. More details will be discussed in the next chapter which mainly about the numerical methods we applied to produce our results.

At the minimum energy $\sqrt{s} = 4m^2_{\tilde{\chi}^0_1}$ where the neutralinos are at rest, the value of $\sigma_\chi v_{\mathrm{lab}} \approx 1.9 \times 10^{-8}$ (GeV) mainly due to the s-wave contribution from t- and u-channel of annihilations into W^+W^- ($\sim 61.4\%$) and ZZ ($\sim 33.7\%$). The annihilation into a pair of top quark-anti top quark plays the crucial role at higher energy, even the main contribution at some phase-space point.

VII
Numerical Results for the Dark Matter Relics

Outline

VII.1 Input parameters

The SM parameters are chosen as follows

$$
\begin{array}{ll}
\alpha_{\text{em}}^{-1} = 127.955 & G_F = 1.16637 \times 10^{-5}\,(\text{GeV})^{-2} \\
\alpha_s = 0.1181 & M_Z = 91.1876\,(\text{GeV}) \\
m_b^{\text{MS}} = 4.18\,(\text{GeV}) & m_t = 172.74\,(\text{GeV}) \\
m_\tau = 1.77682\,(\text{GeV}) & M_W = 80.379\,(\text{GeV}) \\
m_e = 510.99891\,(\text{keV}) & m_\mu = 105.658367\,(\text{MeV}) \\
m_d = 4.7\,(\text{MeV}) & m_u = 2.2\,(\text{MeV}) \\
m_s = 95\,(\text{MeV}) & m_c = 1.274\,(\text{GeV})
\end{array}
$$

Since we have a short time to prepare the numerical analysis, we picked up an already available parameter point in the package **NMSSMCALC** to generate the spectrum used in this thesis. We have not investigated in details the phenomenologies of this point, to see whether it gives any interesting phenomenon. We leave these investigations in future works. The other parameters arising from NMSSM are given below

$M_1 = 644.4699\,(\text{GeV})$	$M_2 = 585.2285\,(\text{GeV})$
$M_3 = 1850\,(\text{GeV})$	$A_t = -1921.717\,(\text{GeV})$
$A_b = -1884.847\,(\text{GeV})$	$A_\tau = 1170.264\,(\text{GeV})$
$A_c = -1921.717\,\text{GeV}$	$A_s = -1884.847\,\text{GeV}$
$A_\mu = 1170.264\,\text{GeV}$	$A_u = -1921.717\,\text{GeV}$
$A_d = -1884.847\,\text{GeV}$	$\mu_{\text{eff}} = 208.7278\,(\text{GeV})$
$\tan\beta = 4.442242$	$M_{H^\pm} = 897.8267\,(\text{GeV})$
$\lambda = 0.301175$	$\kappa = 0.299105$
$\text{Re}A_\kappa = -791.4436\,(\text{GeV})$	$\text{Re}A_\lambda = 627.2\,\text{GeV}$
$M_{\tilde{L}_1} = M_{\tilde{L}_2} = 3000\,(\text{GeV})$	$M_{\tilde{L}_3} = 1368.968\,(\text{GeV})$
$M_{\tilde{E}_1} = M_{\tilde{E}_2} = 3000\,(\text{GeV})$	$M_{\tilde{E}_3} = 2967.018\,(\text{GeV})$
$M_{\tilde{Q}_1} = M_{\tilde{Q}_2} = 3000\,(\text{GeV})$	$M_{\tilde{Q}_3} = 1226.038\,(\text{GeV})$
$M_{\tilde{U}_1} = M_{\tilde{U}_2} = 3000\,(\text{GeV})$	$M_{\tilde{U}_3} = 880.8624\,(\text{GeV})$
$M_{\tilde{D}_1} = M_{\tilde{D}_2} = 3000\,(\text{GeV})$	$M_{\tilde{D}_3} = 2765.234\,(\text{GeV})$

The calculations of the NMSSM mass spectrum is performed by the package **NMSSMCALC**, given in the table below

Higgs bosons	
$M_{h_1} = 87.2381\,(\text{GeV})$	$M_{h_2} = 125.0445\,(\text{GeV})$
$M_{h_3} = 700.044\,(\text{GeV})$	$M_{h_4} = 895.9974\,(\text{GeV})$
$M_{h_5} = 897.8033\,(\text{GeV})$	$M_{H^\pm} = 897.8267\,(\text{GeV})$
Sfermions	
$m_{\tilde{d}_1} = 3000.0930\,(\text{GeV})$	$m_{\tilde{d}_2} = 3000.5330\,(\text{GeV})$
$m_{\tilde{u}_1} = 2999.560\,(\text{GeV})$	$m_{\tilde{u}_2} = 2999.8138\,(\text{GeV})$
$m_{\tilde{s}_1} = 3000.0886\,(\text{GeV})$	$m_{\tilde{s}_2} = 3000.5373\,(\text{GeV})$
$m_{\tilde{c}_1} = 2999.2502\,(\text{GeV})$	$m_{\tilde{c}_2} = 3000.1241\,(\text{GeV})$
$m_{\tilde{b}_1} = 1226.0458\,(\text{GeV})$	$m_{\tilde{b}_2} = 2765.3443\,(\text{GeV})$
$m_{\tilde{t}_1} = 810.9920\,(\text{GeV})$	$m_{\tilde{t}_2} = 1275.6929\,(\text{GeV})$
$m_{\tilde{e}_1} = 3000.2792\,(\text{GeV})$	$m_{\tilde{e}_2} = 3000.3468\,(\text{GeV})$
$m_{\tilde{\mu}_1} = 3000.2789\,(\text{GeV})$	$m_{\tilde{\mu}_2} = 3000.3471\,(\text{GeV})$
$m_{\tilde{\tau}_1} = 1369.729\,(\text{GeV})$	$m_{\tilde{\tau}_2} = 2967.3009\,(\text{GeV})$
$m_{\tilde{\nu}_e} = 2999.3738\,(\text{GeV})$	$m_{\tilde{\nu}_\mu} = 2999.3738\,(\text{GeV})$
$m_{\tilde{\nu}_\tau} = 1367.5953\,(\text{GeV})$	
Mixing of gauginos and higgsinos	
$m_{\tilde{\chi}_1^\pm} = 199.1646\,(\text{GeV})$	$m_{\tilde{\chi}_2^\pm} = 599.4287\,(\text{GeV})$
$m_{\tilde{\chi}_1^0} = 190.7491\,(\text{GeV})$	$m_{\tilde{\chi}_2^0} = 214.6911\,(\text{GeV})$
$m_{\tilde{\chi}_3^0} = 420.9749\,(\text{GeV})$	$m_{\tilde{\chi}_4^0} = 598.2518\,(\text{GeV})$
$m_{\tilde{\chi}_5^0} = 649.0002\,(\text{GeV})$	$m_{\tilde{g}} = 1850.0000\,(\text{GeV})$

Table VII.1: Mass spectrum of new particles arise from NMSSM with the listed input parameters.

With the lighest neutralino mass $\sim 199\,(\text{GeV}) > M_Z$, no resonance can occurs in the s-channel mediated by Z boson. Similarly for the s-channel with Higgs exchanged, from the mass spectrum of Higgs bosons it is clear that only processes mediated by h_3, h_4 and h_5 can have resonances. The total decay width of these resonances are

$$\Gamma_{h_3} = 1.9096\,(\text{GeV}), \quad \Gamma_{h_4} = 7.3494\,(\text{GeV}), \quad \Gamma_{h_5} = 7.1793\,(\text{GeV}), \tag{VII.1.1}$$

given in the output file `slha.out` of **NMSSMCALC**.

VII.2 Numerical Results

VII.2.1 Thermal average $\langle \sigma_\chi v_{\text{Møl}} \rangle$

Performing the single-integral in (II.2.25), one can calculate $\langle \sigma_\chi v_{\text{Møl}} \rangle$ numerically. For efficient purposes, let us manipulate the integral by changing integration variable.

Firstly, note that we must calculate one integration in over kinetic energgy per unit mass ϵ, which ranges from 0 to ∞. There are a variety of methods to deal with this kind of integrals. One possible solution is by considering the asymptotic behaviour of the thermal kernel $\mathscr{K}(x, \epsilon)$, which is constructed using modified Bessel functions of the first and second kind. Recall formula (II.2.29) for expansion of $\sigma_\chi v_{\text{Møl}}$ with respect to ϵ

$$\langle \sigma_\chi v_{\text{Møl}} \rangle \simeq 2\sqrt{\frac{x^3}{\pi}} \int_0^\infty \mathrm{d}\epsilon\, \sqrt{\epsilon} \frac{1+2\epsilon}{(1+\epsilon)^{1/4}} e^{-2x\left(\sqrt{1+\epsilon}-1\right)} \frac{P_1\left(2x\sqrt{1+\epsilon}\right)}{P_2^2(x)} \sigma_\chi v_{\text{lab}}. \tag{VII.2.1}$$

By changing the integration variable from $\epsilon \to y = 2x(1+\epsilon) - 1$ with y ranges from 0 to ∞, Eq. (II.2.29) is transformed into the form

$$\langle \sigma_\chi v_{\text{Møl}} \rangle \simeq \int_0^\infty \mathrm{d}y\, e^{-y} F(x,y). \tag{VII.2.2}$$

Such integral can be evaluated using Gauss-Laguerre quadrature method

$$\int_0^\infty e^{-x} f(x)\mathrm{d}x \approx \sum_{i=1}^n w_i f(x_i), \qquad w_i = \frac{x_i}{(n+1)^2 \left[L_{n+1}(x_i)\right]^2}, \tag{VII.2.3}$$

where x_i is the i-th root of the Laguerre polynomial $L_n(x)$.

A closer look at the integrand of (II.2.25) shows that the cross section experiences sharp peaks at the resonances, and a numerical integration method with arbitrary step size may not produce a good result. One can however applying an importance sampling method to reduce the numerical error, i.e to map the integration variable space to another space where the integrand behaves to be more "equally distributed". The difficulty with importance sampling is that one do not know what is a suitable mapping to use. In fact, from Fig. VI.12 we see that each peak contributes differently to $\sigma_\chi v_{\text{Møl}}$, therefore it should be extremely hard to construct such mapping. From these realizations, we should consider taking the given 2D integral using an adaptive step size method to reduce the integration variance. In practice, we use the VEGAS algorithm (Ref. [54; 55]), a variance reduction of Monte-Carlo method with an adaptive sampling scheme, into numerical integration. This algorithm is available in a library called **CUBA**. This library offers several choice to deal with multidimensional numerical integration, with **C/C++**, **Fortran** and **Mathematica** interfaces. For more information about this library, we reference to the paper [56].

Let us manipulate the integral (II.2.25) into a more useful and practical form for using Monte-Carlo-like integral evaluation. First consider integral of the form

$$I = \int_{x_0}^\infty f(x)\mathrm{d}x, \tag{VII.2.4}$$

by changing from $x \to y = \dfrac{x - x_0}{1 + (x - x_0)} \Rightarrow x = x_0 + \dfrac{y}{1-y}$, the given integral I becomes

$$I = \int_0^1 f\left(\frac{y}{1-y}\right) \frac{\mathrm{d}y}{(1-y)^2}, \tag{VII.2.5}$$

which is the typical form of calculating numerical integration. Let us apply the above tricks to calculate the thermal average of cross-section times Møller velocity

$$\langle \sigma_\chi v_{\text{Møl}} \rangle = \frac{1}{32\pi} \int_0^\infty \mathrm{d}\epsilon \int_0^\pi \mathrm{d}\theta \sin\theta \times \mathscr{K}(x,\epsilon) \times \frac{\beta_f(s)}{s - 2m_\chi^2} \overline{|\mathcal{M}(s,\theta)|^2}, \tag{VII.2.6}$$

where the Mandelstam variable $s = 4m_\chi^2(1+\epsilon)$, and the invariant factor

$$\beta_f = \left[1 - \frac{(m_3+m_4)^2}{s}\right]^{1/2} \left[1 - \frac{(m_3-m_4)^2}{s}\right]^{1/2}. \tag{VII.2.7}$$

Note that when $m_3 = m_4$, β_f coincides with the velocity of the final particles in the center of mass frame. We apply the following change of integration variables

$$\begin{cases} \theta = \pi r_1 & \text{(VII.2.8)} \\ \epsilon \to \dfrac{r_2}{1-r_2} \Rightarrow s = 4m_\chi^2\left(1 + \dfrac{r_2}{1-r_2}\right) = \dfrac{4m_\chi^2}{1-r_2} & \text{(VII.2.9)} \end{cases}$$

Finally we obtain

$\langle \sigma_\chi v_{\text{Møl}} \rangle (x)$

$$= \frac{1}{32\pi}\int_0^1 \frac{\mathrm{d}r_2}{(1-r_2)^2}\int_0^1 \mathrm{d}r_1 \pi \sin(\pi r_1) \times \frac{\mathscr{K}\left(x, \frac{r_2}{1-r_2}\right)}{2m_\chi^2(1+r_2)/(1-r_2)} \times \sum \left\{\beta_f\left(\frac{4m_\chi^2}{1-r_2}\right) \times \overline{\left|\mathcal{M}\left(\frac{4m_\chi^2}{1-r_2}, \pi r_1\right)\right|^2}\right\}$$

$$= \frac{1}{64m_\chi^2}\int_0^1 \mathrm{d}r_2 \int_0^1 \mathrm{d}r_1 \frac{\sin(\pi r_1)}{1-r_2^2} \times \mathscr{K}\left(x, \frac{r_2}{1-r_2}\right) \times \sum \left\{\beta_f\left(\frac{4m_\chi^2}{1-r_2}\right) \times \overline{\left|\mathcal{M}\left(\frac{4m_\chi^2}{1-r_2}, \pi r_1\right)\right|^2}\right\}. \tag{VII.2.10}$$

The summation is taken over all processes of neutralino pair-annihilation to SM-like particles. The numerical result for thermal averaging is shown in Fig. VII.1.

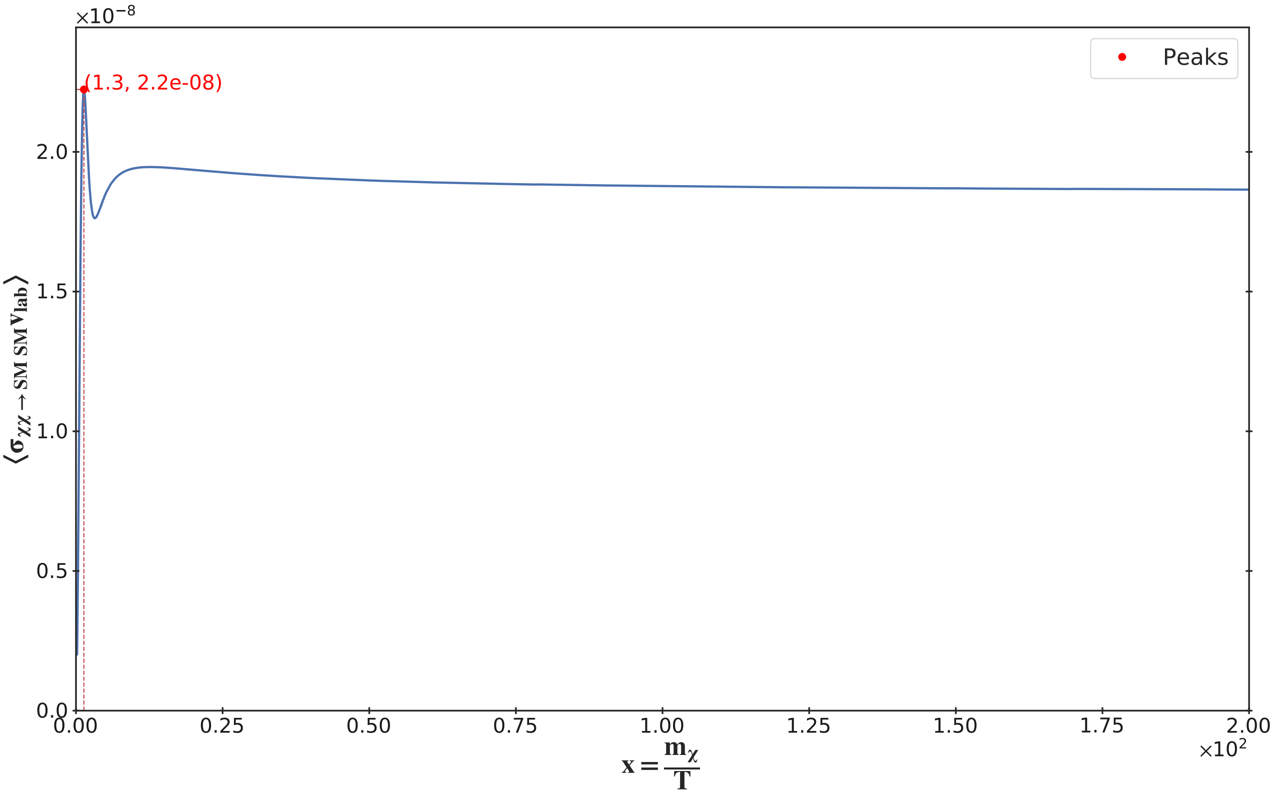


Figure VII.1: The thermal average of velocity-weighted total cross section in the range $x \in (0, 200)$.

VII.2.2 Solving the freeze-out equation

Recall that the freeze-out point x_f is the solution of the freeze-out equation (II.2.42)

$$f(x) \equiv -\left(\frac{\pi}{45G}\right)^{1/2} \frac{g_*^{1/2} m}{x^2} \langle \sigma_\chi v_{\mathrm{M}\o\mathrm{l}} \rangle\, Y^{\mathrm{eq}}\, \delta(\delta+2) + \frac{\mathrm{d}\ln Y^{\mathrm{eq}}}{\mathrm{d}x} = 0. \tag{VII.2.11}$$

This is a typical nonlinear equation, which can be solved by various numerical methods.

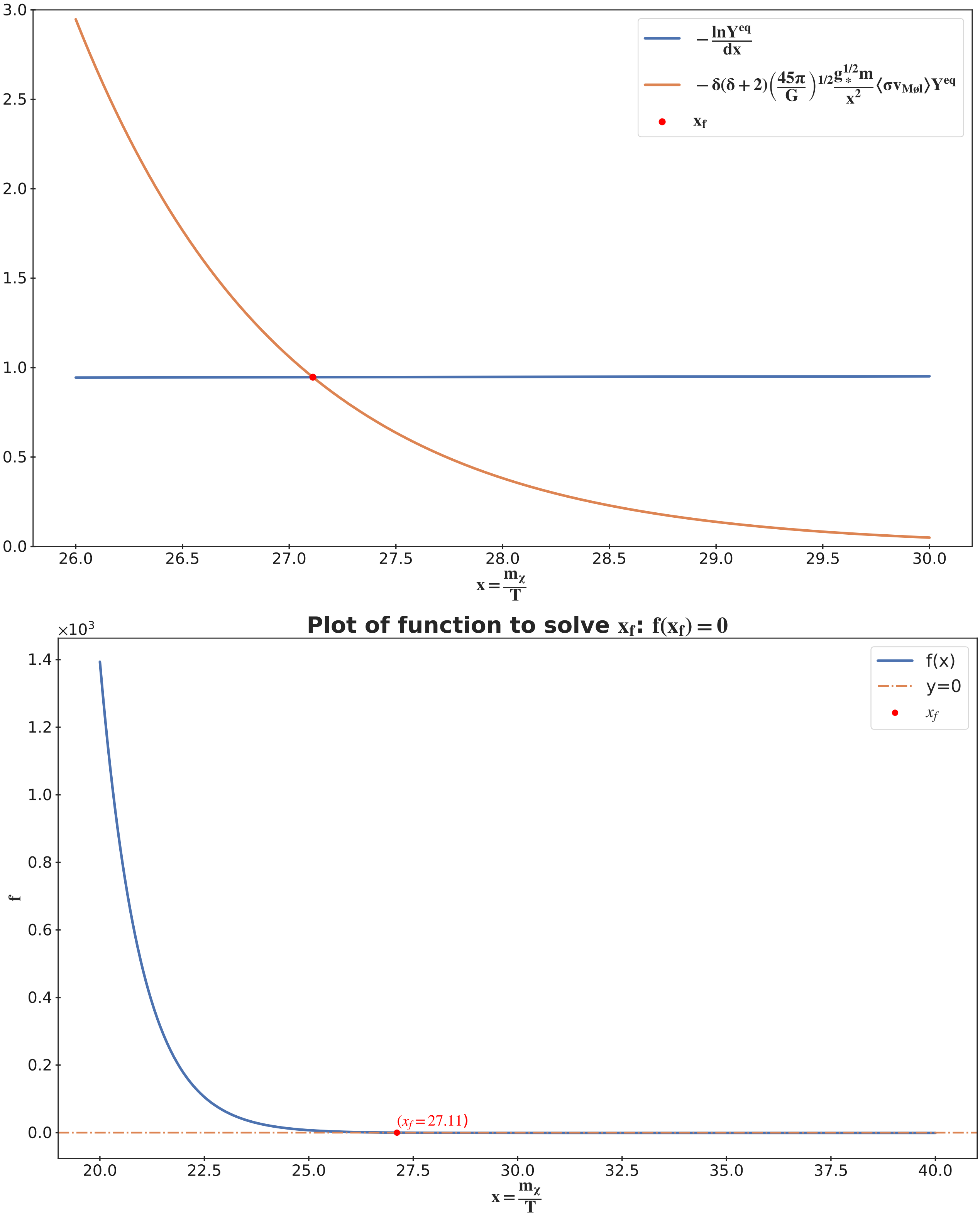


Figure VII.2: Behaviour of equation for freeze-out condition (VII.2.11). The parameter δ is set to 1.5, which is the suggested value in Ref. [**21**, **p. 163**]. The upper figure shows the contribution of each term in Eq. (VII.2.11), which adds up to $f(x)$ in the lower figure.

As we can see from Fig. VII.2, before freezing-out the function $f(x)$ drops substantially due to the factor $1/x^2$ (the derivative term $\mathrm{d}\ln Y^{\mathrm{eq}}/\mathrm{d}x$ decreases logarithmically, thus its contribution to the trend of $f(x)$ can be neglected compared to the contribution of the term $\sim \dfrac{g_*^{1/2} m}{x^2} \langle \sigma_\chi v_{\mathrm{Møl}} \rangle Y^{\mathrm{eq}}$. After freezing-out, the equilibrium yield decreases exponentially to 0 due to the Boltzmann factor e^{-x}, and the main contribution to $f(x)$ comes from the derivative term.

Let us analyze this equation before choosing a appropriate method. We will investigate two class of numerical root-finding methods. The first class is so-called *bracketing methods*, for which the most well-known examples are bisection method, false position method and Brent's method [57] (which is a combination of the bisection method, the secant method and inverse quadratic interpolation). The second class, namely *open methods*, contains root-finding algorithm such as simple fixed point iteration, Newton-Raphson method, secant method. Especially when the cost of computing $\langle \sigma_\chi v_{\mathrm{Møl}} \rangle$ is significantly considerable, the questions of which method should be implemented to solve the freeze-out point becomes important.

Tolerance	Number of iterations		
	Bisection method	Brent method	Secant method
10^{-1}	7	8	13
10^{-2}	10	12	14
10^{-3}	14	12	15
10^{-4}	17	12	16
10^{-5}	20	12	16

Table VII.2: Solutions to freeze-out point x_f with different methods. Both of the bisection and Brent methods belongs to the class of bracketing method, here we set the initial range $[x_{\min}, x_{\max}] = [20, 30]$. The secant method is an open method which requires two initial points, here chosen to be $x_0^{(1)} = 20$ and $x_0^{(2)} = 21$.

With the typical value of WIMP freeze-out $x_f \sim 25$, we can seek for the solution within a specific range by bracketing method easily. From the number of iterations of each methods, the Brent method shows its advantages of rapid convergence rate only at extremely high precision requirement. The tolerance ~ 0.1 is good enough for the calculation of relic density using freeze-out approximation, thus both methods give nearly the same convergence rate. Similar to the Brent method, the scant method shows its advantages when we need high precision result of x_f. It should be noted that due to the stiffness behaviour of the freeze-out equation with $x < x_f$ and the slowly varying trend with $x > x_f$, the secant method depends significantly on the chosen initial points. In our calculations, an attempt of choosing $[x_0^{(1)}, x_0^{(2)}] = [20, 30]$ or $[29, 30]$ fails to converge to the correct solution.

For comparison and completeness, let us represent the non-relativistic limit of the freeze-out condition

$$\sqrt{\frac{\pi}{45G}} \frac{45g}{8\pi^4} \frac{\sqrt{\pi x} e^{-x}}{h_{\mathrm{eff}}(T)} \sqrt{g_{\mathrm{eff}}(T)} m \langle \sigma_\chi v_{\mathrm{Møl}} \rangle \, \delta(\delta + 2) = 1. \tag{VII.2.12}$$

Solving this equation requires approximately the same computational resources compared to its relativistic version, and the same arguments on the suitable numerical-solving method can also be applied here. We represent below the solution of freeze-out equation in both relativistic and non-relativistic limit with varying δ.

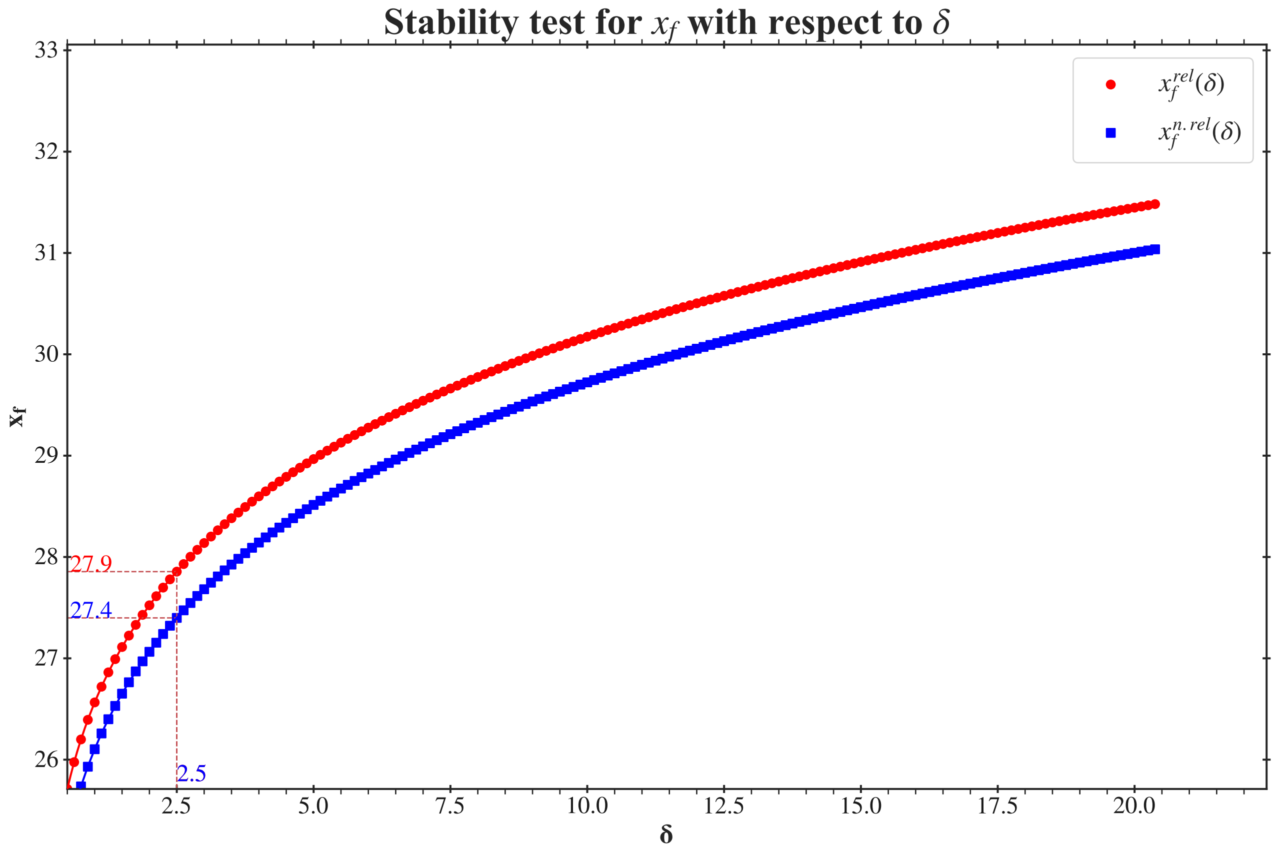


Figure VII.3: Solution of freeze-out point x_f. The red dots represent the numerical solution of relativistic freeze-out condition (VII.2.11). The blue dots represent the numerical solution of the non-relativistic version. The calculation range of δ is chosen from ~ 0 to 20.

The solution for freeze-out criterion thus being insensitive with respect to parameter δ. As we can see in the figure above, with increasing δ the freeze-out point logarithmically rises. This behaviour has been mentioned in [**58**, **Section 5.2**]. In our calculation, we use the value of $\delta = 1.5$ which results in $x_f \approx 26.8$.

VII.2.3 Numerical solution of the comoving number density

The simplest approximation we apply to solve the current comoving density is the *freeze-out approximation*, which has been represented in Subsection II.2.2. Basically, performing the last integration over temperature yields the desired result of Y_0

$$\frac{1}{Y_0} \equiv \frac{1}{Y(T_0)} = \frac{1}{Y_f} + \left(\frac{\pi}{45G}\right)^{1/2} \int_{T_0}^{T_f} g_*^{1/2} \langle \sigma_\chi v_{\text{Møl}} \rangle \, \mathrm{d}T. \tag{VII.2.13}$$

For visualization purposes, let us show the dependence of the integrand $\sqrt{\pi/45G} g_*^{1/2} \langle \sigma_\chi v_{\text{Møl}} \rangle$ over temperature in the figure below.

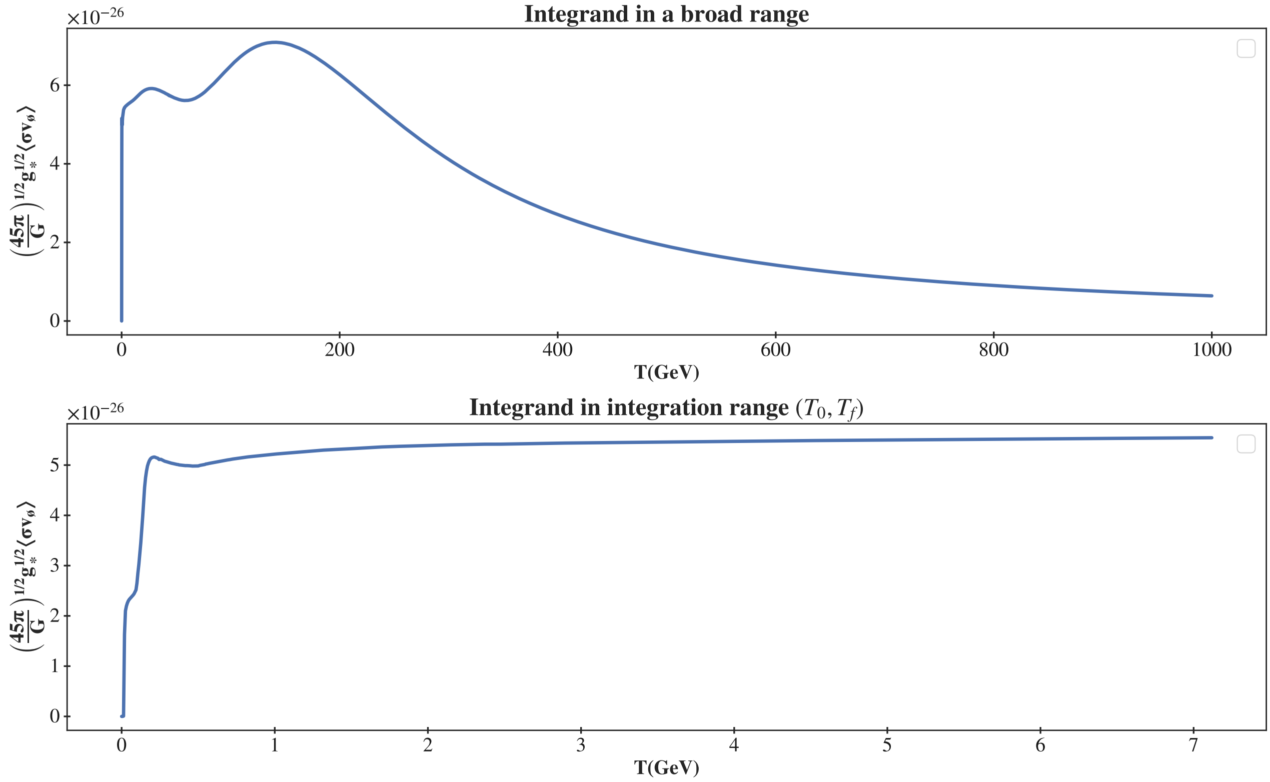


Figure VII.4: Integrand of integration over temperature to determine the current comoving density Y_0. The upper plot shows the integrand in a broad range of temperature while the lower one represents the integrand in the integration region (T_0, T_f). To see the contribution of each process separately, see Appendix F.

The integrand increases stiffly near $T = 0$ (GeV) and remains nearly constant in the integration range, showing the s-wave of $\sigma_\chi v_{\text{lab}}$ is the dominant term. As the temperature rises, the integrand experiences some fluctuation in the range $T \sim 0 \to 200$ (GeV) before gradually decline at higher temperature.

To a quite good precision, the value $1/Y_f$ can be ignored and the lower bound of integration $T_0 \approx 2.726\text{K} \approx 2.35 \times 10^{-13}$GeV is safely set to 0 compared to the upper bound T_f, which in our case ≈ 7 GeV.

At this point, we have two ways to obtain the relic density:

- The first way is by performing a 3-dimensional integration (over scattering angle θ, kinetic energy in a unit mass ϵ and temperature T). It is fully expressed as the equations below

$$Y_0 \approx \frac{1}{\dfrac{1}{Y_f} + \left(\dfrac{\pi}{45G}\right)^{1/2} \displaystyle\int_{T_0}^{T_f} g_*^{1/2} \langle \sigma_\chi v_{\text{Møl}} \rangle \, \mathrm{d}T}, \tag{VII.2.14}$$

  where the integral is performed similar to Eq. (VII.2.10), i.e by changing the integration variables

$$\int_{T_0}^{T_f} g_*^{1/2}(T) \langle \sigma_\chi v_{\text{Møl}} \rangle (m_\chi/T) \, \mathrm{d}T = \frac{(T_f - T_0)}{64 m_\chi^2} \int_0^1 \mathrm{d}r_3 \int_0^1 \mathrm{d}r_2 \int_0^1 \mathrm{d}r_1 \left\{ \frac{\sin(\pi r_1)}{1 - r_2^2} \times \mathscr{K}\left(x, \frac{r_2}{1 - r_2}\right) \right.$$
$$\left. \times\, g_*^{1/2} \left(m_\chi / \left[T_0 + (T_f - T_0) r_3\right]\right) \times \sum \left[\beta_f \left(\frac{4m_\chi^2}{1 - r_2}\right) \times \left| \mathcal{M}\left(\frac{4m_\chi^2}{1 - r_2}, \pi r_1\right) \right|^2 \right] \right\}. \tag{VII.2.15}$$

  Taking the above integral numerically using VEGAS yields the desired result.

- The second method gives us a rough result of Y_0, and works well in our case. The idea is that in cases the integrand $g_*^{1/2} \langle \sigma_\chi v_{\text{Møl}} \rangle$ does not rapidly change (especially does not have sharp peaks), the integration made use of equally spaced grid could be applied. In our code, we implement the composite trapezoidal rule as a default method for calculating the integral from the input data table. We first calculate the thermally averaged cross section times Møller velocity with parameter T in the range $(T_{\min}, T_{\max})$ with

$T_{\min} \leq T_0$ and $T_{max} \geq T_f$, and save these results in the tabular output file. That data can be later loaded to calculate the yield at whatever value of temperature in the range $(T_{\min}, T_{\max})$.

The first method has an advantage that we can control the integration error via the number of sampled points, but in case we need the numerical values of $\langle \sigma_\chi v_{\text{Møl}} \rangle$, we have to perform the integration (VII.2.10) separately. With the second approach, we can obtain both $\langle \sigma_\chi v_{\text{Møl}} \rangle$ and Y in the interested region of temperature. However, the precision of Y depends on the precision of $\langle \sigma_\chi v_{\text{Møl}} \rangle$ at the first place, and also the error of the composite trapezoidal rule. The latter can be control by increasing the number of points at which we calculate $\langle \sigma_\chi v_{\text{Møl}} \rangle$, i.e by expanding the tabular data. With a fixed calculation range $(T_{\min}, T_{\max})$, we can perform such extension by simply sample more points between two neighbor points in the old tabular data. The following figure illustrate such idea:

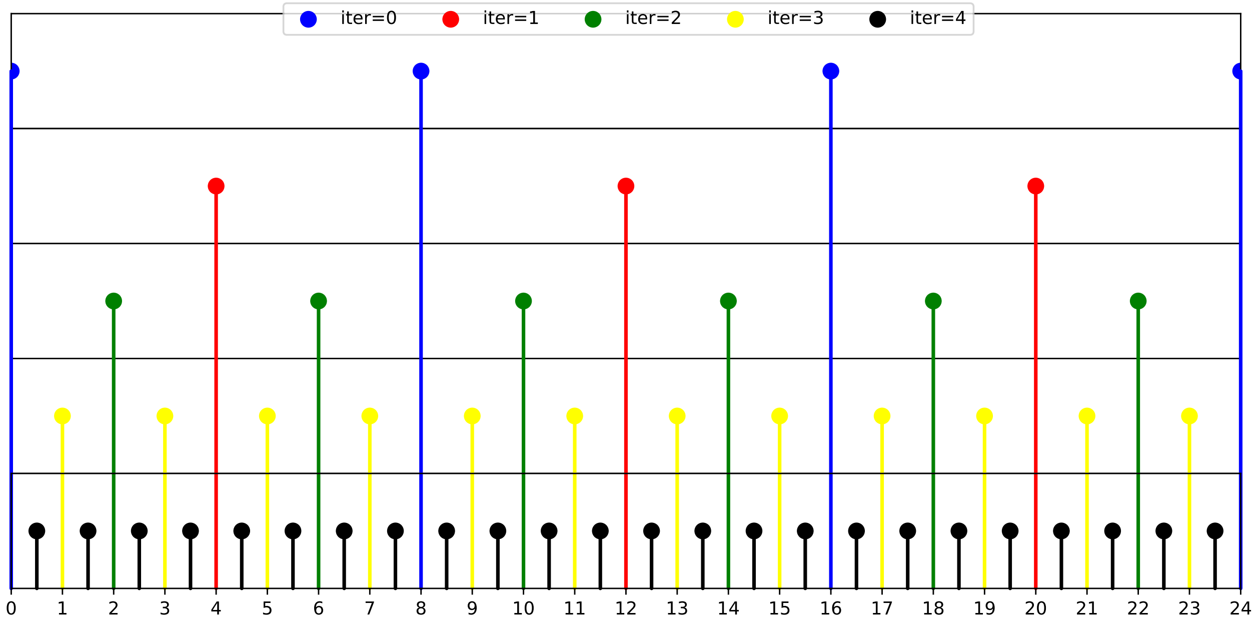


Figure VII.5: Illustration of "gradually denser sampling method". We start (at iter=0) by dividing the given range into n_{sample} equal subregions. After each iteration, a fixed number of new points n_{grid} will be added into each subregion of the old iteration. In this figure we set $n_{\text{sample}} = 3$ and $n_{\text{grid}} = 2$.

Whenever we need to increase the number of points at which $\langle \sigma_\chi v_{\text{Møl}} \rangle$ is calculated, we can keep the old tabular data and proceed the calculation for new data at points in the next iteration. This is a small improvement introduced to the tabular approach we represented above.

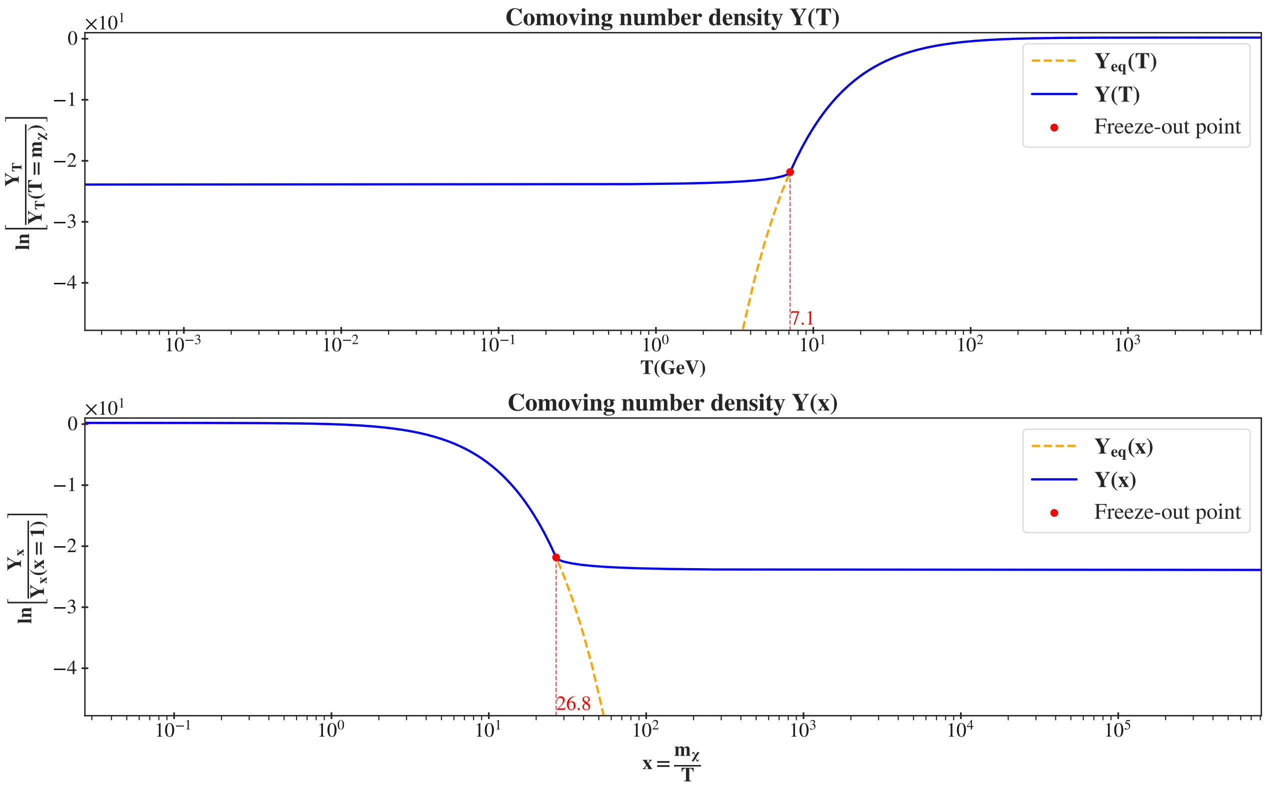


Figure VII.6: Solution of the yield Y with respect to temperature T (the top figure) and to dimensionless parameter x using freeze-out approximation.

For the given set of input parameters, the obtained value of relic density of the lighest neutralino in the NMSSM is

$$\begin{aligned}\Omega_{\tilde{\chi}_1^0} h^2 &\approx 2.8282 \times 10^8 \times m_{\tilde{\chi}_1^0} \times Y(x = x_0) \approx 2.8282 \times 10^8 \times 190.74914 \times 2.334 \times 10^{-13} \\ &\approx 0.01259,\end{aligned} \tag{VII.2.16}$$

compared to the allowed range for the DM relic density from cosmological data [59; 60]

$$0.094 < \Omega_{\mathrm{DM}} h^2 < 0.136, \tag{VII.2.17}$$

our result is smaller than the lower bound of the total relic density of DM. Given our tree-level approximation for the calculation of the annihilation processes and neglecting coannihilation ones, this parameter point does not satisfy the DM relics constraint.

VIII
Summary and Outlooks

Outline

VIII.1 Summary

This thesis has focused on discussing two main parts: the dark matter relic density, and the Next-to-Minimal Supersymmetric Standard Model (NMSSM). What we have done so far is to assume the lightest neutralino in NMSSM to be a component of the dark matter, and by using the freeze-out approximation we are able to calculate the relic density of this particle species. While the results are merely a number of the neutralino abundance at current time, this is important to investigate the DM nature from theoretical point of view since it tells the contribution of the neutralino to the total amount of DM. This part is meant to be a brief summary of what we have represented in this thesis so far:

- Chapter I and III are two short and instructive introduction which aims to answer the two set of questions: *Why does "Dark Matter" matters, and under which mechanism can dark matter be generated?*, and *What is the Standard Model of Physics, and why we need to consider its extension?*. These types of questions are what motivate the construction of appropriate frameworks to describe the calculations involving the density of DM or an supersymmetric extension of the SM.

- Chapter II describes a standard formulation of the Boltzmann equation, with various arguments and approximations have been made in order to simplify the equation that governs the evolution of density of a given species. Some special cases (e.g the asymptotic behaviour of the solution, or techniques for taking the thermal average of a physical quantity) are also investigated.

- Chapter IV and V discussed about the construction of a supersymmetric theory, with two successive applications on extending the SM has been made, namely the MSSM and NMSSM. While the simplest supersymmetric extension MSSM expresses several advantages over the SM, a hierarchy problem so-called μ-problem arises from MSSM. NMSSM is thus proposed to overcome this short-coming of the MSSM by introducing one Higgs singlet into the particle content, thus generate the μ parameter dynamically from electroweak symmetry breaking process. We also cover the detailed derivation of the Lagrangian, as well as analyzing the mass spectrum of both MSSM and NMSSM within these two chapters.

- Finally, the calculation techniques together with the numerical results are represented in the two chapters VI and VII. Careful analysis on each processes of neutralino annihilations at tree-level has been done in Chapter VI, whereas the thermal average of velocity-weighted cross section, the freeze-out point solution and the final integral for calculating the comoving number density have been represented in Chapter VII.

VIII.2 Outlooks

Our private code have been built based on the helicity amplitude method from the basic standpoint instead of taking the advantages of the already built and optimized public packages such as `HELAS` (Ref. [61]). While this gives us a great opportunity to practice on this method as well as to approach modern improvements which boost the calculation speed, the consequence is the cost of slow computation speed since only a few optimization efforts have been made while studying this subject. Most of them have been implemented into our computer code, that mainly related to the decomposition of complicated spinor structure into a sum of simpler ones. Several more complex spinor structures arising in the gauge bosons final states have not been simplified and the Dirac matrix multiplication is applied to produce the data. As a consequence, these two processes requires a longer computation running time in comparison to other processes. Reviewing the spinor chains and optimizing the helicity amplitude method in the code are in the highest priority tasks we want to consider.

More efforts should be put into increasing the precision of the relic density. As being mentioned when deriving the freeze-out approximation with the final DM relic density is calculated via formula (II.2.45); this integral ignores the contribution of equilibrium comoving number density after the DM particle decoupling from thermal bath. Various more modern methods (e.g implicit ODE solving method) have been discussed and succesfully implemented in the public code such as **`DarkSUSY`**, **`micrOMEGAs`** or **`MadDM`**, which we should employ to have more accurate results. Furthermore, the coannihilations (e.g reactions between lightest neutralino and other neutralinos, charginos or sneutrinos) have not been included in the calculations of the relic density in the thesis scope, which can potentially have great impact on the relic density of neutralino annihilation alone. In addition, the sneutrinos in the NMSSM can be a viable DM candidate if it is the lightest supersymmetric particle. We leave the study about this possibility in the future.

Finally, the parameter constraints within the NMSSM context has been studied by many authors. We have not covered the parameter scan using these constraints yet in this thesis. This can be a subject for our future studies. We also plan to perform a comparison of our calculated results with the results obtained from **`micrOMEGAs`**. This is the only published code which allows us to compute the DM relics in the NMSSM.

APPENDICES

Outline

A

Notations, Conventions & Miscellaneous Identities

A.1 Metric conventions and Dirac matrices

We will use mostly the notations from the paper [62, section 1,2], and we summarize them here for ease of reference.

$$\sigma^\mu = (\sigma^0, \boldsymbol{\sigma}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{A.1.1}$$

$$\overline{\sigma}^\mu = (\sigma^0, -\boldsymbol{\sigma}), \quad \sigma^{\mu\nu} \equiv \frac{i}{4}\left(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu\right), \quad \overline{\sigma}^{\mu\nu} \equiv \frac{i}{4}\left(\overline{\sigma}^\mu \sigma^\nu - \overline{\sigma}^\nu \sigma^\mu\right) \tag{A.1.2}$$

Notations in Weyl basis:

$$\gamma^\mu = \begin{pmatrix} \mathbb{O} & \sigma^\mu \\ \overline{\sigma}^\mu & \mathbb{O} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbb{1} & \mathbb{O} \\ \mathbb{O} & \mathbb{1} \end{pmatrix}, \quad \Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & \mathbb{O} \\ \mathbb{O} & \overline{\sigma}^{\mu\nu} \end{pmatrix} \tag{A.1.3}$$

$$\mathcal{C} \equiv i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & \mathbb{O} \\ \mathbb{O} & -i\sigma^2 \end{pmatrix} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \qquad \text{with } \varepsilon^{12} = \varepsilon^{\dot1\dot2} = -\varepsilon_{12} = -\varepsilon_{\dot1\dot2} = 1. \tag{A.1.4}$$

The metric we choose to work on is mostly minus

$$g_{\mu\nu} = \mathrm{diag}(+,-,-,-), \tag{A.1.5}$$

and the Levi-Civita tensor is defined as

$$\varepsilon^{0123} = \varepsilon^{123} = -\varepsilon_{0123} = 1. \tag{A.1.6}$$

A.2 Identities on two-component spinors

This Appendix is a list of useful spinor identities that will be used frequently in the computation above. Fore more detailed information and discussion, see e.g [62]

Two-component spinors product

$$\psi^\alpha\psi^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}\psi\psi, \quad \psi_\alpha\psi_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}\psi\psi, \tag{A.2.1}$$

$$\overline{\psi}^{\dot\alpha}\overline{\psi}^{\dot\beta} = \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\overline{\psi\psi}, \quad \overline{\psi}_{\dot\alpha}\overline{\psi}_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\overline{\psi\psi}, \tag{A.2.2}$$

$$\psi\chi = \chi\psi, \quad \overline{\psi}\overline{\chi} = \overline{\chi}\overline{\psi}, \quad (\psi\chi)^\dagger = \overline{\psi}\overline{\chi}, \tag{A.2.3}$$

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\theta\theta)(\phi\psi) \tag{A.2.4}$$

Identities involving Pauli matrices

$$\chi\sigma^\mu\overline{\psi} = -\overline{\psi}\overline{\sigma}^\mu\chi, \quad \chi\sigma^\mu\overline{\sigma}^\nu\psi = \psi\sigma^\nu\overline{\sigma}^\mu, \tag{A.2.5}$$

$$(\chi\sigma^\mu\overline{\psi})^\dagger = \psi\sigma^\mu\overline{\chi}, \quad (\chi\sigma^\mu\overline{\sigma}^\nu\psi)^\dagger = \overline{\psi}\overline{\sigma}^\nu\sigma^\mu\chi, \tag{A.2.6}$$

$$\frac{1}{2}(\phi\sigma^\mu\overline{\chi})(\psi\sigma_\mu)_{\dot\alpha} = (\phi\psi)\overline{\chi}_{\dot\alpha}, \quad \frac{1}{2}(\phi\sigma^\mu\overline{\chi})(\overline{\eta}\overline{\sigma}_\mu\psi) = -(\phi\psi)(\overline{\chi}\overline{\eta}), \tag{A.2.7}$$

$$(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = \frac{1}{2}g^{\mu\nu}(\theta\theta)(\overline{\theta}\overline{\theta}). \tag{A.2.8}$$

A.3 SuperPoincaré Algebra

The algebra of supersymmetry contains the Poincaré algebra, which we have already familiar with

$$[P^\mu, P^\nu] = 0, \tag{A.3.1}$$

$$[M^{\mu\nu}, P^\sigma] = i\left(\eta^{\nu\sigma}P^\mu - \eta^{\mu\sigma}P^\nu\right), \tag{A.3.2}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\rho}M^{\nu\sigma}\right), \tag{A.3.3}$$

and the identities related to fermionic generators $Q^I_\alpha, Q^J_{\dot\alpha}$ and internal symmetries B_l. Below we will list the (anti) commutation relations one by one, with the proof or clarification follows.

- $\left[M^{\mu\nu}, Q^I_\alpha\right]$.
  Since Q^I_α is a spinor in $(1/2, 0)$ representation, it transforms as

  $$Q^I_\alpha \to \exp\left(\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta \approx Q^I_\alpha + \frac{i}{2}\omega_{\mu\nu}\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta. \tag{A.3.4}$$

  Note that Q^I_α now is also an operator, and thus transform under unitary representation of Lorentz group

  $$\begin{aligned} Q^I_\alpha \to U(\Lambda)Q^I_\alpha U(\Lambda)^{-1} &\approx \left(\mathbb{1} - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) Q^I_\alpha \left(\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \\ &\approx Q^I_\alpha - \frac{i}{2}\omega_{\mu\nu}\left[M^{\mu\nu}, Q^I_\alpha\right]. \end{aligned} \tag{A.3.5}$$

  Equating the matrix corresponding to parameter $\omega_{\mu\nu}$ gives

  $$\boxed{\left[M^{\mu\nu}, Q^I_\alpha\right] = -\left(\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q^I_\beta.} \tag{A.3.6}$$

- $\left[M^{\mu\nu}, \overline{Q}^{I\dot\alpha}\right]$.
  The analysis is totally similar to that of Q_α, with the replacement of $\bar{Q}$ instead of Q, $\overline{\sigma}$ instead of σ and replace the undotted components by doted ones.

  $$\boxed{\left[M^{\mu\nu}, \overline{Q}^{I\dot\alpha}\right] = \left(\overline{\sigma}^{\mu\nu}\right)^{\dot\alpha}_{\ \dot\beta}\,\overline{Q}^{I\dot\beta}.} \tag{A.3.7}$$

  We can obtain this result by simply taking the Hermitian conjugate of $\left[M^{\mu\nu}, Q^I_\gamma\right]$

  $$\begin{aligned} &\left[M^{\mu\nu}, Q^I_\gamma\right]^\dagger = -(\sigma^{\mu\nu}Q^I)_\gamma^{\ \dagger} \Rightarrow \left[M^{\mu\nu}, \overline{Q}^I_{\dot\gamma}\right] = (\overline{Q}^I\overline{\sigma}^{\mu\nu})_{\dot\gamma} = \overline{Q}^I_{\dot\delta}(\overline{\sigma}^{\mu\nu})^{\dot\delta}_{\ \dot\gamma} \\ &\Rightarrow \epsilon^{\dot\alpha\dot\gamma}\left[M^{\mu\nu}, \overline{Q}^I_{\dot\gamma}\right] = \epsilon^{\dot\alpha\dot\gamma}\epsilon_{\dot\delta\dot\beta}(\overline{\sigma}^{\mu\nu})^{\dot\delta}_{\ \dot\gamma}\overline{Q}^{I\dot\beta} \Rightarrow \left[M^{\mu\nu}, \overline{Q}^{I\dot\alpha}\right] = -(\overline{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\overline{Q}^{I\dot\beta}. \end{aligned} \tag{A.3.8}$$

- $\left\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\right\}$
  For later use, we will prove this anticommutator before turning to $\left[P^\mu, Q^I_\alpha\right]$. The anticommutator $\left\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\right\}$ must transform under the tensor product representation of left and right spinor

  $$\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \tag{A.3.9}$$

  which is the four-vector representation of Lorentz group. The only generator that belongs to the same vector representation is generators of translations P^μ. Do to the indices structure, we claim that

  $$\left\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\right\} = A^{IJ}(\sigma^\mu)_{\alpha\dot\beta}P_\mu. \tag{A.3.10}$$

Raising the indices of both sides by multiplying with $\epsilon^{\beta\alpha}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$, we have

$$\left\{Q^{I\beta},\overline{Q}^{J\dot{\alpha}}\right\} = A^{IJ}\epsilon^{\beta\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu = A^{IJ}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \tag{A.3.11}$$

Note that both the commutator and anticommutator respect the unitary transformation, which allows us to diagonalize $A^{IJ} = \mathcal{A}\delta^{IJ}$. For now, we could not fix $\mathcal{A}$ using any mathematical constraint, and by convention A is set by 2

$$\boxed{\left\{Q^I_\alpha,\overline{Q}^J_{\dot{\beta}}\right\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu.} \tag{A.3.12}$$

- $[P^\mu,Q^I_\alpha]$ and $\left[P^\mu,\overline{Q}^{I\dot{\alpha}}\right]$.
  First note that P^μ is an operator working in the vector representation $(1/2,1/2)$ and Q^I_α belongs to $(1/2,0)$. We analyze the tensor product of those two representations

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(\frac{1}{2},0\right) = \left(0,\frac{1}{2}\right)\oplus\left(1,\frac{1}{2}\right). \tag{A.3.13}$$

Due to a theorem stated by Haag, Lopuszanski and Sohnius, the only allowed fermionic generators in the superalgebra are supersymmetry generators with spin $1/2$, and the representation $(1,1/2)$ is not justified. The first representation in the direct product tells that $[P^\mu,Q^I_\alpha]$ must be build out of Q^I_α. One of the most general expression for this commutator is

$$[P^\mu,Q^I_\alpha] = C^I{}_J\sigma^\mu_{\alpha\dot{\beta}}\overline{Q}^{J\dot{\beta}} = C^I{}_J\left(\sigma^\mu\overline{Q}^J\right)_\alpha \tag{A.3.14}$$

Taking the Hermitian conjugation at both sides of Eq. (A.3.14)

$$\begin{aligned}
&-\left[P^{\mu\dagger},(Q^I_\alpha)^\dagger\right] = \left(C^I{}_J\left(\sigma^\mu\overline{Q}^J\right)_\alpha\right)^\dagger \Rightarrow -\left[P^\mu,\overline{Q}^I_{\dot{\alpha}}\right] = (C^*)^I{}_J\left(\sigma^\mu\overline{Q}^J\right)^\dagger_{\dot{\alpha}}\\
&\Rightarrow -\left[P^\mu,\overline{Q}^I_{\dot{\alpha}}\right] = (C^*)^I{}_J\left(Q^J\sigma^\mu\right)_{\dot{\alpha}} = (C^*)^I{}_JQ^{J\alpha}\sigma^\mu_{\alpha\dot{\alpha}} = (C^*)^I{}_J\epsilon^{\alpha\beta}\sigma^\mu_{\alpha\dot{\alpha}}Q^J_\beta\\
&\Rightarrow -\left[P^\mu,\epsilon^{\dot{\beta}\dot{\alpha}}\overline{Q}^I_{\dot{\alpha}}\right] = -(C^*)^I{}_J\left(\epsilon^{\beta\alpha}\epsilon^{\dot{\beta},\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\right)Q^J_\beta = -(C^*)^I{}_J\bar{\sigma}^{\dot{\beta}\beta}Q^J_\beta\\
&\Rightarrow \left[P^\mu,\overline{Q}^{I\dot{\beta}}\right] = (C^*)^I{}_J\bar{\sigma}^{\dot{\beta}\beta}Q^J_\beta.
\end{aligned} \tag{A.3.15}$$

Determining the matrix C requires the use of generalized Jacobi identities for (Q,P,P) and (Q,Q,P) systems. The first one states

$$\begin{aligned}
&\left[\left[Q^I_\alpha,P^\mu\right],P^\nu\right] + \left[\left[P^\mu,P^\nu\right],Q^I_\alpha\right] + \left[\left[P^\nu,Q^I_\alpha\right],P^\mu\right] = 0\\
&\Rightarrow -C^I{}_J\sigma^\mu_{\alpha\dot{\beta}}\left[\overline{Q}^{J\dot{\beta}},P^\nu\right] + 0 + C^I{}_J\sigma^\nu_{\alpha\dot{\beta}}\left[\overline{Q}^{J\dot{\beta}},P^\mu\right] = 0\\
&\Rightarrow C^I{}_J(C^*)^J{}_K\sigma^\mu_{\alpha\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\beta}Q^K_\beta - C^I{}_J(C^*)^J{}_K\sigma^\nu_{\alpha\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}Q^K_\beta\\
&\Rightarrow C^I{}_J(C^*)^J{}_K\left(\sigma^\mu_{\alpha\dot{\beta}}\bar{\sigma}^{\nu\dot{\beta}\beta} - \sigma^\nu_{\alpha\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}\right)Q^K_\beta = 0\\
&\Rightarrow C^I{}_J(C^*)^J{}_K(\sigma^{\mu\nu})_\alpha{}^\beta Q^K_\beta = 0.
\end{aligned} \tag{A.3.16}$$

In general $(\sigma^{\mu\nu})_\alpha{}^\beta \neq 0$, and we obviously do not want to consider the trivial case where $Q^K_\beta = 0$ (which make no differences between Poincaré and SuperPoincaré group), thus implies the constraint

$$\boxed{C^I{}_J(C^*)^J_K = 0.} \tag{A.3.17}$$

To continue, we need to use the anticommutator $\{Q^I_\alpha,Q^J_\beta\}$. This product belongs to the representation

$$\left(\frac{1}{2},0\right)\otimes\left(\frac{1}{2},0\right) = (0,0)\oplus(1,0), \tag{A.3.18}$$

which means this anticommutator can be build from the sum of a Lorentz scalar and a tensor of rank 2:

$$\{Q^I_\alpha,Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ} + \epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma M_{\mu\nu}Y^{IJ}. \tag{A.3.19}$$

We note that since $\epsilon_{\alpha\beta}$ is antisymmetric under exchanging the indices, Z^{IJ} can be chosen to be antisymmetric under $I \leftrightarrow J$. On the contrary, $\epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma$ is symmetric under $\beta \leftrightarrow \alpha$ (to see that, first exchange $\alpha \leftrightarrow \gamma$, next $\gamma \leftrightarrow \beta$; each exchanging contribute a minus sign, and two minus sign cancel), implying that Y^{IJ} can be chosen to be symmetric under $I \leftrightarrow J$. The generalized Jacobi identity can be written as

$$
\begin{aligned}
&\left[\left\{Q^I_\alpha, Q^J_\beta\right\}, P^\mu\right] = \left\{Q^I_\alpha, \left[Q^J_\beta, P^\mu\right]\right\} + \left\{Q^J_\beta, \left[Q^I_\alpha, P^\mu\right]\right\} \\
&\Rightarrow \epsilon^{\alpha\beta}\left[\left\{Q^I_\alpha, Q^J_\beta\right\}, P^\mu\right] = \epsilon^{\alpha\beta}\left\{Q^I_\alpha, \left[Q^J_\beta, P^\mu\right]\right\} + \epsilon^{\alpha\beta}\left\{Q^J_\beta, \left[Q^I_\alpha, P^\mu\right]\right\} \\
&\Rightarrow \epsilon^{\alpha\beta}\left[\epsilon_{\alpha\beta}Z^{IJ} + \epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma M_{\mu\nu}Y^{IJ}, P^\mu\right] = -\epsilon^{\alpha\beta}C^J{}_K\sigma^\mu_{\beta\dot\beta}\left\{Q^I_\alpha, \bar Q^{K\dot\beta}\right\} - \epsilon^{\alpha\beta}C^I{}_K\sigma^\mu_{\alpha\dot\alpha}\left\{Q^J_\beta, \overline{Q}^{K\dot\alpha}\right\} \\
&\Rightarrow 2\left[Z^{IJ}, P^\mu\right] = C^J{}_K\sigma^\mu_{\beta\dot\beta}\left\{Q^{I\beta}, \bar Q^{K\dot\beta}\right\} - C^I{}_K\sigma^\mu_{\alpha\dot\alpha}\left\{Q^{J\alpha}, \overline{Q}^{K\dot\alpha}\right\} \\
&\Rightarrow 2\left[Z^{IJ}, P^\mu\right] = C^J{}_K\sigma^\mu_{\alpha\dot\alpha}2\delta^{IK}\bar\sigma^{\nu\dot\alpha\alpha}P_\nu - C^I{}_K\sigma^\mu_{\alpha\dot\alpha}2\delta^{JK}\bar\sigma^{\nu\dot\alpha\alpha}P_\nu \\
&\Rightarrow \left[Z^{IJ}, P^\mu\right] = \left(C^J{}_I - C^I{}_J\right)\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha}P_\nu.
\end{aligned} \tag{A.3.20}
$$

The fact that Z^{IJ} belongs to $(0,0)$implies Z^{IJ} are generators an internal symmetry group (and possibly a linear combination of generators B_l of some internal symmetry groups). This means

$$\left[Z^{IJ}, P^\mu\right] = 0. \tag{A.3.21}$$

Plugging into Eq. (A.3.20) gives

$$0 = \left(C^J{}_I - C^I{}_J\right)\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha}P_\nu. \tag{A.3.22}$$

Once again $\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\alpha}P_\nu$ in generally not vanishes, which requires

$$C^J{}_I = C^I{}_J \Rightarrow C^T = C. \tag{A.3.23}$$

Eq. (A.3.17) together with (A.3.23) imply

$$CC^\dagger = 0. \tag{A.3.24}$$

Obviously $CC^\dagger$ is Hermitian, thus diagonalizable. A matrix have diagonal form equals zero can only be a zero matrix. Plugging back into Eq. (A.3.14) and (A.3.15), we conclude

$$C^{IJ} = 0, \forall I, J \in \{0, 1, \dots, N\} \Rightarrow \boxed{\left[P^\mu, Q^I_\alpha\right] = 0,} \tag{A.3.25}$$

and similarly for the bar conjugation part

$$\boxed{\left[P^\mu, \overline{Q}^{I\dot\alpha}\right] = 0.} \tag{A.3.26}$$

- $\{Q^I_\alpha, Q^J_\beta\}$ and $\{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\}$.
Let us turn back to the formula (A.3.19),

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ} + \epsilon_{\beta\gamma}(\sigma^{\mu\nu})_\alpha{}^\gamma M_{\mu\nu}Y^{IJ},$$

which is quite general but we can do more with the symmetric part. Consider the generalized Jacobi identity for the system (Q, Q, P) in Eq. (A.3.20). The RHS vanishes due to the commutation of Q and P, and we are left with

$$
\begin{aligned}
&\left[\{Q^I_\alpha, Q^J_\beta\}, P^\mu\right] = \left[\epsilon_{\alpha\beta}Z^{IJ} + \epsilon_{\beta\gamma}(\sigma^{\rho\sigma})_\alpha{}^\gamma M_{\rho\sigma}Y^{IJ}, P^\mu\right] = 0 \\
&\Rightarrow \epsilon_{\beta\gamma}(\sigma^{\rho\sigma})_\alpha{}^\gamma Y^{IJ}\left[M_{\rho\sigma}, P^\mu\right] = 0.
\end{aligned} \tag{A.3.27}
$$

Since P^μ does not commute with $M^{\rho\sigma}$, the matrix elements Y^{IJ} is required to be zero for all I, J, thus gives

$$\boxed{\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ}.} \tag{A.3.28}$$

Taking the Hermitian conjugate automatically yields

$$\boxed{\{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^*.} \tag{A.3.29}$$

- Identities related to internal symmetry generators B_l.
  Assuming we have the internal symmetry groups, which is generated by B_l and

$$[B_l, B_m] = i f^n_{lm} B_n. \tag{A.3.30}$$

By definition, the internal symmetries are independent of spacetime and thus commute with the Poincaré generators

$$[B_l, P^\mu] = 0, \tag{A.3.31}$$
$$[B_l, M^{\mu\nu}] = 0. \tag{A.3.32}$$

We have seen that the central charge is related by the anticommutator of two $Q's$, and that it should be built out of internal symmetries generators. This implies the commutator between $Q's$ and B_l is not trivial. We expect

$$\left[Q^I_\alpha, B_l\right] = (b_l)^I{}_J Q^J_\alpha, \tag{A.3.33}$$
$$\Rightarrow \left[\overline{Q}^I_{\dot\alpha}, B_l\right] = -(b^*_l)^I{}_J \overline{Q}^J_{\dot\alpha} = -(b_l)^J{}_I \overline{Q}^J_{\dot\alpha}, \tag{A.3.34}$$

where the second equality of Eq. (A.3.34) is because B_l are all Hermitian, and the set b_l belongs to the N dimensional representation of the corresponding internal symmetry group, implying each b_l is Hermitian. The largest possible internal symmetries group can act on fermionic generators is thus $U(N)$, generated by $N \times N$ Hermitian matrices b_l. The proof is straightforward by using generalized Jacobi identity for (Q, B, B)

$$\begin{aligned}
&\left[\left[Q^I_\alpha, B_l\right], B_m\right] + \left[\left[B_l, B_m\right], Q^I_\alpha\right] + \left[\left[B_m, Q^I_\alpha\right], B_l\right] = 0\\
&\Rightarrow b^I_{lJ}\left[Q^J_\alpha, B_m\right] + i f^n_{lm}\left[B_n, Q^I_\alpha\right] - b^I_{mJ}\left[Q^J_\alpha, B_l\right] = 0\\
&\Rightarrow b^I_{lJ} b^J_{mK} + i f^n_{lm} b^I_{nK} - b^I_{mJ} b^J_{lK} = 0 \Rightarrow \left[b_l, b_m\right]^I{}_K = i f^n_{lm} b^I_{nK}\\
&\Rightarrow \boxed{[b_l, b_m] = i f^n_{lm} b_n,}
\end{aligned} \tag{A.3.35}$$

i.e the matrices b_l has the same Lie algebraic structure as B_l does.

In conclusion, we have derived the following

SuperPoincaré algebra for arbitrary N

$$[P^\mu, P^\nu] = 0, \tag{A.3.36a}$$
$$[M^{\mu\nu}, P^\sigma] = i\left(\eta^{\nu\sigma} P^\mu - \eta^{\mu\sigma} P^\nu\right), \tag{A.3.36b}$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\rho} M^{\nu\sigma}\right), \tag{A.3.36c}$$
$$\left[P^\mu, Q^I_\alpha\right] = 0, \tag{A.3.36d}$$
$$\left[P^\mu, \overline{Q}^{I\dot\alpha}\right] = 0, \tag{A.3.36e}$$
$$\left[M^{\mu\nu}, Q^I_\alpha\right] = -\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta Q^I_\beta, \tag{A.3.36f}$$
$$\left[M^{\mu\nu}, \overline{Q}^{I\dot\alpha}\right] = -\left(\overline{\sigma}^{\mu\nu}\right)^{\dot\alpha}{}_{\dot\beta} \overline{Q}^{I\dot\beta}, \tag{A.3.36g}$$
$$\left\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\right\} = 2\delta^{IJ}(\sigma^\mu)_{\alpha\dot\beta} P_\mu, \tag{A.3.36h}$$
$$\left\{Q^I_\alpha, Q^J_\beta\right\} = \epsilon_{\alpha\beta} Z^{IJ}, \tag{A.3.36i}$$
$$\left\{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\right\} = \epsilon_{\dot\alpha\dot\beta}(Z^{IJ})^*, \tag{A.3.36j}$$
$$[B_l, B_m] = i f^n_{lm} B_n, \tag{A.3.36k}$$
$$\left[Q^I_\alpha, B_l\right] = (b_l)^I{}_J Q^J_\alpha, \tag{A.3.36l}$$
$$\left[\overline{Q}^I_{\dot\alpha}, B_l\right] = -(b_l)^J{}_I \overline{Q}^J_{\dot\alpha}. \tag{A.3.36m}$$

Here and afterward, the uppercase Latin indices is the index of fermionic generators and ranges from 1 to N. Those letters at the beginning of Greek alphabet $(\alpha, \beta, \gamma, \dots)$ is the spinor indices, whose values are either 1 or 2. Other Greek symbols will be used to denote the spacetime indices.

Z^{IJ} as *Central Charge*

Of all generators of SuperPoincaré group we have introduced at the beginning, there is one more operator arises when considering the anticommutator of $Q's$, which is the *central charge operator* Z^{IJ}. As shown above, this operator belongs to the $(0,0)$ representation of Poincaré group. We also claimed that the central charge should be built out of generators of internal symmetry groups G, i.e

$$Z^{IJ} = a^{l|IJ} B_l. \tag{A.3.37}$$

The reason for the name is because Z commute with all of the generators. Obviously Z commute with all Poincaré group generators

$$\left[Z^{IJ}, P^{\mu}\right] = 0, \tag{A.3.38}$$
$$\left[Z^{IJ}, M^{\mu\nu}\right] = 0, \tag{A.3.39}$$

due to the fact that Z is composed of internal generators and these are invariant under boosts, rotations and translations. Our tasks now is to prove that Z commute with B_l, Q, $\overline{Q}$ and with elements of itself. First consider the generalized Jacobi identity for (Q, Q, B) [1]

$$\begin{aligned}
&\left[\left\{Q^I_\alpha, Q^J_\beta\right\}, B_l\right] = \left\{\left[Q^J_\beta, B_l\right], Q^I_\alpha\right\} + \left\{\left[B_l, Q^I_\alpha\right], Q^J_\beta\right\} \\
&\Rightarrow \epsilon_{\alpha\beta}\left[Z^{IJ}, B_l\right] = b^J_{lK}\left\{Q^K_\beta, Q^I_\alpha\right\} - b^I_{lK}\left\{Q^K_\alpha, Q^J_\beta\right\} = b^J_{lK}\epsilon_{\beta\alpha} Z^{KI} - b^I_{lK}\epsilon_{\alpha\beta} Z^{KJ} \\
&\Rightarrow \left[Z^{IJ}, B_l\right] = b^J_{lK} Z^{IK} - b^I_{lK} Z^{KJ}.
\end{aligned} \tag{A.3.40}$$

And it is realized that Z^{IJ} form a subalgebra of the bosonic symmetry group generated by B_l (which, as we are going to confirm, is an *Abelian subalgebra*). By contracting both sides with $a^{l|KL}$, we have

$$\left[Z^{IJ}, Z^{KL}\right] = a^{l|KL}\left(b^J_{lK} Z^{IK} - b^I_{lK} Z^{KJ}\right). \tag{A.3.41}$$

Next we consider the Jacobi identity for the trio (Q, Q, Q)

$$\begin{aligned}
&\left[\left\{Q^I_\alpha, Q^J_\beta\right\}, Q^K_\gamma\right] + \left[\left\{Q^J_\beta, Q^K_\gamma\right\}, Q^I_\alpha\right] + \left[\left\{Q^K_\gamma, Q^I_\alpha\right\}, Q^J_\beta\right] = 0 \\
&\Rightarrow \epsilon_{\alpha\beta}\left[Z^{IJ}, Q^K_\gamma\right] + \epsilon_{\beta\gamma}\left[Z^{JK}, Q^I_\alpha\right] + \epsilon_{\gamma\alpha}\left[Z^{KI}, Q^J_\beta\right] = 0 \\
&\Rightarrow a^{l|IJ} b^K_{lL} Q^L_\gamma + \epsilon_{\beta\gamma} a^{l|JK} b^I_{lL} Q^L_\alpha + \epsilon_{\gamma\alpha} a^{l|KI} b^J_{lL} Q^L_\beta = 0.
\end{aligned} \tag{A.3.42}$$

This identity hold for arbitrary (α, β, γ). We consider two following cases

- $(\alpha, \beta, \gamma) = (1, 2, 1)$.
$$-a^{l|IJ} b^K_{lL} Q^L_1 + a^{l|JK} b^I_{lL} Q^L_1 = 0. \tag{A.3.43}$$

- $(\alpha, \beta, \gamma) = (2, 1, 2)$.
$$a^{l|IJ} b^K_{lL} Q^L_2 - a^{l|JK} b^I_{lL} Q^L_2 = 0. \tag{A.3.44}$$

Writing in a more compact form

$$a^{l|IJ} b^K_{lL} Q^L_\alpha = a^{l|JK} b^I_{lL} Q^L_\alpha \Rightarrow \boxed{a^{l|IJ} b^K_{lL} = a^{l|JK} b^I_{lL}.} \tag{A.3.45}$$

From Eq. (A.3.41), simultaneously exchanging $(I, J) \leftrightarrow (K, L)$ gives

$$\left[Z^{IJ}, Z^{KL}\right] = -a^{l|IJ}\left(b^K_{lM} Z^{ML} + b_{lM} Z^{KM}\right) = -a^{l|JK} b^I_{lM} Z^{ML} - a^{l|LI} b^J_{lM} Z^{KM}. \tag{A.3.46}$$

[1] From now on, for simple notation we will denote $(b_l)^I{}_J$ as b^I_{lJ}.

Equating the coefficients of b^I_{lM} and b^L_{lM} in Eq. (A.3.41) and (A.3.46), we obtain

$$\boxed{\begin{cases} a^{l|KL}Z^{MJ} = -a^{l|JK}Z^{ML} & \text{(A.3.47a)} \\ a^{l|KL}Z^{IM} = -a^{l|LI}Z^{KM} & \text{(A.3.47b)} \end{cases}}$$

We have shown the effect of exchanging indices between a, b and a, Z as in Eq. (A.3.45) and (A.3.47). To prove (A.3.40), we need to find the value of product b, Z. The scheme is as follows

$$a^{l|KL}b^I_{lM}Z^{MJ} = a^{l|LI}b^K_{lM}Z^{MJ} = -a^{l|JL}b^K_{lM}Z^{MI} = -a^{l|KL}b^J_{lM}Z^{IM}$$

$$\Rightarrow \boxed{b^I_{lM}Z^{MJ} = -b^J_{lM}Z^{IM} = b^J_{lM}Z^{MI}.} \tag{A.3.48}$$

Plugging this identity back to Eq. (A.3.40), we can easily see that this commutator vanishes

$$\left[Z^{IJ}, B_l\right] = b^J_{lK}Z^{IK} - b^I_{lK}Z^{KJ} = b^J_{lK}Z^{IK} - b^J_{lK}Z^{KI} = 0, \tag{A.3.49}$$

and thus implying commutator of matrix elements of Z yields 0, since Z is a linear combination of B_l

$$\boxed{\left[Z^{IJ}, Z^{LK}\right] = 0.} \tag{A.3.50}$$

Consider the generalized Jacobi identity for $(Q, Q, \overline{Q})$

$$\left[\left\{Q^I_\alpha, Q^J_\beta\right\}, \overline{Q}^K_{\dot\gamma}\right] + \left[\left\{Q^J_\beta, \overline{Q}^K_{\dot\gamma}\right\}, Q^I_\alpha\right] + \left[\left\{\overline{Q}^K_{\dot\gamma}, Q^I_\alpha\right\}, Q^J_\beta\right] = 0$$

$$\Rightarrow \epsilon_{\alpha\beta}\left[Z^{IJ}, \overline{Q}^K_\gamma\right] + 2\delta^{IJ}\sigma^\mu_{\beta\dot\gamma}\left[P_\mu, Q^I_\alpha\right] + 2\delta^{KI}\sigma^\mu_{\alpha\dot\gamma}\left[P_\mu, Q^J_\beta\right] = 0$$

$$\Rightarrow \epsilon_{\alpha\beta}\left[Z^{IJ}, \overline{Q}^K_\gamma\right] + 0 + 0 = 0 \Rightarrow \boxed{\left[Z^{IJ}, \overline{Q}^K_\gamma\right] = 0,} \tag{A.3.51}$$

where the second and the third term in Jacobi identity vanishes due to the fact Q commute with P. We cannot obtain the same relation for $\overline{Q}$ by directly taking Hermitian conjugate since Z is a complex matrix in general. Consider the Jacobi identity involving $(Z, Q, \overline{Q})$:

$$\left\{\left[Z^{IJ}, Q^K_\alpha\right], \overline{Q}^L_{\dot\beta}\right\} + \left\{\left[\overline{Q}^L_{\dot\beta}, Z^{IJ}\right], Q^K_\alpha\right\} = \left[\left\{Q^K_\alpha, \overline{Q}^L_{\dot\beta}\right\}, Z^{IJ}\right]$$

$$\Rightarrow \left\{\left[Z^{IJ}, Q^K_\alpha\right], \overline{Q}^L_{\dot\beta}\right\} + 0 = 2\delta^{KL}\sigma^\mu_{\alpha\dot\beta}\left[P_\mu, Z^{IJ}\right]$$

$$\Rightarrow \left\{\left[Z^{IJ}, Q^K_\alpha\right], \overline{Q}^L_{\dot\beta}\right\} = 0. \tag{A.3.52}$$

Since Z is made of anticommutator of Qs, the $\left[Z^{IJ}, Q^K_\alpha\right]$ also belongs to $(0, 1/2)$ representation, and can be expanded as a linear combination of the Qs

$$\left[Z^{IJ}, Q^K_\alpha\right] = M^{IJK}_N Q^N_\alpha,$$

$$\Rightarrow \left\{\left[Z^{IJ}, Q^K_\alpha\right], \overline{Q}^L_{\dot\beta}\right\} = 0 = M^{IJK}_N\left\{Q^N_\alpha, \overline{Q}^L_{\dot\beta}\right\}$$

$$\Rightarrow M^{IJK}_L \sigma^\mu_{\alpha\dot\beta}P_\mu = 0. \tag{A.3.53}$$

Again, the operator $\sigma^\mu_{\alpha\dot\beta}P_\mu$ is generally non-vanished, which require $M^{IJK}_L = 0$. Plugging back in the expansion of $\left[Z^{IJ}, Q^K_\alpha\right]$ gives

$$\boxed{\left[Z^{IJ}, Q^K_\alpha\right] = 0.} \tag{A.3.54}$$

A.4 Grassmann numbers

Grassmann numbers play an essential role in building the supersymmetric Lagrangian. Let us introduce a set of Grassmann numbers θ_α, $\overline{\theta}_{\dot{\alpha}}$ that anticommute with itself.

$$\{\theta^\alpha, \theta^\beta\} = 0, \quad \{\,\overline{\theta}_{\dot{\alpha}}, \overline{\theta}_{\dot{\beta}}\} = 0, \quad \{\theta^\alpha, \overline{\theta}_{\dot{\beta}}\} = 0. \tag{A.4.1}$$

The anticommutation relations in SuperPoincaré algebra reads

$$\left\{Q_\alpha, \overline{Q}_{\dot{\beta}}\right\} = 2\sigma_{\alpha\dot{\beta}}P_\mu, \quad \{Q_\alpha, Q_\beta\} = \left\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\right\} = 0. \tag{A.4.2}$$

The Grassmann variables allow us to transform these anticommutation relations into commutation relations. Specifically

$$\begin{aligned}\left[\theta Q, \overline{\theta Q}\right] &= \left[\theta^\alpha Q_\alpha, \overline{\theta}_{\dot{\alpha}}\overline{Q}^{\dot{\alpha}}\right] = \theta^\alpha\left\{Q_\alpha, \overline{\theta}_{\dot{\alpha}}\right\}\overline{Q}^{\dot{\alpha}} - \theta^\alpha\overline{\theta}_{\dot{\alpha}}\left\{Q_\alpha, \overline{Q}^{\dot{\alpha}}\right\} - \overline{\theta}_{\dot{\alpha}}\left\{\overline{Q}^{\dot{\alpha}}, \theta^\alpha\right\}Q_\alpha + \left\{\overline{\theta}_{\dot{\alpha}}, \theta^\alpha\right\}\overline{Q}^{\dot{\alpha}}Q_\alpha \\ &= -\theta^\alpha\overline{\theta}_{\dot{\alpha}}(2\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\alpha\dot{\beta}}P_\mu) = 2\theta^\alpha\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}P_\mu = 2\theta\sigma^\mu\overline{\theta}P_\mu.\end{aligned} \tag{A.4.3}$$

Similarly, we can expand the commutation relations $[\theta Q, \theta Q]$ and $\left[\overline{\theta Q}, \overline{\theta}, \overline{Q}\right]$ in terms of sum of anticommutation relations of each pair of operators, and noting that these operators are pairwise anticommute, thus vanishes identically

$$[Q\theta, Q\theta] = \left[\overline{Q\theta}, \overline{Q\theta}\right] = 0. \tag{A.4.4}$$

Let us list some properties of the Grassmann variables, which will becomes extremely useful when dealing with superspace and superfields. Firstly, the derivatives work in analogy to the Minkowski coordinates:

$$\partial_\alpha\theta^\beta = \frac{\partial\theta^\beta}{\partial\theta^\alpha} = \delta^\alpha_\beta, \quad \overline{\partial}_{\dot{\alpha}}\overline{\theta}^{\dot{\beta}} = \frac{\partial\overline{\theta}^{\dot{\beta}}}{\partial\overline{\theta}^{\dot{\alpha}}} = \delta^{\dot{\beta}}_{\dot{\alpha}}. \tag{A.4.5}$$

We can further derive the raising-lowering mechanism on the Grassmann derivatives, and the independent of θ and $\overline{\theta}$

$$\partial^\alpha = -\varepsilon^{\alpha\beta}\partial_\beta, \quad \overline{\partial}^{\dot{\alpha}} = -\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{\partial}_{\dot{\beta}}, \tag{A.4.6}$$

$$\partial_\alpha\overline{\theta}_{\dot{\beta}} = 0, \quad \overline{\partial}^{\dot{\alpha}}\theta^\beta = 0. \tag{A.4.7}$$

The integration over Grassmann variables is defined as follows

$$\int d\theta = 0, \quad \int d\theta\ \theta = 1, \tag{A.4.8}$$

$$\Rightarrow \int d\theta\ (f_0 + \theta f_1) = f_1 = \frac{\partial f_1}{\partial\theta} \longrightarrow \int = \partial, \quad \theta = \delta(\theta). \tag{A.4.9}$$

The multiple integration can be defined similarly

$$\int d^2\theta\ \theta\theta = \int d^2\overline{\theta}\ \overline{\theta\theta} = 1. \tag{A.4.10}$$

One notes that $1 = \frac{1}{4}\delta^\alpha_\alpha = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta(\theta\theta)$, thus again identifies the integration and differentiation

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta, \quad d^2\overline{\theta} = -\frac{1}{4}\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{\partial}_{\dot{\alpha}}\overline{\partial}_{\dot{\beta}} \tag{A.4.11}$$

B

Transformation of Parameters in the NMSSM

The electromagnetic coupling constant, the mixing angle β and the masses of W- and Z-bosons are defined as usual:

$$e = g' c_w = g s_w, \tag{B.0.1}$$

$$\tan\beta = v_u/v_d, \tag{B.0.2}$$

$$M_Z^2 = \frac{\left(g^2 + g'^2\right)\left(v_u^2 + v_d^2\right)}{4}, \tag{B.0.3}$$

$$M_W^2 = \frac{g\left(v_u^2 + v_d^2\right)}{4}. \tag{B.0.4}$$

The transformation laws for each set of new parameters read

$\{v_u, v_d, g, g', M_{H_u}^2, M_{H_d}^2, b, \xi\} \longrightarrow \{M_Z, M_W, e, m_A, \tan\beta, T_h, T_H, T_A\}$

$$v_d = \frac{2\cos\beta s_w c_w M_Z}{e}, \tag{B.0.5}$$

$$v_u = \frac{2\sin\beta s_w c_w M_Z}{e}, \tag{B.0.6}$$

$$g = e/s_w, \tag{B.0.7}$$

$$g' = e/c_w, \tag{B.0.8}$$

$$b\sin\xi = -\frac{eT_A}{2c_w s_w M_Z \cos(\beta - \beta_n)}, \tag{B.0.9}$$

$$\begin{aligned} b\cos\xi = & \frac{m_A^2 \cos\beta\ \sin\beta}{\cos^2(\beta - \beta_n)} \\ & + \frac{T_h e}{4c_w s_w M_Z \cos^2(\beta - \beta_n)}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\cos(2\beta_n)\right] \\ & - \frac{T_H e}{4c_w s_w M_Z \cos^2(\beta - \beta_n)}\left[\sin(\alpha - \beta) + \sin(\alpha + \beta)\cos(2\beta_n)\right], \end{aligned} \tag{B.0.10}$$

$$\begin{aligned} M_{H_d}^2 = & -\frac{1}{2}M_Z^2\cos(2\beta) + \frac{m_A^2\sin^2\beta}{\cos^2(\beta - \beta_n)} \\ & - \frac{T_h e}{2c_w s_w M_Z \cos^2(\beta - \beta_n)}\left[\sin(\alpha - \beta)\cos^2\beta_n + \sin\alpha\sin\beta\sin(2\beta_n)\right] \end{aligned}$$

$$- \frac{T_H e}{2c_w s_w M_Z \cos^2(\beta - \beta_n)} \left[\cos(\alpha - \beta)\cos^2\beta_n + \cos\alpha \sin\beta \sin(2\beta_n)\right], \tag{B.0.11}$$

$$\begin{aligned} M_{H_u}^2 = &\frac{M_Z^2(1-\cos 2\beta)\cos 2\beta}{4\sin^2\beta} + \frac{m_A^2 \cos^2\beta}{\cos^2(\beta-\beta_n)} \\ &+ \frac{T_h e \sin\beta_n}{4c_w s_w M_Z \cos^2(\beta-\beta_n)} [\cos(\alpha+\beta-\beta_n) + 2\cos(\alpha-\beta+\beta_n) + \cos(\alpha+\beta+\beta_n)] \\ &+ \frac{T_H e \sin\beta_n}{4c_w s_w M_Z \cos^2(\beta-\beta_n)} [\sin(\alpha+\beta-\beta_n) + 2\sin(\alpha-\beta+\beta_n) + \sin(\alpha+\beta+\beta_n)]. \end{aligned} \tag{B.0.12}$$

Instead of using m_A^2 as a parameter, we choose $m_{H^\pm}$. The transformation laws read

$v_u, v_d, g, g', M_{H_u}^2, M_{H_d}^2, b, \xi \longrightarrow M_Z, M_W, e, m_{H^\pm}, \tan\beta, T_h, T_H, T_A$

$$v_d = \frac{2\cos\beta s_w c_w M_Z}{e}, \tag{B.0.13}$$

$$v_u = \frac{2\sin\beta s_w c_w M_Z}{e}, \tag{B.0.14}$$

$$g = e/s_w, \tag{B.0.15}$$

$$g' = e/c_w, \tag{B.0.16}$$

$$b\sin\xi = -\frac{eT_A}{2c_w s_w M_Z \cos(\beta-\beta_n)}, \tag{B.0.17}$$

$$\begin{aligned} b\cos\xi = &-c_w^2 M_Z^2 \sin\beta\cos\beta + \frac{m_{H^\pm}^2 \sin\beta\cos\beta}{\cos^2(\beta-\beta_c)} \\ &+ \frac{T_h e}{4c_w s_w M_Z \cos^2(\beta-\beta_c)} [\cos(\alpha-\beta) + \cos(\alpha+\beta)\cos 2\beta_c] \\ &- \frac{T_H e}{4c_w s_w M_Z \cos^2(\beta-\beta_c)} [\sin(\alpha-\beta) + \sin(\alpha+\beta)\cos 2\beta_c], \end{aligned} \tag{B.0.18}$$

$$\begin{aligned} M_{H_d}^2 = &\frac{1}{2} M_Z^2 \left(-c_w^2 - s_w^2 \cos 2\beta\right) + \frac{m_{H^\pm}^2 \sin^2\beta}{\cos^2(\beta-\beta_c)} \\ &- \frac{T_h e \cos\beta_c}{2c_w s_w M_Z \cos^2(\beta-\beta_c)} [\cos\beta_c \sin(\alpha-\beta) + 2\sin\alpha\sin\beta\sin\beta_c] \\ &- \frac{T_H e \cos\beta_c}{2c_w s_w M_Z \cos^2(\beta-\beta_c)} [\cos\beta_c \cos(\alpha-\beta) + 2\cos\alpha\sin\beta\sin\beta_c], \end{aligned} \tag{B.0.19}$$

$$\begin{aligned} M_{H_u}^2 = &\frac{1}{2} M_Z^2 \left(-c_w^2 + s_w^2 \cos 2\beta\right) + \frac{m_{H^\pm}^2 \cos^2\beta}{\cos^2(\beta-\beta_c)} \\ &- \frac{T_h e \sin\beta_c}{4c_w s_w M_Z \cos^2(\beta-\beta_c)} [\cos(\alpha+\beta-\beta_c) + 2\cos(\alpha-\beta+\beta_c) + \cos(\alpha+\beta+\beta_c)] \\ &+ \frac{T_H e \sin\beta_c}{4c_w s_w M_Z \cos^2(\beta-\beta_c)} [\sin(\alpha+\beta-\beta_c) + 2\sin(\alpha-\beta+\beta_c) + \sin(\alpha+\beta+\beta_c)]. \end{aligned} \tag{B.0.20}$$

C

$2 \to 2$ scattering cross section

C.1 $\boldsymbol{\sigma}\mathbf{v}_{\text{lab}}$ in neutralino pair-annihilation

We first need to calculate the velocity of one of the incoming particle in the lab frame based on the energy and its mass in the laboratory frame. Let us consider the two incoming particles with masses m_1 and m_2, as in the below figure, with the z-direction along the velocity of particle 1.

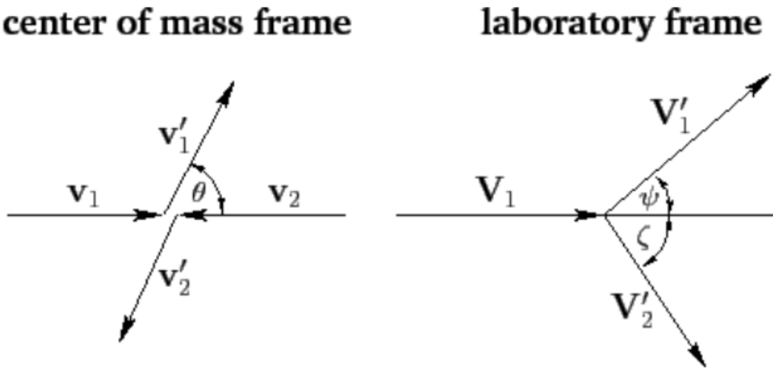


Figure C.1: $2 \to 2$ scattering. Source: http://farside.ph.utexas.edu/teaching/336k/Newtonhtml/node52.html

The velocity of particle 2 in CMS frame is

$$v_z = -\frac{p_z^*}{E_2^*} = -\frac{\sqrt{E_2^{*2} - m_2^2}}{E_2^*}, \tag{C.1.1}$$

thus the boost parameters are calculated as usual

$$\gamma = \frac{1}{\sqrt{1 - v_z^2}} = \frac{1}{\sqrt{1 - \dfrac{E_2^{*2} - m_2^2}{E_2^2}}} = \frac{E_2^*}{m_2}, \tag{C.1.2}$$

$$\gamma\beta = \frac{E_2^*}{m_2}\left(-\frac{|\mathbf{p}^*|}{E_2}\right) = -\frac{|\mathbf{p}^*|}{m_2}. \tag{C.1.3}$$

In the rest frame of particle 2 ($\mathbf{p}_{2\ \text{lab}} = 0$), the momentum of particle 1 is obtained by a boost along z direction

$$p_{\text{lab}} = p_{1\ \text{lab}} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} E_1^* \\ |\mathbf{p}^*| \end{pmatrix} = \frac{1}{m_2}\begin{pmatrix} E_2^* & |\mathbf{p}^*| \\ |\mathbf{p}^*| & E_2^* \end{pmatrix}\begin{pmatrix} E_1^* \\ |\mathbf{p}^*| \end{pmatrix} = \frac{1}{m_2}\begin{pmatrix} E_1^* E_2^* + |\mathbf{p}^*|^2 \\ |\mathbf{p}^*|\left(E_1^* + E_2^*\right) \end{pmatrix} \tag{C.1.4}$$

$$\Rightarrow v_{\text{lab}} = \frac{|\mathbf{p}^*|\left(E_1^* + E_2^*\right)}{E_1^* E_2^* + |\mathbf{p}^*|^2}. \tag{C.1.5}$$

Using the formula of cross section in the center of mass frame, the concerning quantity is calculated as

$$\sigma v_{\text{lab}} = \int \mathrm{d}\Omega \frac{1}{64\pi^2} \frac{|\mathbf{p}^{*'}|}{|\mathbf{p}^*|} \overline{|\mathcal{M}|}^2 \frac{|\mathbf{p}^*| \left(E_1^* + E_2^*\right)}{E_1^* E_2^* + |\mathbf{p}^*|^2}. = \int \mathrm{d}\Omega \frac{\overline{|\mathcal{M}|}^2}{64\pi^2 s} \frac{|\mathbf{p}^{*'}|}{E_1^* E_2^* + |\mathbf{p}^*|^2}. \tag{C.1.6}$$

Recall that

$$E_1^* + E_2^* = s, \tag{C.1.7}$$

$$|\mathbf{p}^{*'}| = \frac{1}{2\sqrt{s}} \sqrt{\lambda\left(s, m_3^2, m_4^2\right)} = \frac{1}{2}\beta_f, \tag{C.1.8}$$

$$E_1^* E_2^* + |\mathbf{p}^*|^2 = E_1^* \left(E_1^* + E_2^*\right) - m_1^2 = \frac{s + m_1^2 - m_2^2}{2} - m_1^2 = \frac{s - m_1^2 - m_2^2}{2}. \tag{C.1.9}$$

Plugging all the expression into Eq. (C.1.6) gives

$$\sigma v_{\text{lab}} = \int \mathrm{d}\Omega \; \frac{\beta_f \overline{|\mathcal{M}|}^2}{64\pi^2 \left(s - m_1^2 - m_2^2\right)}, \tag{C.1.10}$$

where

$$\beta_f = \sqrt{\frac{\lambda(s, m_3^2, m_4^2)}{s}} = \left[1 - \frac{(m_3 + m_4)^2}{s}\right]^{1/2} \left[1 - \frac{(m_3 - m_4)^2}{s}\right]^{1/2}. \tag{C.1.11}$$

In case of neutralino pair-annihilation, the two incoming states is of the same species, thus Eq. (C.1.10) reduces to [**21**, **Eq. (3.23)**]

$$\sigma v_{\text{lab}} = \frac{\beta_f}{64\pi^2 \left(s - 2m^2\right)} \int \mathrm{d}\Omega \overline{|\mathcal{M}|^2}. \tag{C.1.12}$$

$\mathcal{D}$

The Friedmann, Robertson-Walker (FRW) Models in Cosmology

D.1 The Friedmann–Lemaître–Robertson–Walker (FLRW) metric

Consider a homogeneous, isotropic, expanding (or otherwise, contracting) Universe; this can be described using the FLRW metric

$$(\mathrm{d}s)^2 = (\mathrm{d}t)^2 - R(t)^2 \mathrm{d}\mathbf{\Sigma}^2, \tag{D.1.1}$$

with the time-dependent spatial coefficient $R(t)$ is so-called the scale factor. In reduced-circumference polar coordinates the spatial metric has the form

$$\mathrm{d}\mathbf{\Sigma} = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{D.1.2}$$

with parameter k represent the curvature of space.

D.2 The Friedmann equations

The Friedmann equations are a set of differential equations deduced from the Einstein's field equations for the FLRW metric above. This set of equations describe the expansion of homogeneous and isotropic Universe in the context of general relativity. They are

$$\frac{\dot{R}^2 + k}{R^2} = \frac{8\pi G\rho + \Lambda}{3}, \tag{D.2.1a}$$

$$\frac{\ddot{R}}{R} = \frac{-4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3}, \tag{D.2.1b}$$

where G is the Newtons gravitational constant, Λ is the cosmological constant. ρ and P are the density and pressure. A simplified form of the second equation is obtained by using the first one, which reads

$$\dot{\rho} = -3H(\rho + p), \tag{D.2.2}$$

which expresses the conservation of energy-momentum ternsor: $T^{\mu\nu}{}_{;\nu} = 0$

Related conservation laws

Below we sketch a proof of the conservation of entropy by considering the Friedmann equations the first law of thermal dynamics, which reads [2]

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V$$

[2] In this proof, we shall assume the vanishing of the chemical potential.

$$\Rightarrow \mathrm{d}S = \frac{\mathrm{d}U}{T} + \frac{P}{T}\mathrm{d}V = \frac{\mathrm{d}(\rho V)}{T} + \frac{P}{T}\mathrm{d}V = \frac{V}{T}\mathrm{d}\rho + \frac{\rho + P}{T}\mathrm{d}V$$
$$\Rightarrow \frac{\mathrm{d}S}{\mathrm{d}t} = \frac{V}{T}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho + P}{T}\frac{\mathrm{d}V}{\mathrm{d}t} = -3HV\left(\frac{\rho + P}{T}\right) + \frac{\rho + P}{T}\frac{\mathrm{d}V}{\mathrm{d}t}. \tag{D.2.3}$$

In a cosmological volume $V \propto R^3 \Rightarrow \mathrm{d}V/\mathrm{d}t = 3HV$, the two terms in (D.2.3) cancels and thus yields the conservation of total entropy

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0. \tag{D.2.4}$$

Define the entropy density $s \equiv S/V$ where V is the comoving volume, the first law of thermal dynamics is rewritten as

$$T\mathrm{d}(sV) = \mathrm{d}(\rho V) + P\mathrm{d}V \Rightarrow \mathrm{d}\rho - T\mathrm{d}s = (Ts - \rho - P)\,\frac{\mathrm{d}V}{V}. \tag{D.2.5}$$

In equilibrium, the intensive variables such as ρ, P and s is a function of temperature only. Hence the above equation with LHS contains differential dT and RHS has differential dV must vanish separately. The RHS thus implies the formula for calculating entropy density

$$s = \frac{\rho + P}{T}. \tag{D.2.6}$$

$\mathcal{E}$ Modified Bessel Function of the Second Kind

E.1 Relevant Properties

Bessel functions and the related special functions have been known since the 18th century and play an important role in solving various type of differential equations. As in this thesis, the modified Bessel functions of the second kind are the solution to integrals involving Boltzmann distribution. The modified Bessel functions were first introduced by A. B. Basset and H. M. MacDonald as a solution to the modified Bessel differential equation

$$z^2 w''(z) + zw'(z) - (z^2 + \nu^2)w(z) = 0 \Rightarrow w(z) = c_1 I_\nu(z) + c_2 K_\nu(z). \tag{E.1.1}$$

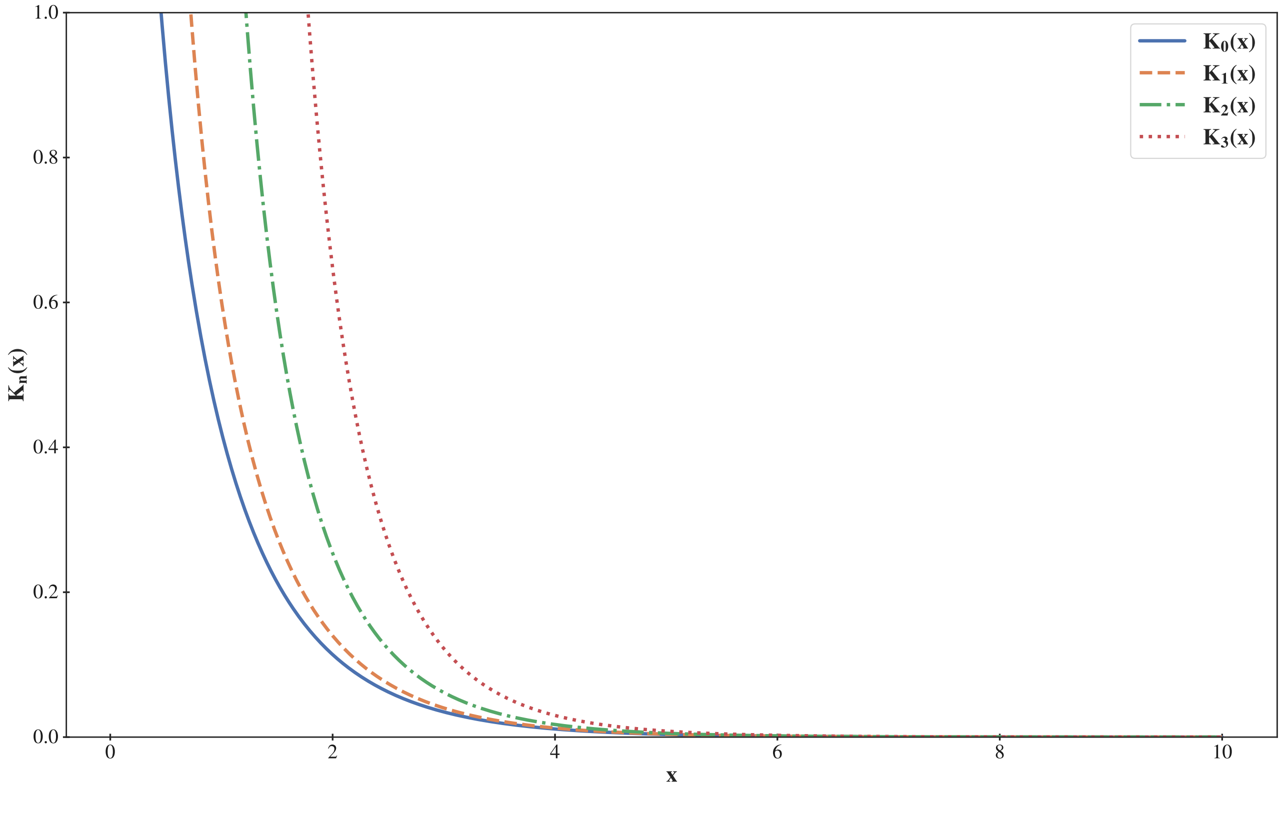


Figure E.1: Plot of modified Bessel functions of the second kind, with parameter n ranges from 0 to 3.

There are tons of properties of these special functions, we shall here mention only the necessary properties that will be used in the scope of the thesis.

- The first one is the integral representation of modified Bessel function of the second kind $K_\nu(z)$, which helps us in calculating a specific class of integrals

$$K_\nu(z) = \frac{\pi^{1/2}(z/2)^\nu}{\Gamma\left(\nu + \dfrac{1}{2}\right)} \int_1^\infty e^{-zy}\left(y^2 - 1\right)^{\nu - \dfrac{1}{2}} \, dy, \tag{E.1.2}$$

$$K_\nu(z) = \frac{\pi^{1/2}(z/2)^\nu}{\Gamma\left(\nu + \dfrac{1}{2}\right)} \int_0^\infty e^{-z\cosh y} (\sinh y)^{2\nu} \, dy, \tag{E.1.3}$$

$$K_\nu(z) = \frac{\pi^{1/2}(z/2)^\nu}{\Gamma\left(\nu + \dfrac{1}{2}\right)} \int_0^\infty e^{-zy} e^{-z\cosh y} \cosh(\nu y) \, dy. \tag{E.1.4}$$

- The second thing we want to mention is the recurrence relation of $K_\nu(z)$; these identities would be handful in simplifying expressions containing several modified Bessel functions (e.g in the calculation of equilibrium density $n^{\rm eq}$)

$$K_\nu(z) = \frac{z}{2\nu}\left[K_{\nu+1}(z) - K_{\nu-1}(z)\right]. \tag{E.1.5}$$

This relation can be directly proved by integrating Eq. (E.1.4) by parts

$$\begin{aligned} K_\nu(z) &= \frac{\sinh(\nu y)}{\nu} e^{-z\cosh y}\bigg|_0^\infty + \frac{z}{\nu}\int_0^\infty dy\, e^{-z\cosh y}\sinh(\nu y)\sinh(y) \\ &= \frac{z}{2\nu}\int_0^\infty dy\, e^{-z\cosh y}\Big(\cosh[(\nu+1)y] - \cosh[(\nu-1)y]\Big) \\ &= \frac{z}{2\nu}\left[K_{\nu+1}(z) - K_{\nu-1}(z)\right]. \end{aligned} \tag{E.1.6}$$

- In practice, we prefer to use the exponential scaled modified Bessel functions. Let us show first the asymptotic behaviour when $z \to\gg 1$

$$K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z} P_\nu(z), \tag{E.1.7}$$

where $P_\nu(z)$ is an asymptotic series given by (see Ref. [63, **Eq. (9.7.2)**])

$$P_\nu(z) = 1 + \frac{4\nu^2 - 1^2}{1!(8z)} + \frac{\left(4\nu^2 - 1^2\right)\left(4\nu^2 - 3^2\right)}{2!(8z)^2} + \ldots. \tag{E.1.8}$$

This asymptotic expansion can be obtain as follows: we consider the integrand of (E.1.4)

$$\begin{aligned} \cosh(\nu y) e^{-z\cosh y} &= e^{-z} e^{\dfrac{-zy^2}{2}} \cosh(\nu y) e^{-z\left(\cosh y - 1 - \dfrac{y^2}{2}\right)} \\ &= e^{-z} e^{\dfrac{-zy^2}{2}}\left[1 + \frac{m}{2}y^2 + \frac{\nu^2 - z}{24}y^4 + \mathcal{O}(y^6)\right]. \end{aligned} \tag{E.1.9}$$

Performing a sequence of integration of the type $\int_0^\infty e^{-zy^2/2} y^{2n}\, dy$ (which results in an expression of Gamma function), one can obtain the desired asymptotic series. For instance, up to the 4th order of y in (E.1.9) gives

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z}\left[1 + \frac{4m^2 - 1}{8z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right]. \tag{E.1.10}$$

A sketch of the exponential property of $K_\nu(z)$ is given below.

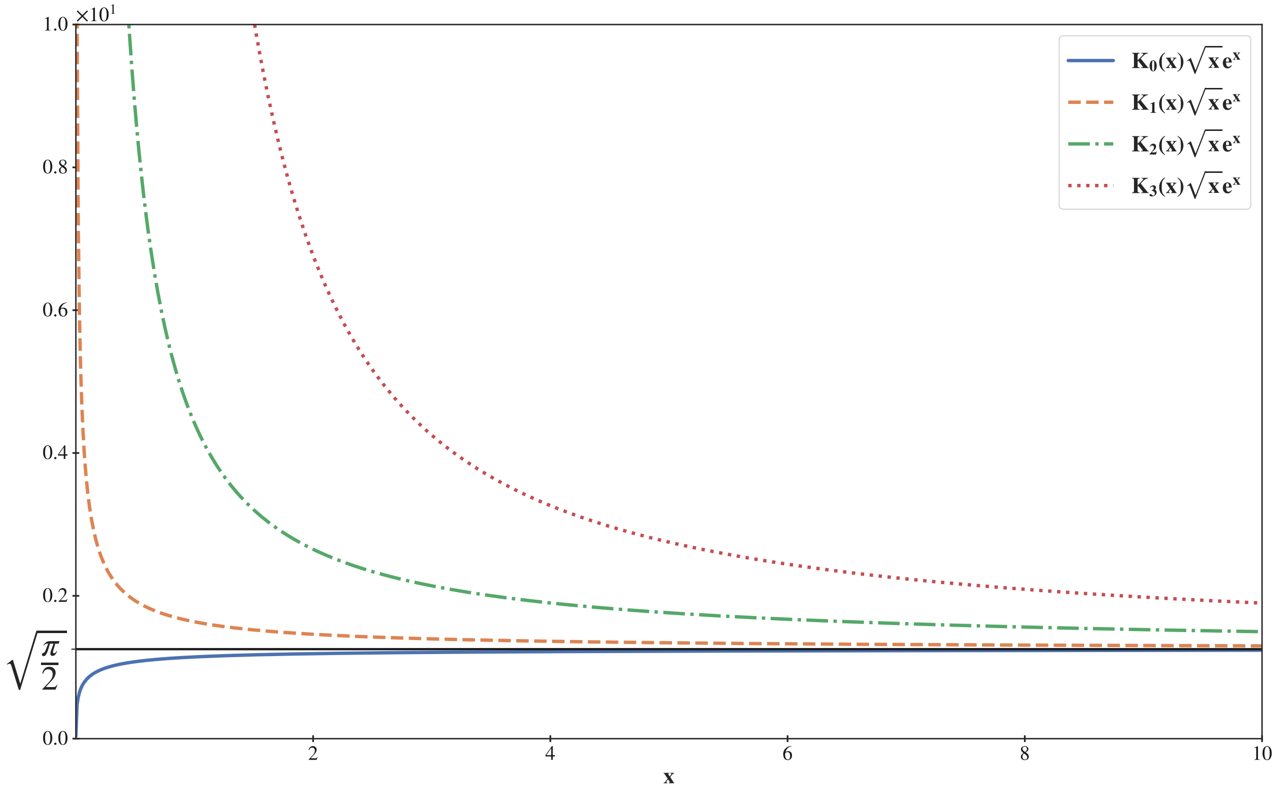


Figure E.2: asymptotic behaviour of modified Bessel functions of the second kind.

In practice, we prefer to use the exponentially scaled modified Bessel function $e^z K_\nu(z)$. Especially when performing calculations with large value of argument, applying the scaled modified Bessel functions give better performance. For instance when performing the integration where the integrand containing the thermal kernel Eq. (II.2.26), the exponential terms on the numerator and denominator partially cancel out. Keeping these terms can cause the error of dividing by zero error.

E.2 Related applications

This section aims to show some applications of the modified Bessel function of the second kind, mainly using the integral representation in calculating some physical quantities. Let us recall that a relativistic, homogeneous and isotropic macroscopic system of particles (bosons or fermions) in thermal equilibrium can be described using the corresponding statistics:

$$f_\eta(x,p;T) = f_\eta(T,E) = \frac{g}{(2\pi)^3}\frac{1}{e^{E/T}+\eta} \implies \begin{cases} \text{Bose-Einstein distribution: } \eta = -1 \to \text{bosons} \\ \text{Fermi-Dirac distribution: } \eta = +1 \to \text{fermions} \\ \text{Boltzmann distribution: } \eta = 0 \end{cases}, \tag{E.2.1}$$

where g counts the number of internal degrees of freedom. Macroscopic quantities such as number, momentum and energy densities can be constructed from the given distribution by taking integration over momentum space, specifically

- Number density

$$n = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, f_\eta(x,p;T). \tag{E.2.2}$$

- Energy density

$$\rho = \frac{g}{(2\pi)^3}\int \mathrm{d}^3p\, E f_\eta(x,p;T). \tag{E.2.3}$$

- Pressure

$$P = \frac{1}{3}\frac{g}{(2\pi)^3}\int \mathrm{d}^3p\,\frac{p^2}{E}f_\eta(x,p;T). \tag{E.2.4}$$

Due to the isotropic property: $\mathrm{d}^3p = 4\pi p^2\mathrm{d}p$. To work with the dimensionless quantities, we make the following change of variables

$$x = \frac{m}{T}, \qquad \xi = \frac{p}{T}, \tag{E.2.5}$$

and rewrite (E.2.2), (E.2.3) and (E.2.4) as

$$n^{\mathrm{eq}} = \frac{gT^3}{2\pi^2}\int_0^\infty \mathrm{d}\xi\,\frac{\xi^2}{e^{\sqrt{\xi^2+x^2}}+\eta} = \frac{gT^3}{2\pi^2}A_\eta, \tag{E.2.6}$$

$$\rho^{\mathrm{eq}} = \frac{gT^4}{2\pi^2}\int_0^\infty \mathrm{d}\xi\,\frac{\xi^2\sqrt{\xi^2+x^2}}{e^{\sqrt{\xi^2+x^2}}+\eta} = \frac{gT^4}{2\pi^2}B_\eta, \tag{E.2.7}$$

$$P^{\mathrm{eq}} = \frac{gT^4}{6\pi^2}\int_0^\infty \mathrm{d}\xi\,\frac{\xi^4}{\sqrt{\xi^2+x^2}\left(e^{\sqrt{\xi^2+x^2}}+\eta\right)} = \frac{gT^4}{6\pi^2}C_\eta. \tag{E.2.8}$$

Let us consider the case $\eta = 0$, i.e the equilibrium system follows the Boltzmann distribution, and calculate the integrals A_0, B_0, C_0. We make another change of variables to bring these integrals into one of the representation (E.1.2), (E.1.3), (E.1.4):

$$\cosh y = \sqrt{\xi^1/x^2+1}, \sinh y = \xi/x \quad\Rightarrow\quad \mathrm{d}\xi = x\cosh y\mathrm{d}y. \tag{E.2.9}$$

The first integral is

$$\begin{aligned} A_0 = \int_0^\infty \mathrm{d}\xi\,\frac{\xi^2}{e^{\sqrt{\xi^2+x^2}}} &= x^3\int_0^\infty \mathrm{d}y\,e^{-x\cosh y}\sinh^2 y\cosh y \\ &= \frac{x^3}{4}\int_0^\infty \mathrm{d}y\,e^{-x\cosh y}\big[\cosh(3y)-\cosh y\big] \\ &= \frac{x^3}{4}\big[K_3(x)-K_1(x)\big] = x^2K_2(x), \end{aligned} \tag{E.2.10}$$

where we used the integral representation (E.1.4) and the recurrence relation $[K_3(x)-K_1(x)]\,x/4 = K_2(x)$, derived directly from (E.1.6). The next integral to calculate is that of energy density

$$\begin{aligned} B_0 = \int_0^\infty \mathrm{d}y\,\frac{\xi^2\sqrt{\xi^2+x^2}}{e^{\sqrt{\xi^2+x^2}}} &= x^4\int_0^\infty \mathrm{d}y\,e^{-x\cosh y}\sinh^2 y\cosh^2 y \\ &= \frac{x^4}{8}\int_0^\infty \mathrm{d}y\,e^{-x\cosh y}\big[\cosh(4y)-1\big] \\ &= \frac{x^4}{8}\left[K_4(x)-K_0(x)\right]. \end{aligned} \tag{E.2.11}$$

Applying repeatedly the recurrence relation (E.1.6), we can rewrite the above result as

$$B_0 = x^2\big[3K_2(x)+xK_1(x)\big]. \tag{E.2.12}$$

Finally, the calculation on the integral C_0 proceed similarly as follows

$$\begin{aligned} C_0 = \int_0^\infty \mathrm{d}\xi\,\frac{\xi^4}{\sqrt{\xi^2+x^2}\left(e^{\sqrt{\xi^2+x^2}}\right)} &= x^4\int_0^\infty \mathrm{d}y\,e^{-x\cosh y}\sinh^4(y) \\ &= 3x^2K_2(x). \end{aligned} \tag{E.2.13}$$

where Eq. (E.1.3) with $\nu = 2$ is used in the final equality. Substitute the expression A_0, B_0 and C_0 back to Eq. (E.2.6), (E.2.7) and (E.2.8) respectively gives

$$n^{\mathrm{eq}} = \frac{gT^3}{2\pi^2}x^2K_2(x), \tag{E.2.14}$$

$$\rho^{\mathrm{eq}} = \frac{gT^4}{2\pi^2} x^2 \big[3K_2(x) + xK_1(x)\big], \tag{E.2.15}$$

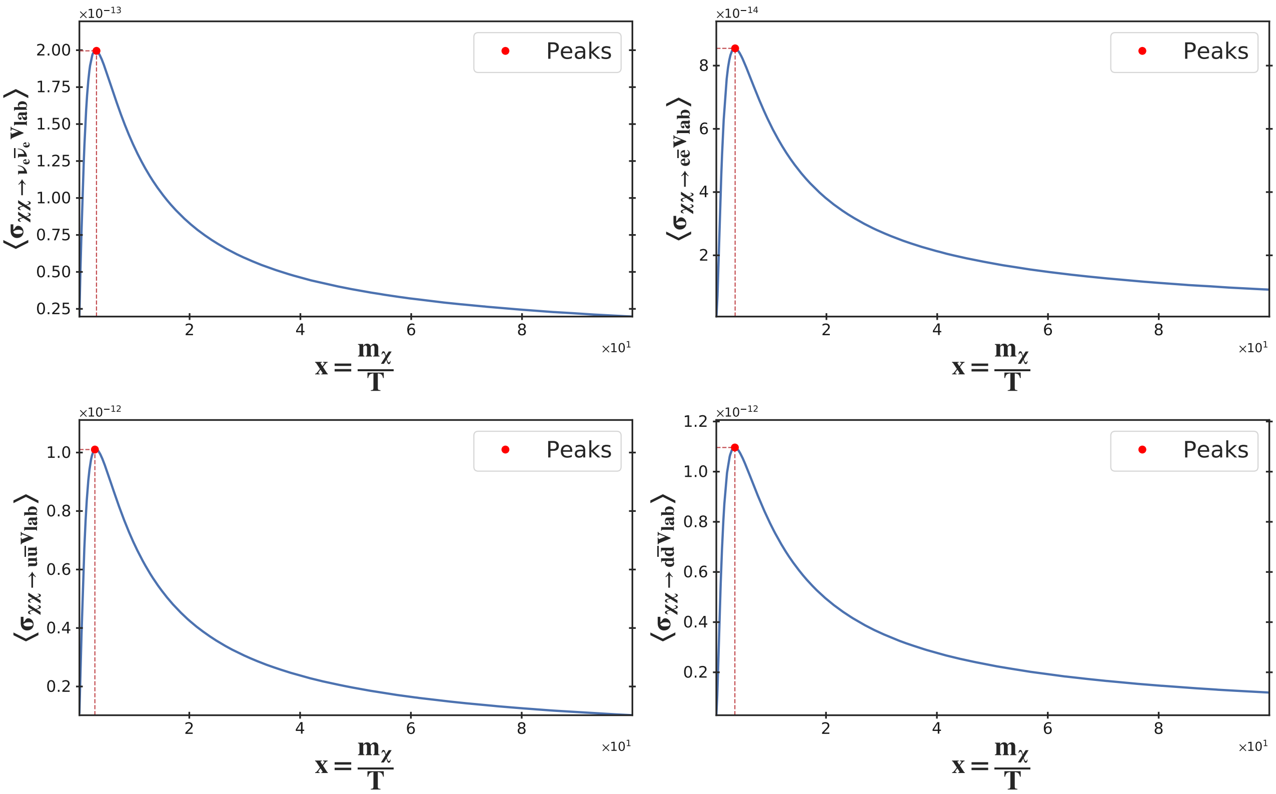
$$P^{\mathrm{eq}} = \frac{gT^4}{2\pi^2} x^2 K_2(x). \tag{E.2.16}$$

$$\tag{E.2.17}$$

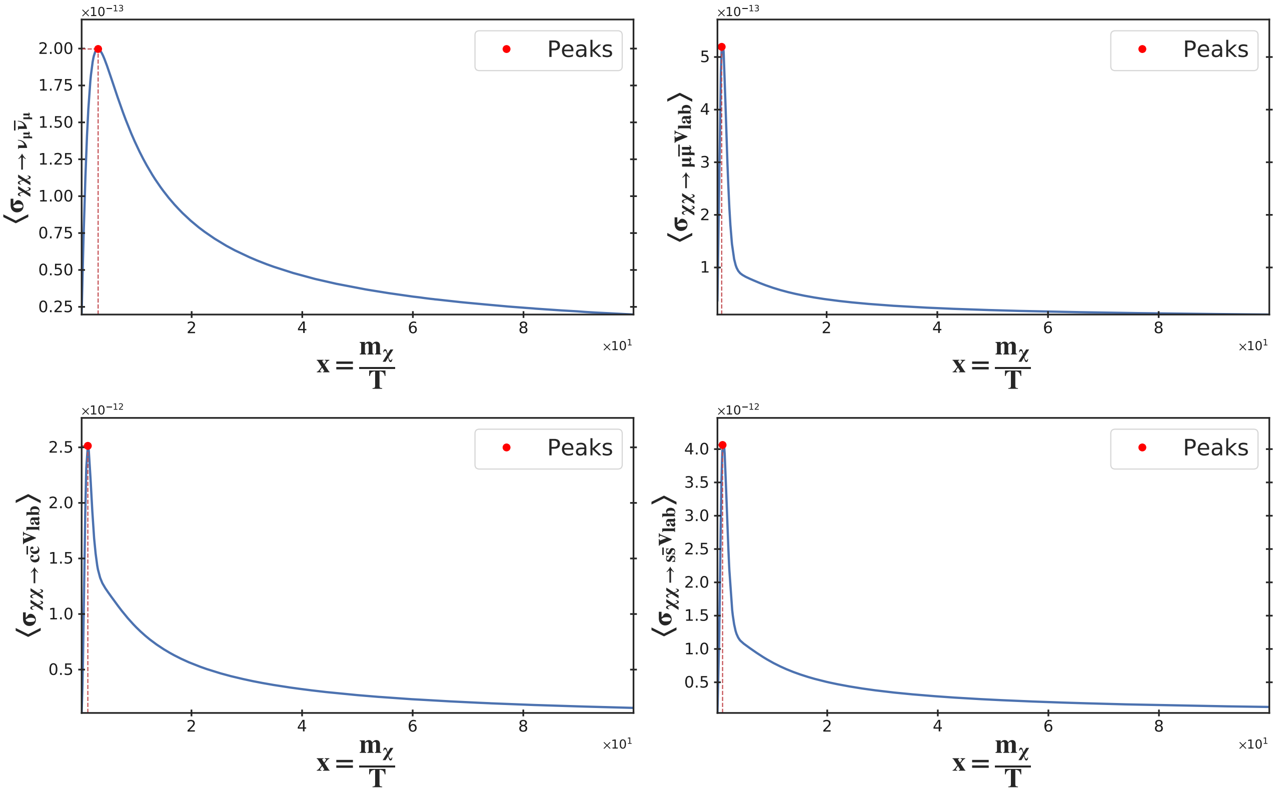
F

Thermal average of velocity-weighted cross section

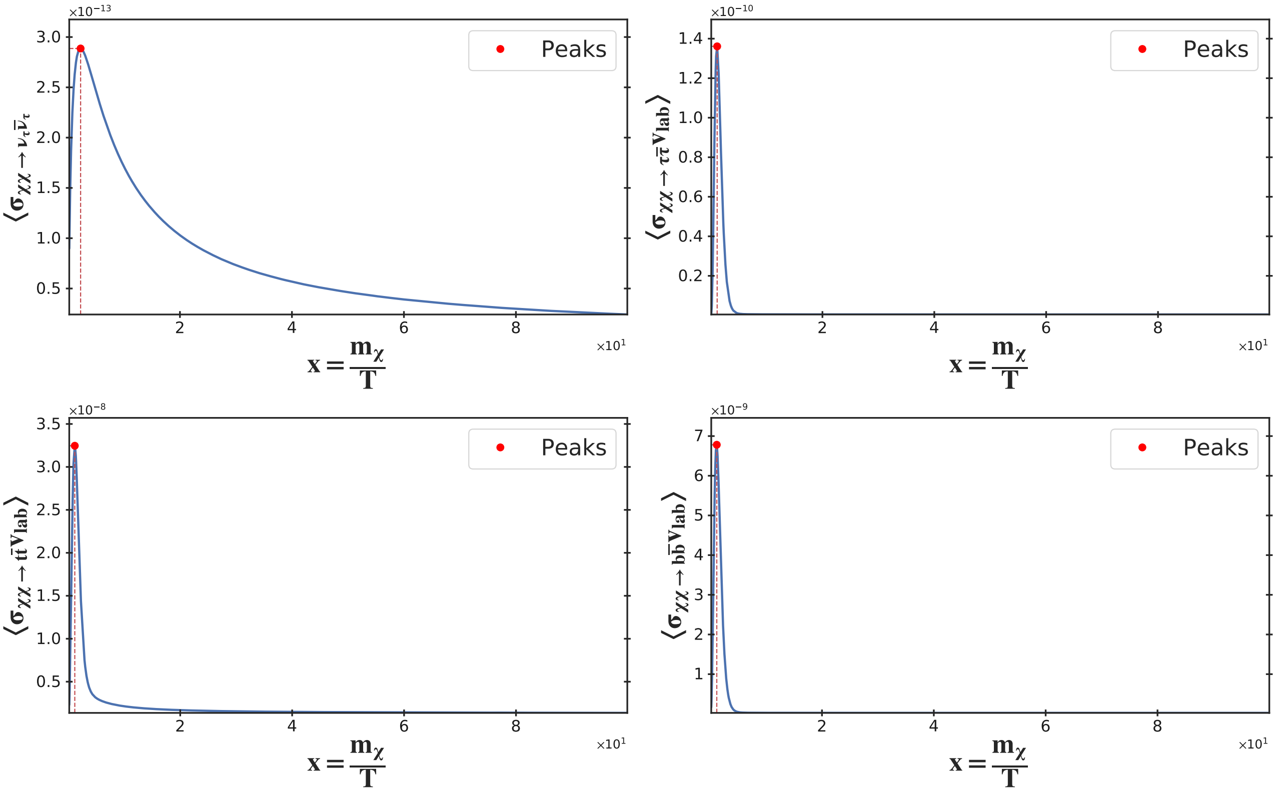
In the present appendix we attempt to show the plot of velocity-weighted cross section of each tree-level neutralino annihilation process.



(a) Fermionic final states of 1st generation.



(b) Fermionic final states of 2nd generation.



(c) Fermionic final states of 3rd generation.

Figure F.1: $\langle \sigma v_{\mathrm{lab}} \rangle$ of neutralino pair annihilation with weak fermionic final states.

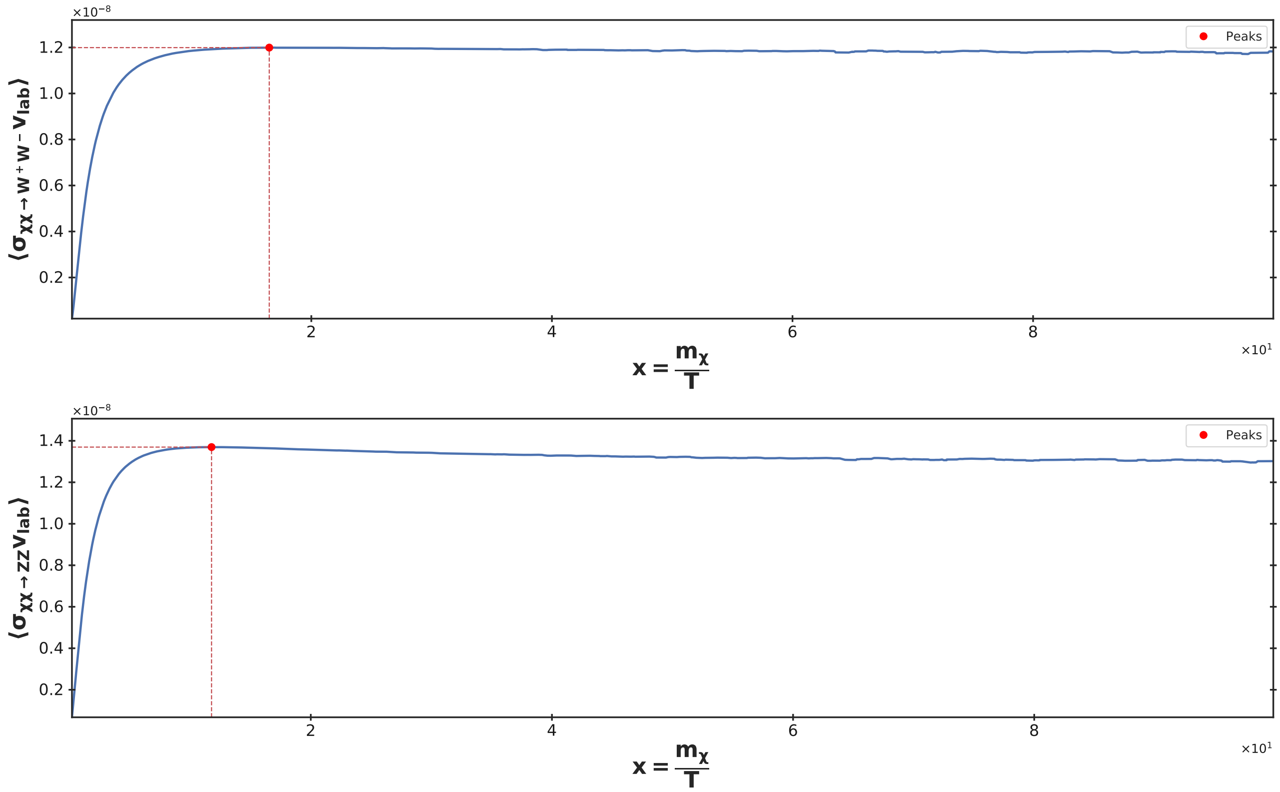


Figure F.2: $\langle \sigma v_{\rm lab} \rangle$ of neutralino pair annihilation with weak gauge bosons final states.

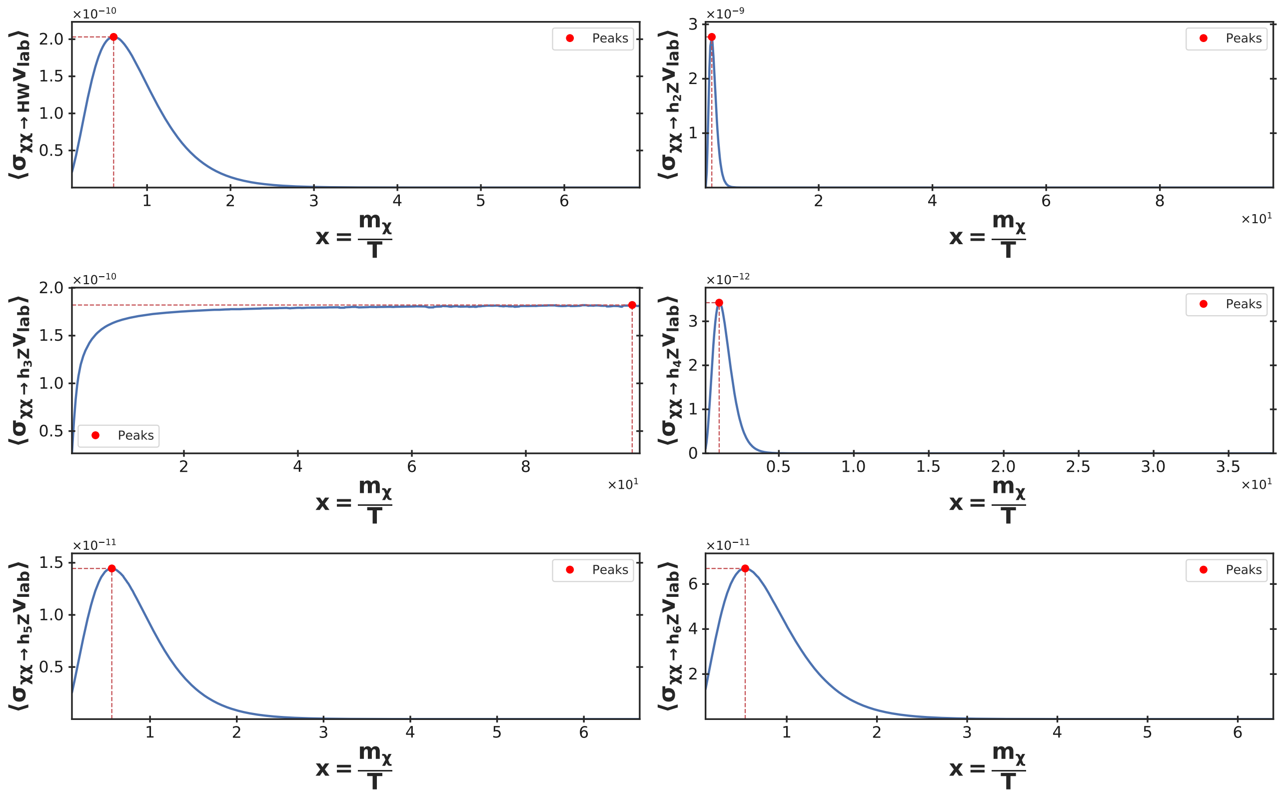


Figure F.3: $\langle \sigma v_{\rm lab} \rangle$ of neutralino pair annihilation with one Higgs boson and one weak gauge boson final states.

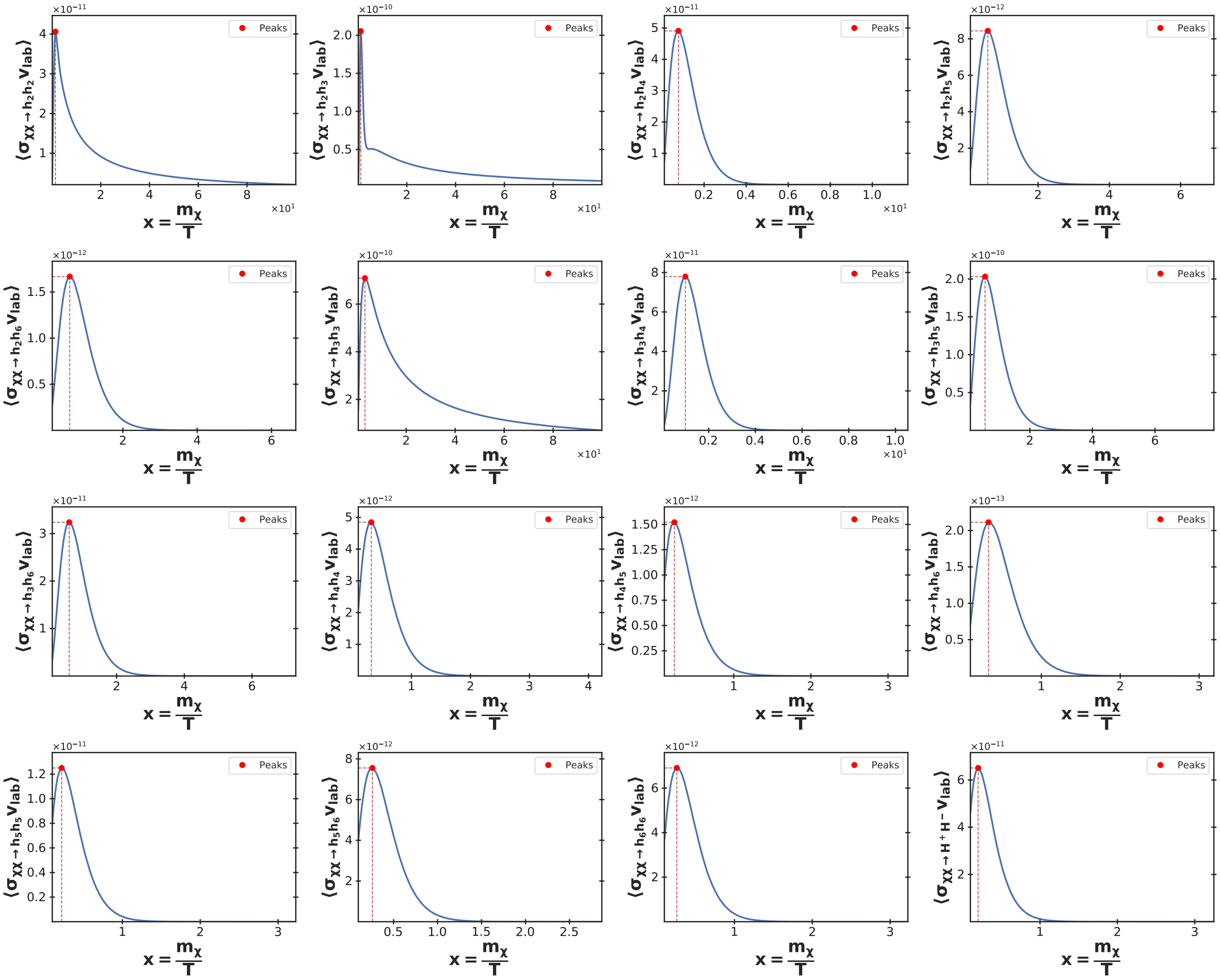


Figure F.4: $\langle \sigma v_{\mathrm{lab}} \rangle$ of neutralino pair annihilation with Higgs bosons final states.

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