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Bachelor's Degree thesis

Scattering process $e^+ + e^- \to \mu^+ + \mu^-$ in Standard Model Effective Field Theory and some applications for the spin-density matrix of Z boson

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Introduction

The first two chapters in this thesis were followed the first two chapters of Hong Minh's thesis[1]. I have checked all the calculations there and just put essential things to my thesis. In that, I have provided a well-known method to find the total cross-section and some distributions of the given process $e^+ + e^- \to \mu^+ + \mu^-$ in both Quantum Electrodynamics (QED) and the Standard Model (SM) cases. Many essential calculated steps will be introduced in those chapters such as how to find vertex factors, the propagators, Feynman amplitude... Some comparisons between two theories shall be implemented to help us to see how different they are.

Nevertheless, the main purpose of this thesis is studying dimension-six operators using Standard Model Effective Field Theory (SMEFT) framework and then apply it in our process. Although SM is the successful theory which has amazing agreement with experiments, many physicists believe that it is just the leading order terms of an exhaustive theory. Experimental errors of measurements are becoming smaller when the LHC reaches the higher level of energy, and it can help to find new physics beyond SM. Therefore, in Chap. 3, we will derive many deviations of relations from the SM especially the total cross-section result of $e^+ + e^- \to \mu^+ + \mu^-$ process in SMEFT. Also, some main features of physics such as gauge invariant of dimension-six operators, the independent of Feynman amplitude on gauge-fixing parameters... will be checked.

In the final chapter, I have re-produced the relations between the Z boson decay angular distributions and the spin-density matrix elements of the Z boson (it is noted that Z bosons produced at e^+e or pp collision are polarized). Then I applied these results to the process $e^+ + e^- \to Z \to \mu^+ + \mu^-$ and find the density matrix for Z boson in two different ways. First, I used the helicity-amplitude method to calculate the density matrix. In another way, I found the normalized distribution of the cross-section using the amplitude-squared method, then I compared with the relation produced above to obtain density matrix elements. After that, I find the normalized angular distribution for the process $e^+ + e^- \to Z + \gamma \to \mu^+ + \mu^-$ which mediated by Z boson and photon. Using the same manner, I was able to derive the so-called "density matrix" elements (it is not actually the density matrix since it has two mediators) and then saw the difference between two density matrix. Moreover, after using the SMEFT framework, I will be able to calculate the effects of dimension-six operators on the above spin observables and make a comparison with that of SM.

1

Scattering process $e^- + e^+ \to \mu^- + \mu^+$ in QED

1.1 Feynman rules in QED

First, the main difference between many models in particle physics is the change of the Lagrangian density (Lagrangian for short). In QED, only electromagnetic interaction, which mediated by the photon, exists. The QED Lagrangian, therefore, contains only the field of photon A_μ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi_A}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{D} - m)\psi, \tag{1.1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. D_μ is the covariant derivative, which is: $D_\mu = \partial_\mu - ieA_\mu$. The term $-\dfrac{1}{2\xi_A}(\partial_\mu A^\mu)^2$ is the gauge fixing term which contains the gauge fixing parameter ξ_A. As you will see later, gauge-fixing term is a technique to calculate the photon propagator.
Note that in Lagrangian (1.1), the terms which contain one field only are called the free Lagrangian terms, except for the case $-m\bar{\psi}\psi$ is called the mass term. While the terms containing more than one field are the interaction terms.

1.1.1 Photon propagator

Applying the principle of least action for the free electromagnetic field part of the Lagrangian in (1.1), the Euler-Lagrange equation is of the form:

$$\partial_\sigma\left(\frac{\partial\mathcal{L}_E}{\partial(\partial_\sigma A_\lambda)}\right) - \frac{\partial\mathcal{L}_E}{\partial A_\lambda} = 0. \tag{1.2}$$

It is obvious that the second term is equal to zero, the first term will be calculated step by step, we have

$$\Rightarrow \frac{\partial\mathcal{L}_E}{\partial(\partial_\sigma A_\lambda)} = \partial^\lambda A^\sigma - \partial^\sigma A^\lambda - \frac{1}{\xi_A}(\partial_\mu A^\mu)g^{\lambda\sigma}.$$

Substituting that result into Eq.(1.2), the equation of motion for photon field becomes

$$\left[\Box g^{\mu\rho} - \left(1 - \frac{1}{\xi_A}\right)\partial^\mu\partial^\rho\right]A_\rho = 0. \tag{1.3}$$

The propagator of the vector field $D_{\rho\nu}(x-y)$ is the solution of the inhomogeneous equation of motion above with a point-like source:

$$\left[\Box g^{\mu\rho} - \left(1 - \frac{1}{\xi_A}\right)\partial^\mu\partial^\rho\right] D_{\rho\nu}(x-y) = g^\mu_\nu \delta^4(x-y). \tag{1.4}$$

Using the Fourier transformation for both sides of the Eq.(1.4), we have:

$$\left[\Box g^{\mu\rho} - \left(1 - \frac{1}{\xi_A}\right)\partial^\mu\partial^\rho\right] \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} D_{\rho\nu}(q) = g^\mu_\nu \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)},$$

$$\left[-q^2 g^{\mu\rho} + \left(1 - \frac{1}{\xi_A}\right) q^\mu q^\rho\right] D_{\rho\nu}(q) = g^\mu_\nu. \tag{1.5}$$

Based on the Green function method, we know that $D_{\mu\nu}$ is the photon propagator and the general form of it is

$$D_{\rho\nu}(q) = A(q^2) q_\rho q_\nu + B(q^2) g_{\rho\nu} \tag{1.6}$$

To determine two coefficients above, we have to insert the general form of photon propagator into Eq.(1.5), it then becomes

$$\left[-q^2 g^{\mu\rho} + \left(1 - \frac{1}{\xi_A}\right) q^\mu q^\rho\right] \left[A(q^2) q_\rho q_\nu + B(q^2) g_{\rho\nu}\right] = g^\mu_\nu.$$

We can easily derive $B(q^2) = \dfrac{-1}{q^2}$ and $A(q^2) = \dfrac{1-\xi_A}{q^4}$. Hence, we can rewrite the photon propagator in term of:

$$D_{\mu\nu}(q) = \frac{1-\xi_A}{q^4} q_\mu q_\nu - \frac{g_{\mu\nu}}{q^2}. \tag{1.7}$$

An interesting thing is that the gauge-fixing term can help us to derive the photon propagator, but itself is not gauge invariant. Let's now discuss about this special term. We can see that the gauge fixing term depends only on $(\partial_\mu A^\mu)^2$ and independent of the fermion field. Thus, the gauge fixing term can only affect the photon propagator, which is the internal line. while the external fermion lines still unchanged. Notice that the internal line is not a physical observable, so it is not important if this propagator is not gauge invariant. Therefore, the Feynman rules in the intermediate steps are important but it is no need to be gauge invariance. Nevertheless, the final results must be gauge invariant because it is the physical observables.

As I have mentioned above, the gauge-fixing term is a technique to calculate the photon propagator. Because if we do not have that term, we have derived an invalid identity $1/q^2 = 0$. Thus, we could not obtain the coefficient $A(q^2)$ in the general form of the photon propagator.

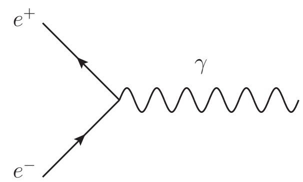
So that, we need to introduce gauge fixing term as a trick. But finally, the physical observables must be independent of gauge-fixing parameters. And so far in my thesis, you will see that no physical observable depend on that parameters. Because of that reason, we can arbitrarily choose the value of gauge-fixing parameters. If we choose $\xi = 1$, we will call it Feynman gauge. Unitary gauge for $\xi = \infty$. And the R_ξ gauge which still remains ξ.

1.1.2 Vertex factor of $ee\gamma$

One of the useful methods to find the vertex factor is looking for the interaction terms in Lagrangian, then remove all fields and multiply by imaginary unit. In Eq.(1.1), the interaction term is:

$$\bar{\psi} i \not{D} \psi \supset \bar{\psi} i(-ie\gamma^\mu A_\mu)\psi = \bar{\psi} e\gamma^\mu A_\mu \psi. \tag{1.8}$$

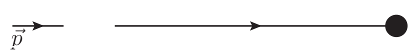
Thus, the vertex factor reads



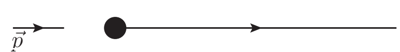
$$= ie\gamma^\mu. \tag{1.9}$$

1.1.3 Feynman rules in QED

Now we will introduce the Feynman rules in QED, notice that we only focus on the rules for diagrams at tree level only. External lines contribute a factor as follows:



$$= u(p),$$



$$= \bar{u}(p),$$

$$= \bar{v}(p),$$

$$= v(p).$$

The essential feature of Feynman rules is that the energy-momentum conservation law must be obeyed at each vertex.

1.2 Feynman amplitude

Based on the Feynman rules in QED that we had already introduced, we can write down the Feynman amplitude for the scattering process of $e^- + e^+ \to \mu^- + \mu^+$ in Fig. (1.1) as follows:

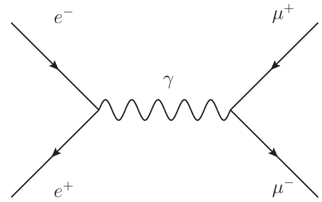


Figure 1.1: Feynman diagram of process $e^+ + e^- \to \mu^+\mu^-$ in QED

$$\mathcal{M} = [\bar{v}_{s'}(p')(ie\gamma^\mu)u_s(p)]\left[\frac{(1-\xi)q_\mu q_\nu}{q^4} - \frac{g_{\mu\nu}}{q^2}\right][\bar{u}_r(k)(ie\gamma^\nu)v_{r'}(k')]$$
$$= -\frac{(1-\xi)e^2}{q^4}[\bar{v}_{s'}(p')\not{q}u_s(p)][\bar{u}_r(k)\not{q}v_{r'}(k')] + \frac{e^2}{q^2}[\bar{v}_{s'}(p')\gamma^\mu u_s(p)][\bar{u}_r(k)\gamma_\mu v_{r'}(k')]$$

We have denoted the four-momenta and spin indices of e^-, e^+, μ^-, μ^+ to be (p,s), (p',s'), (k,r), (k',r'), respectively. Using the Dirac equations

$$\begin{cases} \not{p}u_s(p) = m_e u_s(p), \\ \bar{v}_{s'}(p')\not{p}' = -m_e\bar{v}_{s'}(p'), \end{cases}$$

the Feynman amplitude becomes:

$$\mathcal{M} = \frac{e^2}{q^2}[\bar{v}_{s'}(p')\gamma^\mu u_s(p)][\bar{u}_r(k)\gamma_\mu v_{r'}(k')].$$

You can see that $\mathcal{M}$ is now independent of ξ. The squared amplitude, therefore, have the form

$$\mathcal{M}^2 = \mathcal{M}^\dagger\mathcal{M} = \frac{e^4}{q^4}[\bar{v}_{s'}(p')\gamma^\nu u_s(p)][\bar{u}_s(p)\gamma^\mu v_{s'}(p')][\bar{v}_{r'}(k')\gamma_\mu u_r(k)][\bar{u}_r(k)\gamma_\nu v_{r'}(k')].$$

In the case of unpolarized beam, we have to average the cross-section over the initial spin state since we do not know the initial particles spin. However, we must sum the cross-section over the final state because we accept all final particles in the detector and do not measure their spin state. Summing over the spin states, $v_{s'}(p')\bar{v}_{s'}(p')$ can be replaced by $\not{p}' - m_e$. Similarly for another electron and the final muon. Now look at the squared amplitude, we see that it is a scalar and can be rewritten as traces. So we have:

$$\Rightarrow |\mathcal{M}_0|^2 = \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4q^4}Tr[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu]\ Tr[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu]$$
$$= \frac{8e^4}{q^4}[(p.k')(p'.k) + (p'.k')(p.k) + m_\mu^2(p.p') + m_e^2(k.k') + 2m_e^2m_\mu^2],$$

where the trace above were tricked as follows

$$\begin{aligned} Tr[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu] &= Tr(\not{p}'\gamma^\mu\not{p}\gamma^\nu) - Tr(m_e\gamma^\mu m_e\gamma^\nu) \\ &= p'_\alpha p_\beta Tr(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu) - m_e^2 Tr(\gamma^\mu\gamma^\nu) \\ &= p'_\alpha p_\beta 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}) - 4m_e^2 g^{\mu\nu} \\ &= 4(p^\mu p'^\nu - pp'g^{\mu\nu} + p^\nu p'^\mu - m_e^2 g^{\mu\nu}), \end{aligned}$$

$$\begin{aligned} Tr[(\not{k}' + m_\mu)\gamma_\mu(\not{k} + m_\mu)\gamma_\nu] &= Tr(\not{k}'\gamma_\mu\not{k}\gamma_\nu) - Tr(m_\mu\gamma_\mu m_\mu\gamma_\nu) \\ &= k'^\alpha k^\beta Tr(\gamma_\alpha\gamma_\mu\gamma_\beta\gamma_\nu) - m_\mu^2 Tr(\gamma_\mu\gamma_\nu) \\ &= k'^\alpha k^\beta 4(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\nu}g_{\mu\beta}) - m_\mu^2 4g_{\mu\nu} \\ &= 4(k'_\mu k_\nu - k'kg_{\mu\nu} + k'_\nu k_\mu - m_\mu^2 g_{\mu\nu}). \end{aligned}$$

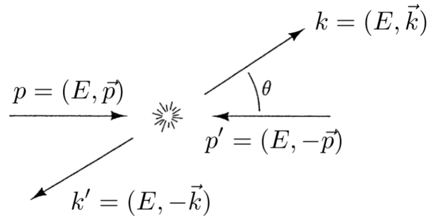


Figure 1.2: The process $ee \to \mu\mu$ in CM frame

Let's introduce a quantity out going angle $\theta = (\mathbf{k}, Oz)$ and work in Center of Mass Frame (CMF) Fig.(1.2). Note that we shall use the approximation $me = 0$. In order to write the squared-amplitude in term of energy E and angle θ. We must derive some identities below:

$$\begin{cases} p.k = p'.k' = E^2 - \mathbf{k}.\mathbf{p} = E^2 - |\mathbf{p}||\mathbf{k}|\cos\theta = \mathrm{E}^2 - \mathrm{E}|\mathbf{k}|\cos\theta, \\ p.k' = p'.k = E^2 + \mathbf{k}.\mathbf{p} = E^2 + |\mathbf{p}||\mathbf{k}|\cos\theta = \mathrm{E}^2 + \mathrm{E}|\mathbf{k}|\cos\theta, \\ q^2 = (p + p')^2 = (2E, \mathbf{p} - \mathbf{p})^2 = 4E^2, \\ p.p' = E^2 + |\mathbf{p}|^2 = E^2 + E^2 = 2E^2, \\ k.k' = E^2 + |\mathbf{k}|^2. \end{cases} \tag{1.10}$$

The squared amplitude then reads

$$|\mathcal{M}_0|^2 = \frac{e^4}{E^2}\left[(E^2 + m_\mu^2) + (E^2 - m_\mu^2)\cos^2\theta\right]. \tag{1.11}$$

1.3 Some distributions of $e^- + e^+ \to \mu^- + \mu^+$ process

1.3.1 Total cross-section

Based on [2], the cross section in the center of mass frame for two final-state particles is of the form:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{k}|}{|\mathbf{p}|}|\mathcal{M}_0|^2, \tag{1.12}$$

where $\sqrt{s} = 2E$ is the total energy in CM frame.

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{|\mathbf{k}|e^4}{256\pi^2 E^5}\left[(E^2 + m_\mu^2) + (E^2 - m_\mu^2)\cos^2\theta\right]. \tag{1.13}$$

Integrating the differential cross-section over all directions, we can derive the total cross-section:

$$\sigma_T = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{|\mathbf{k}|e^4}{48\pi E^5}\left(E^2 + \frac{1}{2}m_\mu^2\right) \tag{1.14}$$

Fig.(1.3) indicates the dependence of the total cross-section on $\sqrt{s} = 2E$ which is the total energy in CM frame. As you can see, the plot begin at $\sqrt{s} = 2m_\mu$. That is a reasonable result since based on the energy conservation, we must have the total energy at least equal to the mass of a pair of muons in order to create these particles.

1.3.2 Angular distribution of muon

The angular distribution is the important distribution, it helps us to predict which angle is more sensitive to the muon. So that we will know where to put the detectors. From Eq.(1.13), we easily obtain the angular distribution

$$\frac{d\sigma}{d\theta} = \frac{|\mathbf{k}|e^4}{256\pi^2 E^5}\int_0^{2\pi}\Big[(E^2+m_\mu^2)+(E^2-m_\mu^2)\cos^2\theta\Big]\sin\theta.\mathrm{d}\varphi,$$
$$= \frac{|\mathbf{k}|e^4}{128\pi E^5}\Big[(E^2+m_\mu^2)+(E^2-m_\mu^2)\cos^2\theta\Big]\sin\theta.$$

Fig.(1.4) illustrates the angular distribution of muon where the total energy is 20 GeV. We can see the plot is symmetry between the backward and forward side, and it has two peaks at $\theta = 0.95$ (rad) and $\theta = 2.19$ (rad) corresponding to the two most sensitive angles.

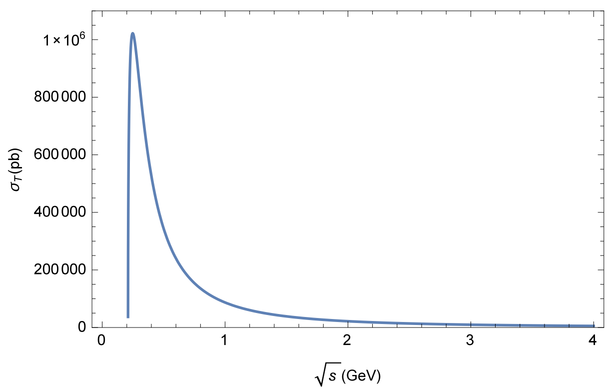


Figure 1.3: Total cross-section

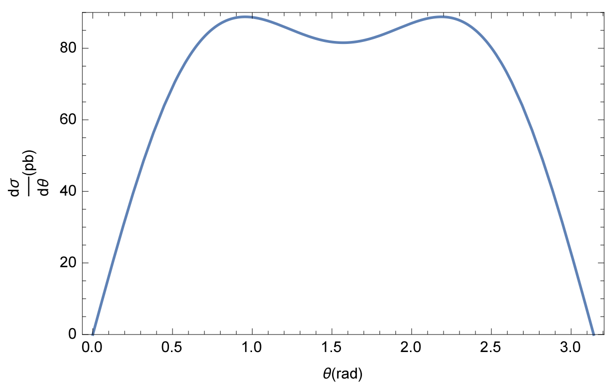


Figure 1.4: Angular distribution of muon

1.3.3 Transverse momentum and longitudinal momentum distributions of muon

In general, if we wish to change from distribution $f(x)$ to distribution $g(y)$, where y is the function of x, we can use the formation

$$g(y) = \sum_{x_i} f(x)\left|\frac{dx}{dy}\right|_{x=x_i}. \tag{1.15}$$

In Eq.(1.15), the term $|dx/dy|$ is called the Jacobian. It must be understood that x_i on the right handside should be written in terms of y via the inverse function.
The next popular quantities are transverse and longitudinal momentum which are denoted by k_t and k_l, respectively. First, the transverse momentum reads

$$k_t = |\mathbf{k}|\sin\theta \Rightarrow \begin{cases} \theta_1 = arcsin\dfrac{k_t}{|\mathbf{k}|}, \\ \theta_2 = \pi - arcsin\dfrac{k_t}{|\mathbf{k}|}. \end{cases} \tag{1.16}$$

The Jacobians are derived as:

$$\left|\frac{d\theta}{dk_t}\right| = \frac{1}{|\mathbf{k}||\cos\theta|} \Rightarrow \left|\frac{d\theta}{dk_t}\bigg|_{\theta=\theta_1}\right| = \left|\frac{d\theta}{dk_t}\bigg|_{\theta=\theta_2}\right| = \frac{1}{|\mathbf{k}|\sqrt{1-\sin^2\theta_1}} = \frac{1}{\sqrt{|\mathbf{k}|^2 - k_t^2}} \tag{1.17}$$

Using Eq. (1.15) above, the transverse momentum distribution becomes

$$\frac{d\sigma}{dk_t} = \sum_i \frac{d\sigma}{d\theta}\bigg|_{\theta=\theta_i} \left|\frac{d\theta}{dk_t}\bigg|_{\theta=\theta_i}\right| = \frac{k_t\left(2E^2 - k_t^2\right)e^4}{64\pi E^5\sqrt{|\mathbf{k}|^2 - k_t^2}} \tag{1.18}$$

Similarly, we could find the longitudinal momentum distribution of muon.

$$\frac{d\sigma}{dk_l} = \frac{e^4}{128\pi E^5}\left(E^2 + m_\mu^2 + k_l^2\right) \tag{1.19}$$

where the longitudinal momentum is defined as: $k_l = |\mathbf{k}|\cos\theta$. Fig. (1.5) show the transverse and longitudinal momentum distributions of muon in the case that CM energy is 10 GeV.

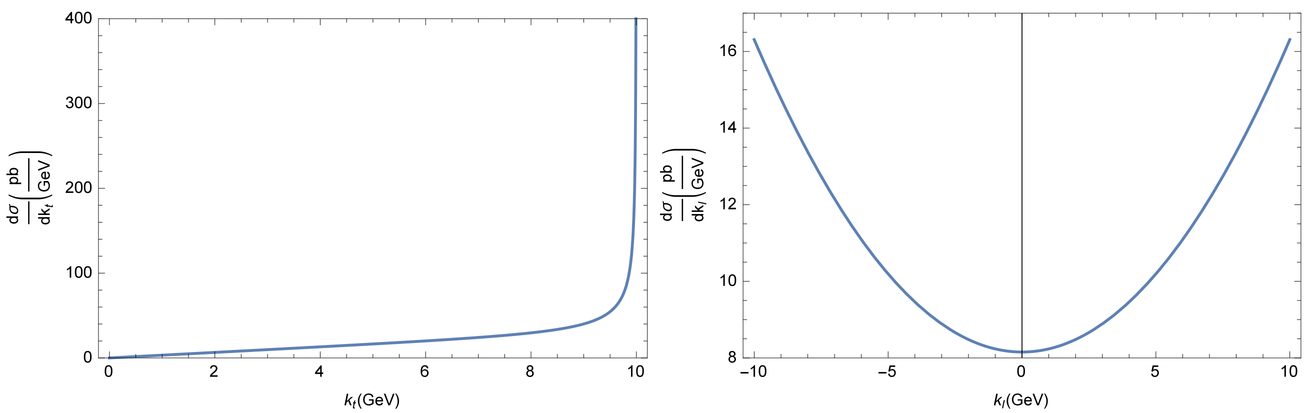


Figure 1.5: Transverse- and longitudinal-momentum distribution of muon

2

Scattering process $e^- + e^+ \to \mu^- + \mu^+$ in Standard Model

2.1 An overview of Standard Model

Standard Model (SM) is the most successful theory ever which is able to describe three of four fundamental interactions: electromagnetic, weak and strong interactions. Gravity interaction is not considered in this model. This model also helps us to classify all currently known elementary particles which are shown in Fig.(2.1)[1]. You can see that there are twelve particles of matter (quarks and leptons), governed by three forces that are caused by the exchange of four Gauge boson particles (photon, Z, W boson and gluon). The other particle in SM is the famous Higgs boson, which is thought to give mass to the other massive particles. The paramount property of SM is using the local $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge symmetry. Experiments with high energy particles at accelerators have completed our knowledge about SM with amazing precision.

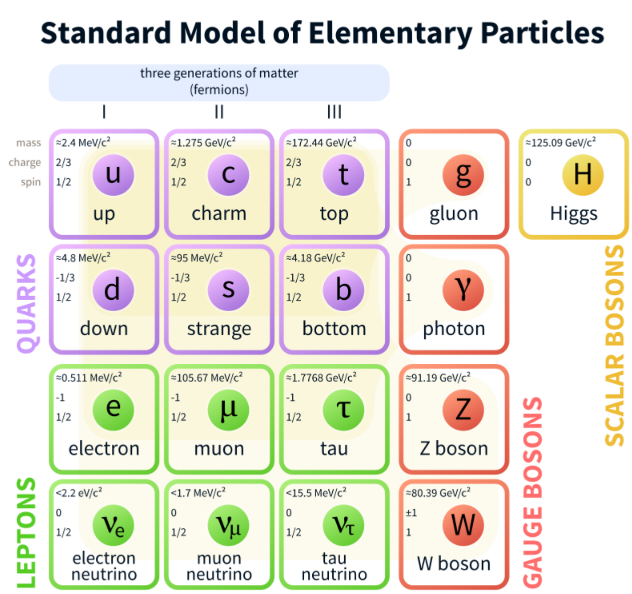


Figure 2.1: Elementary particles in SM

[1]The source for the picture is from https://en.wikipedia.org/wiki/Standard_Model

The Lagrangian of SM is invariant under that transformation and have the form:

$$\mathcal{L} = \mathcal{L}_{fermion} + \mathcal{L}_{gauge} + \mathcal{L}_{gf} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} + \mathcal{L}_{ghost}. \tag{2.1}$$

In this chapter we only work with three first terms of SM Lagrangian, the other terms we will consider in the later chapter. Another thing to note that we only focus on the transformation of $SU(2)_L \otimes U(1)_Y$ since we do not have strong interaction in $ee \to \mu\mu$ process.
One of an essential point in SM that you must keep in mind that left- and right-handed fields have different transformations because left-handed fields and right-handed fields are arranged in doublet and singlet, respectively. Therefore, the right-handed fields are not affected by the transformation of $SU(2)_L$ group. Left- and right-handed under the transformations of $SU(2)_L \otimes U(1)_Y$ are:

$$\begin{cases} \psi_L \to \psi'_L = \exp\left\{ \mathrm{ig}\frac{\tau^{\mathrm{i}}}{2}\alpha_{\mathrm{i}}(\mathrm{x}) + \mathrm{ig}'\frac{\mathrm{Y}}{2}\beta(\mathrm{x}) \right\}\psi_{\mathrm{L}} \\ \psi_R \to \psi'_R = \exp\left\{ \mathrm{ig}'\frac{\mathrm{Y}}{2}\beta(\mathrm{x}) \right\}\psi_{\mathrm{R}} \end{cases} \tag{2.2}$$

Where τ^i are the Pauli matrices and they are also the generators of $SU(2)_L$. Y is the hypercharge which has different values for each particle according to the Gell-Mann Nishijima formula:

$$Q = I^3 + \frac{Y}{2}, \tag{2.3}$$

with Q is the charge of the particle. I^3 is the eigenvalue of $\tau^3/2$ and it is called the isospin.

2.1.1 Fermion term of SM

The first term of SM Lagrangian in Eq. (2.1) is:

$$\mathcal{L}_{fermion} = \sum_f i\bar{\psi}_L \not{D}\psi_L + \sum_f i\bar{\psi}_R \not{D}\psi_R. \tag{2.4}$$

The general covariant derivative is:

$$D_\mu = \partial_\mu - ig'\frac{Y}{2}B_\mu - ig\frac{\tau^i}{2}W^i_\mu - ig_s G^a_\mu \frac{\lambda^a}{2} \tag{2.5}$$

Where λ^a stand for Gell-Mann matrices and G^a_μ, W^i_μ, B_μ are the corresponding gauge fields of $SU(3)_C$, $SU(2)_L$, and $U(1)_Y$, respectively. But these gauge fields are not physical fields (expect for G^a_μ but we do not talk about it more) and we call them as weak-eigenstate basic. The mass-eigenstate basic contains the physical fields which could be written as combinations of gauge fields:

$$\begin{cases} W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu) \\ Z_\mu = c_W W^3_\mu - s_W B_\mu \\ A_\mu = s_W W^3_\mu + c_W B_\mu \end{cases} \tag{2.6}$$

Here, we have denoted $c_W = \cos\theta_{\mathrm{W}}$ and $s_W = \sin\theta_{\mathrm{W}}$, where θ_W is the weak mixing angle defined as $\theta_W = \arctan(\mathrm{g}'/\mathrm{g})$.

2.1.2 Gauge fields in SM

The SM gauge field Lagrangian is of the form:

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu}. \tag{2.7}$$

$W^a_{\mu\nu}$ and $F_{\mu\nu}$ are called the field tensors corresponding to the gauge fields of $SU(2)_L$ and $U(1)_1$ respectively.

$$\begin{cases} W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon_{abc}W^b_\mu W^c_\nu, \\ F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \end{cases} \tag{2.8}$$

2.2 Vertex factor of Z-Boson

Note that the interaction term of Z boson and fermions is hidden in the kinetic terms in Eq.(2.4). Since Z_μ is a linear combination of W_μ and B_μ, we will work with the part that contains the gauge fields W^3_μ and B_μ only. Let's start with Lagrangian in Eq.(2.4).

$$\begin{aligned} \mathcal{L}_{fermion} &\supset i\bar{\psi}_L\gamma^\mu D_\mu\psi_L + i\bar{\psi}_R\gamma^\mu D_\mu\psi_R & (2.9)\\ &\supset i\bar{\psi}_L\gamma^\mu\left(-ig\frac{\tau^3}{2}W^3_\mu - ig'\frac{Y}{2}B_\mu\right)\psi_L + i\bar{\psi}_R\gamma^\mu\left(-ig'\frac{Y}{2}B_\mu\right)\psi_R. & (2.10) \end{aligned}$$

Now we want to find the terms contain the Z boson field, thus we have to change the weak- to mass-eigenstate basic. From Eq. (2.6), we can deduce the inverse transformations:

$$\begin{cases} W^3_\mu = c_W Z_\mu + s_W A_\mu, \\ B_\mu = -s_W Z_\mu + c_W A_\mu. \end{cases} \tag{2.11}$$

Inserting (2.11) into (2.10) and only keep the terms with Z boson field, we have:

$$\mathcal{L}_{fermion} \supset i\bar{\psi}_L\gamma^\mu\left[-\frac{ig}{c_W}\left(c_W^2\frac{\tau^3}{2} - s_W^2\frac{Y}{2}\right)Z_\mu\right]\psi_L + i\bar{\psi}_R\gamma^\mu\left(\frac{ig}{c_W}s_W^2\frac{Y}{2}Z_\mu\right)\psi_R. \tag{2.12}$$

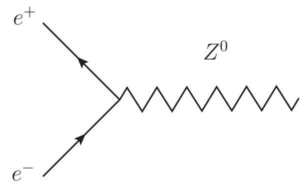
Since we do not have neutrino in the process we care about, hence we only keep e_L in doublet $\psi_L = \begin{pmatrix}\nu_e & e\end{pmatrix}_L$ and, of course, e_R in singlet ψ_R. Using Gell-Mann Nishijima formula in Eq.(2.3), we are able to obtain hypercharge $Y = -1$ and $Y = -2$ for left- and right-handed electrons respectively. The interaction term of e^- and e^+ mediated by Z boson could be written as:

$$\mathcal{L}^{\bar{e}eZ}_{int} = g_L Z_\mu(\bar{e}_L\gamma^\mu e_L) + g_R Z_\mu(\bar{e}_R\gamma^\mu e_R). \tag{2.13}$$

Where $g_L = \frac{g}{c_W}\left(-\frac{1}{2} + s_W^2\right)$ and $g_R = \frac{g}{c_W}s_W^2$. For any Dirac spinor ψ, we can always write it in term of: $\psi = P_L\psi + P_R\psi = \frac{1-\gamma^5}{2}\psi + \frac{1+\gamma^5}{2}\psi$. Based on that identity, Eq.(2.13) can rewrite as

$$\mathcal{L}^{\bar{e}eZ}_{int} = \bar{e}\left(g_L\gamma^\mu P_L + g_R\gamma^\mu P_R\right)eZ_\mu = \frac{1}{2}Z_\mu\bar{e}\left(g_V\gamma^\mu - g_A\gamma^\mu\gamma^5\right)e, \tag{2.14}$$

where $g_V = g_L + g_R$ and $g_A = g_L - g_R$. Thus, the vertex factor of eeZ is as follows:



$$= \frac{i}{2}\left(g_V\gamma^\mu - g_A\gamma^\mu\gamma^5.\right) \tag{2.15}$$

2.3 Propagator of Z-boson

To find the propagator of Z-boson, we have to insert (2.11) into the gauge field Lagrangian in (2.7) and pull out the kinetic term for Z_μ. Notice that we just interested in the Lagrangian of free Z_μ only. So we will do step by step to find what terms contain Z_μ only, we start with

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu(-s_W Z_\nu + c_W A_\nu) - \partial_\nu(-s_W Z_\mu + c_W A_\mu) \tag{2.16}$$

$$\Rightarrow -\frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} = -\frac{1}{4}s_W^2(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu). \tag{2.17}$$

We have used the notation $F^Z_{\mu\nu}$ for the terms of $F_{\mu\nu}$ that contain Z_μ only. Similarly, we can find the terms with only Z_μ of $W^a_{\mu\nu}$:

$$-\frac{1}{4}W^{Za}_{\mu\nu}W^{Za\mu\nu} = -\frac{1}{4}c_W^2(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu). \tag{2.18}$$

Thus, we can obtain the Lagrangian of free Z_μ only:

$$\mathcal{L}_Z^{kin} = -\frac{1}{4}F^Z_{\mu\nu}F^{Z\mu\nu} - \frac{1}{4}W^{Za}_{\mu\nu}W^{Za\mu\nu} \tag{2.19}$$

$$= -\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu). \tag{2.20}$$

However, this is not yet the complete Lagrangian of free Z_μ, we have to add two more terms, which are the mass term of Z_μ and the gauge fixing term. In the first chapter, we do not have the mass terms since photon is massless,

$$\Rightarrow \mathcal{L}_Z = -\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu) + \frac{1}{2}m_Z^2 Z_\mu Z^\mu - \frac{1}{2\xi_Z}(\partial^\mu Z_\mu)^2. \tag{2.21}$$

Based on the Lagrangian above, using the same manner which has been introduced in chapter 1, We are able to derive the form of the propagator of Z boson $D_{\mu\nu}(q)$ as:

$$D_{\mu\nu}(q) = \frac{1}{q^2 - m_Z^2}\left[-g_{\mu\nu} + (1-\xi_Z)\frac{q_\mu q_\nu}{q^2 - \xi_Z m_Z^2}\right]. \tag{2.22}$$

Let's have a look at the above propagator, the denominator becomes zero when $q^2 = m_Z^2$ which will cause the divergence in the total cross-section later. This is because of the approximation that we make when we consider only the free Z_μ Lagrangian. So if we take many higher terms into account, we will obtain the corresponding Breit-Wigner propagator in Feynman gauge:

$$D_{\rho\nu}(q) = \frac{-g_{\rho\nu}}{q^2 - m_Z^2 + i\Gamma_Z m_Z}. \tag{2.23}$$

Where Γ_Z is the decay width of Z Boson and its experimental value is approximate 2.452 GeV. The manner to find it by theory will be introduced in later section.

2.4 Feynman amplitude

Basically, the Feynman rules in SM is similar to that of QED. Except for the internal line since we have one diagram in addition. The process currently has two mediators which are illustrated by two Feynman diagrams in Fig.(2.2) The Feynman amplitude can be write down as:

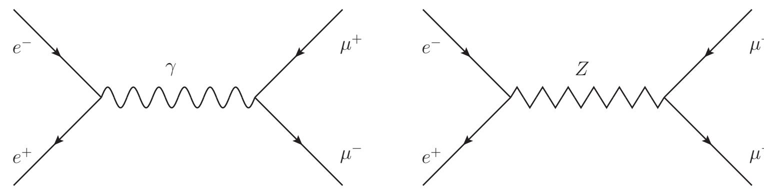


Figure 2.2: Feynman diagrams of process $e^+ + e^- \to \mu^+\mu^-$ in SM

$$\mathcal{M} = \mathcal{M}_\gamma + \mathcal{M}_Z, \tag{2.24}$$

Where $\mathcal{M}_\gamma$ and $\mathcal{M}_Z$ are the Feynman amplitude for the diagram which is mediated by photon and Z boson respectively. They have the form:

$$\mathcal{M}_\gamma = \left[\bar{v}_{s'}(p')(-ie\gamma^\mu)u_s(p)\right]\left(-\frac{g_{\mu\nu}}{q^2}\right)\left[\bar{u}_r(k)(-ie\gamma^\nu)v_{r'}(k')\right], \tag{2.25}$$

$$\begin{aligned}\mathcal{M}_Z = &\left[\bar{v}_{s'}(p')\frac{i}{2}(g_V\gamma^\mu - g_A\gamma^\mu\gamma^5)u_s(p)\right]\left(-\frac{g_{\mu\nu}}{q^2 - m_Z^2 + i\Gamma_Z m_Z}\right)\\ &\times\left[\bar{u}_r(k)\frac{i}{2}(g_V\gamma^\nu - g_A\gamma^\nu\gamma^5)v_{r'}(k')\right].\end{aligned} \tag{2.26}$$

After long calculations, we can derive the squared-amplitude as follows:

$$|\mathcal{M}_0|^2 = \frac{1}{4}\sum_{spin}\mathcal{M}\mathcal{M}^\dagger = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2, \tag{2.27}$$

where the explicit forms are:

$$\begin{aligned}|\mathcal{M}_1|^2 = &\frac{1}{(s-m_Z^2)^2 + \Gamma_Z^2 m_Z^2}\left[\frac{1}{2}g_V^4(p.k')(p'.k) + \frac{1}{2}g_V^4(p.k)(p'.k')\right.\\ &+3g_A^2g_V^2(p.k')(p'.k) - g_A^2g_V^2(p.k)(p'.k') + \frac{1}{2}g_A^4(p.k')(p'.k)\\ &\left.+\frac{1}{2}g_A^4(p.k)(p'.k') + \frac{1}{2}m_\mu^2g_V^4(p.p') - \frac{1}{2}m_\mu^2g_A^4(p.p')\right],\end{aligned} \tag{2.28}$$

$$\begin{aligned}|\mathcal{M}_2|^2 = &\frac{e^2}{s(s-m_Z^2+i\Gamma_Z m_Z)}\left[2g_V^2(p.k')(p'.k) + 2g_V^2(p.k)(p'.k')\right.\\ &\left.+2g_A^2(p.k')(p'.k) - 2g_A^2(p.k)(p'.k') + 2m_\mu^2g_V^2(p.p')\right]\\ &+\frac{e^2}{s(s-m_Z^2-i\Gamma_Z m_Z)}\left[2g_V^2(p.k')(p'.k) + 2g_V^2(p.k)(p'.k')\right.\\ &\left.+2g_A^2(p.k')(p'.k) - 2g_A^2(p.k)(p'.k') + 2m_\mu^2g_V^2(p.p')\right],\end{aligned} \tag{2.29}$$

$$|\mathcal{M}_3|^2 = \frac{e^4}{q^4}\left[8(p.k')(p'.k) + 8(p.k)(p'.k') + 8m_\mu^2(p.p')\right]. \tag{2.30}$$

Similar to Chap.1, now we will work in CM frame and re-use the results in Eqs.(1.10). From that, we can obtain $|\mathcal{M}_1|^2$, $|\mathcal{M}_2|^2$ and $|\mathcal{M}_3|^2$:

$$|\mathcal{M}_1|^2 = \frac{E^2}{(s-m_Z^2)^2+\Gamma_Z^2 m_Z^2}\left[(g_V^2+g_A^2)^2\left(E^2+|\mathbf{k}|^2\cos^2\theta\right)+(g_V^4-g_A^4)m_\mu^2+8g_A^2g_V^2E|\mathbf{k}|\cos\theta\right], \tag{2.31}$$

$$|\mathcal{M}_2|^2 = \frac{8e^2E^2(s-m_Z^2)}{s\left[(s-m_Z^2)^2+\Gamma_Z^2m_Z^2\right]}\left[g_V^2\left(E^2+|\mathbf{k}|^2\cos^2\theta+\mathrm{m}_\mu^2\right)+2g_A^2E|\mathbf{k}|\cos\theta\right], \tag{2.32}$$

$$|\mathcal{M}_3|^2 = \frac{16e^4E^2}{s^2}\left(E^2+m_\mu^2+|\mathbf{k}|^2\cos^2\theta\right). \tag{2.33}$$

Let's now denote $\mu = m_\mu^2/E^2$ and then introduce the factor χ_0 to make the squared-amplitude shorter:

$$\chi_0(s) = \frac{s}{4e^2(s-m_Z^2+i\Gamma_Z m_Z)} \Rightarrow \begin{cases} \dfrac{1}{(s-m_Z^2)^2+\Gamma_Z^2m_Z^2} = \dfrac{16e^4}{s^2}|\chi_0(s)|^2, \\ \dfrac{s-m_Z^2}{(s-m_Z^2)^2+\Gamma_Z^2m_Z^2} = \dfrac{4e^2}{s}\mathrm{Re}\chi_0(\mathrm{s}). \end{cases} \tag{2.34}$$

For convenient, we will denote $G_1(s)$,$G_1(s)$ and $G_1(s)$ as follow:

$$G_1(s) = (g_v^2+g_A^2)^2|\chi_0(s)|^2 + 2g_V^2\mathrm{Re}\chi_0(\mathrm{s}) + 1, \tag{2.35}$$

$$G_2(s) = \left[(g_v^2+g_A^2)^2+(g_V^4-g_A^4)\mu\right]|\chi_0(s)|^2 + 2g_V^2\mathrm{Re}\chi_0(\mathrm{s})(\mu+1)+\mu+1, \tag{2.36}$$

$$G_3(s) = 2g_A^2g_V^2|\chi_0(s)|^2 + g_A^2\mathrm{Re}\chi_0(\mathrm{s}). \tag{2.37}$$

Then, we will be able to write down a more compact form of $|\mathcal{M}_0|^2$:

$$|\mathcal{M}_0|^2 = \frac{16e^4E^2}{s^2}\left[G_1(s)|\mathbf{k}|^2\cos^2\theta + \mathrm{G_2(s)E^2} + \mathrm{4G_3(s)E}|\mathbf{k}|\cos\theta\right]. \tag{2.38}$$

2.5 Some distributions

2.5.1 Total cross-section

First, we have to derive the total cross-section. Using Eq.(1.12), we can find

$$\frac{d\sigma}{d\Omega} = \frac{e^4|\mathbf{k}|}{16\pi^2Es^2}\left[G_1(s)|\mathbf{k}|^2\cos^2\theta + \mathrm{G_2(s)E^2} + \mathrm{4G_3(s)E}|\mathbf{k}|\cos\theta\right]. \tag{2.39}$$

Integrating over all values of $\cos\theta$ and ϕ, we can obtain the total cross-section as

$$\sigma_T = \frac{e^4|\mathbf{k}|}{4\pi Es^2}\left[\frac{1}{3}G_1(s)|\mathbf{k}|^2 + G_2(s)E^2\right]. \tag{2.40}$$

In Fig.(2.3), the total cross-section of SM and QED are almost the same in low energy range. However, because of the factor $\frac{1}{(s-m_Z^2)^2+\Gamma_Z^2m_Z^2}$ in $|\mathcal{M}_0|^2$, we see the peak at about 90GeV, this is also the sign of Z-boson.

About the angular distribution, afer some simple calculations, we obtain:

$$\frac{d\sigma}{d\theta} = \frac{e^4|\mathbf{k}|}{8\pi Es^2}\left[G_1(s)|\mathbf{k}|^2\cos^2\theta + \mathrm{G_2(s)E^2} + \mathrm{4G_3(s)E}|\mathbf{k}|\cos\theta\right]\sin\theta \tag{2.41}$$

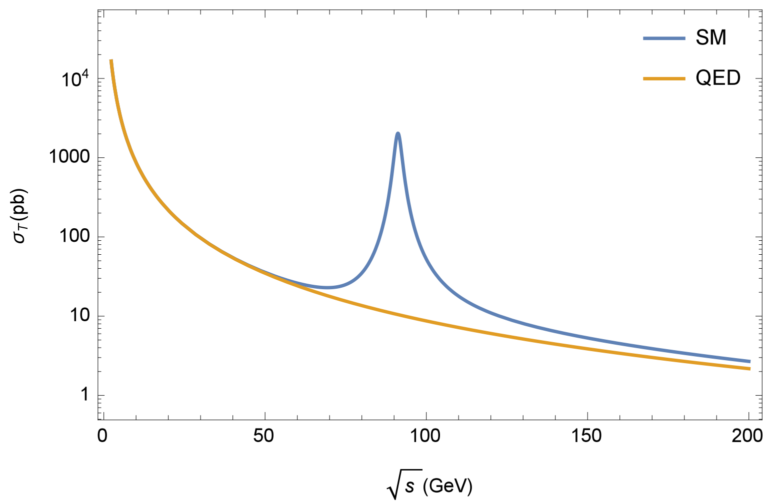


Figure 2.3: Total cross-section of muon

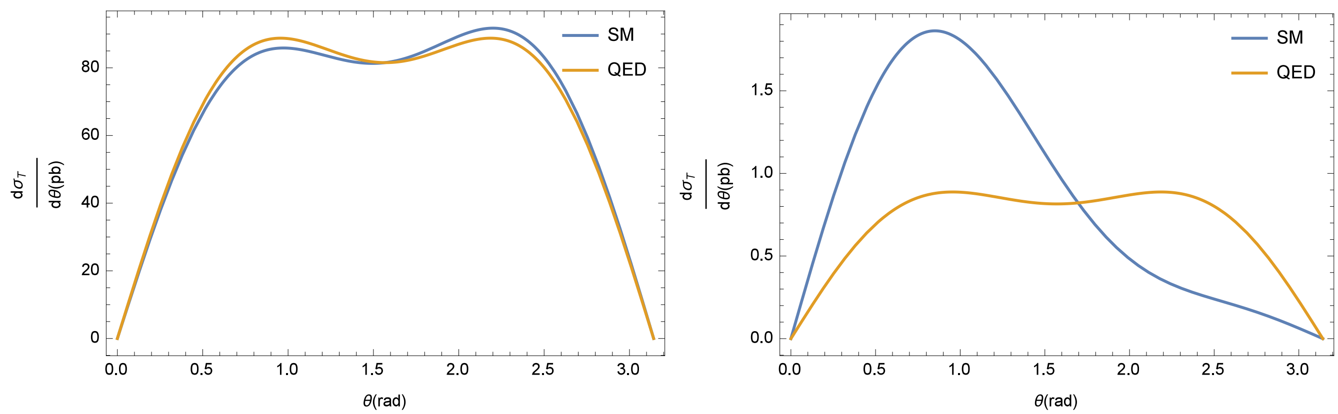


Figure 2.4: The angular distribution, $\sqrt{s} = 20$ GeV (left) and $\sqrt{s} = 200$ GeV (right)

2.5.2 Forward-backward asymmetry

In the angular distribution, which is not similar to QED, the forward- and backward-side are not symmetry. And we can see it is clear if we enhance the initial total energy. In order to see how asymmetry it is, we introduce the forward-backward asymmetry(A_{FB}) as follows:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{\sigma_{FB}}{\sigma_T}, \tag{2.42}$$

where:

$$\sigma_F = \int_0^{\frac{\pi}{2}} \frac{d\sigma}{d\theta} d\theta = \frac{e^4|\mathbf{k}|}{8\pi E s^2}\left[\frac{1}{3}G_1(s)|\mathbf{k}|^2 + G_2(s)E^2 + 2G_3(s)E|\mathbf{k}|\right], \tag{2.43}$$

$$\sigma_B = \int_{\frac{\pi}{2}}^{\pi} \frac{d\sigma}{d\theta} d\theta = \frac{e^4|\mathbf{k}|}{8\pi E s^2}\left[\frac{1}{3}G_1(s)|\mathbf{k}|^2 + G_2(s)E^2 - 2G_3(s)E|\mathbf{k}|\right], \tag{2.44}$$

$$\Rightarrow A_{FB} = \frac{\sigma_{FB}}{\sigma_T} = \frac{6G_3(s)\sqrt{1-\mu}}{G_1(s)(1-\mu) + 3G_2(s)}. \tag{2.45}$$

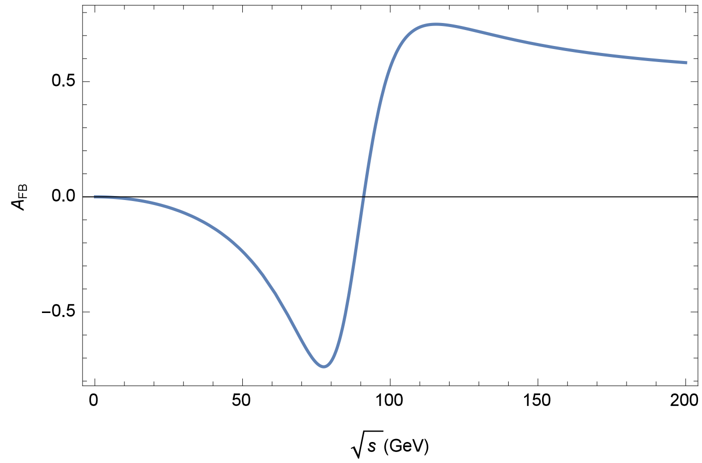


Figure 2.5: A_{FB} with respect to $\sqrt{s}$

Fig.(2.5) shows the value of the forward-backward asymmetry. In the case of low energy, we can see A_{FB} approximate zero which is the property of QED. Thus, we are able to consider QED is just a special case of SM.

2.5.3 Transverse and longitudinal momentum distributions

Using the same manner as we have mentioned in chapter 1, we can find the transverse and longitudinal momentum distributions are of the form:

$$\frac{d\sigma}{dk_t} = \sum_i \frac{d\sigma}{d\theta}\bigg|_{\theta=\theta_i} \left| \frac{d\theta}{dk_t}\bigg|_{\theta=\theta_i} \right| = \frac{e^4}{4\pi E s^2} \frac{k_t}{\sqrt{|\mathbf{k}|^2 - k_t^2}} \left[G_1(s)(|\mathbf{k}|^2 - k_t^2) + G_2(s)E^2 \right], \tag{2.46}$$

$$\frac{d\sigma}{dk_l} = \sum_i \frac{d\sigma}{d\theta}\bigg|_{\theta=\theta_i} \left| \frac{d\theta}{dk_l}\bigg|_{\theta=\theta_i} \right| = \frac{e^4}{8\pi E s^2} \left[G_1(s)k_l^2 + G_2(s)E^2 + 4G_3(s)Ek_l \right]. \tag{2.47}$$

Figures (2.6) and (2.7) indicate the transverse and longitudinal momentum distributions with $\sqrt{s} = 200$GeV.

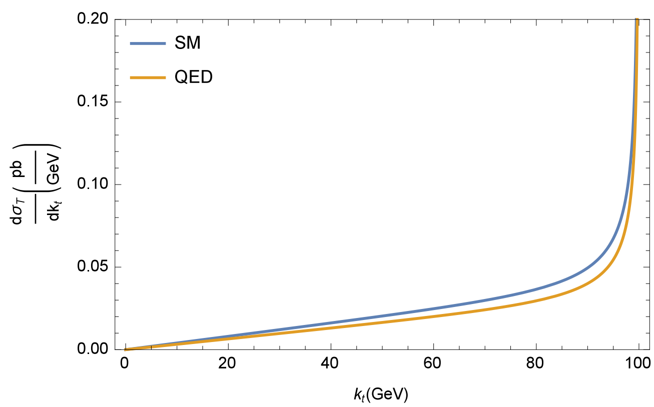


Figure 2.6: Transverse momentum distribution

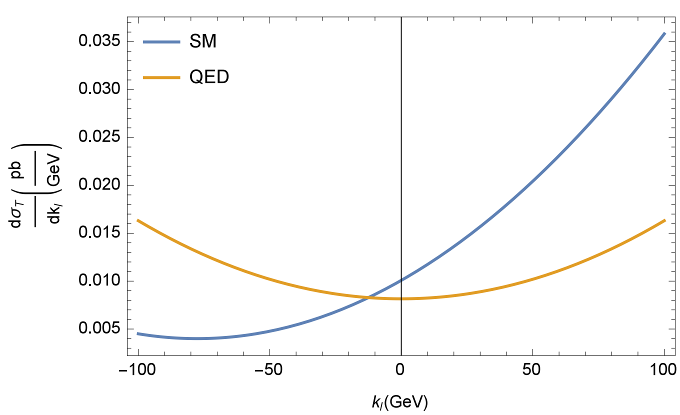


Figure 2.7: Longitudinal momentum distribution

3

Scattering process $e^- + e^+ \to \mu^- + \mu^+$ in SMEFT

3.1 An overview of Standard Model Effective Field Theory(SMEFT)

Although Standard Model has much success to describe strong and electroweak interactions, there are still many issues that have not been solved. Such as right-handed neutrinos, the mass of neutrinos. Thus many physicists believe there still have a new physics beyond the SM to explain such phenomena.

There are many ways to establish Models beyond SM, and all of them have the critical rule is changing the Lagrangian of SM with addition terms. SMEFT is not an exception to that rule, which has considered SM as a part of an effective theory and using higher dimensional operators in addition. Thus the SM is considered as the leading order terms of an exhaustive theory

In SMEFT, we introduce one new parameter (Λ), which is the typical energy scale of SMEFT. Notice that in SM, the typical energy scale is the Electroweak Scale $v = 246$GeV. A field theory valid above Λ has to obey the requirements which have been listed in [3]:

- Its gauge group should contain $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ of the SM.
- All the SM degrees of freedom should be incorporated either as fundamental or composite fields.
- At low-energies, it should reduce to the SM, provided no undiscovered but weakly coupled light particles exist, like axions or sterile neutrinos.

The Lagrangian in SMEFT is given by

$$\mathcal{L}_{SMEFT} = \mathcal{L}_{SM} + \frac{1}{\Lambda}\sum_k C_k^{(5)} Q_k^{(5)} + \frac{1}{\Lambda^2}\sum_k C_k^{(6)} Q_k^{(6)} + \mathcal{O}\left(\frac{1}{\Lambda^3}\right), \tag{3.1}$$

where $\mathcal{L}_{SM}$ is the part of the SM Lagrangian which is renormalizable. In the rest terms, $Q_k^{(n)}$, $C_k^{(n)}$ stand for dimension-n operators and dimensionless Wilson coefficients, respectively. The appearances of the energy scale dimension-one Λ in many terms help us to reduce the dimension of those terms become dimension-4 which is the inevitable feature of Lagrangian Density.

In this thesis, we shall work on dimension-four and -six operators only and ignore all higher dimension operators. The dimension-six operators were firstly introduced by Buchmuller and

Wyler [4] in 1985. Such operators must be invariant under Lorentz and gauge transformations but in Ref. [4], they do not totally independent of each other and some of them violate the baryon number conservation. Therefore in 2010, the updated list of dimension-six operators was newly revised by the Warsaw University group [3]. They have a comment that I think it interesting: "It is really amazing that no author of almost 600 papers that quoted Ref. [4] over 24 years has ever decided to rederive the operator basis from the outset to check its correctness."

3.1.1 Notation and conventions

We will mainly based on the notation and conventions of Ref. [3], which have introduced some main things:
The form of SM Lagrangian before Spontaneous Symmetry Breaking is

$$\begin{aligned}\mathcal{L}_{SM} = &-\frac{1}{4}G^A_{\mu\nu}G^{A\mu\nu} - \frac{1}{4}W^I_{\mu\nu}W^{I\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) + m^2\varphi^\dagger\varphi - \frac{1}{2}\lambda(\varphi^\dagger\varphi)^2 \\ &+ i\left(\bar{l}\not{D}l + \bar{e}\not{D}e + \bar{q}\not{D}q + \bar{u}\not{D}u + \bar{d}\not{D}d\right) - \left(\bar{l}\Gamma_e e\varphi + \bar{q}\Gamma_u u\widetilde{\varphi} + \bar{q}\Gamma_d d\varphi + \text{h.c.}\right), \end{aligned} \tag{3.2}$$

where $\Gamma_{e,u,d}$ are the Yukawa couplings. The matter field and all their corresponding Hypercharges are listed in Tab. 3.1

Fields	Notations	Hypercharge Y
Left-handed lepton doublets	l^j_p	-1/2
Right-handed charged leptons	e_p	-1
Left-handed quark doublets	$q^{\alpha j}_p$	1/6
Right-handed quarks	u^α_p	2/3
Right-handed quarks	d^α_p	-1/3
Higgs boson doublet	φ^j	1/2

Table 3.1: The SM matter content

Note that the indices $j = 1, 2$, $\alpha = 1, 2, 3$, $p = 1, 2, 3$ stand for isospin, color and generation indices, respectively. In Eq.(3.2), the notation $\widetilde{\varphi}^j$ stand for $\epsilon_{jk}(\varphi^k)^*$, where ϵ_{jk} is the Levi-Civita tensor with $\epsilon_{12} = 1$. For the case of covariant derivatives, there is a deviation from Chap.(2) which is the sign convention:

$$D_\mu = \partial_\mu + ig' B_\mu Y + igW^I_\mu S^I + ig_s G^A_\mu T^A. \tag{3.3}$$

The generators of $SU(3)$ and $SU(2)$ are denoted by $T^S = \frac{1}{2}\lambda^A$ and $S^I = \frac{1}{2}\tau^I$, respectively. For later convenience, we will use the notation:

$$\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi \equiv i\varphi^\dagger\left(D_\mu - \overleftarrow{D}^\dagger_\mu\right)\varphi, \qquad \varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi \equiv i\varphi^\dagger\left(\tau^I D_\mu - \overleftarrow{D}^\dagger_\mu\tau^I\right)\varphi. \tag{3.4}$$

The gauge field strength tensors and their covariant derivatives are of the forms

$$\begin{aligned} G^A_{\mu\nu} &= \partial_\mu G^A_\nu - \partial_\nu G^A_\mu - g_s f^{ABC} G^B_\mu G^C_\nu, & (D_\rho G_{\mu\nu})^A &= \partial_\rho G^A_{\mu\nu} - g_s f^{ABC} G^B_\rho G^C_{\mu\nu}, \\ W^I_{\mu\nu} &= \partial_\mu W^I_\nu - \partial_\nu W^I_\mu - g\epsilon^{IJK} W^J_\mu W^K_\nu, & (D_\rho W_{\mu\nu})^I &= \partial_\rho W^I_{\mu\nu} - g\epsilon^{IJK} W^J_\rho W^K_{\mu\nu}, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, & D_\rho B_{\mu\nu} &= \partial_\rho B_{\mu\nu}. \end{aligned} \tag{3.5}$$

All dimension-six operators, which were listed in [3], are introduced in Table (3.2) and Table (3.3). Those operators must independent with each other and have to obey the Standard Model gauge symmetries. Before go to next section, one convention will be used that we will re-denote the Wilson coefficients as

$$\frac{C_{\text{Wilson}}}{\Lambda^2} \equiv C_{Wilson}, \tag{3.6}$$

it leads to the dimension-"minus two" of notation C_{Wilson} and the approximation $\frac{1}{\Lambda^4} = 0$ becomes $C^2_{Wilson} = 0$.

X^3		φ^6 and $\varphi^4 D^2$		$\psi^2\varphi^3$	
Q_G	$f^{ABC}G^{A\nu}_\mu G^{B\rho}_\nu G^{C\mu}_\rho$	Q_φ	$(\varphi^\dagger\varphi)^3$	$Q_{e\varphi}$	$(\varphi^\dagger\varphi)(\bar{l}_p e_r \varphi)$
$Q_{\tilde{G}}$	$f^{ABC}\tilde{G}^{A\nu}_\mu G^{B\rho}_\nu G^{C\mu}_\rho$	$Q_{\varphi\Box}$	$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi)$	$Q_{u\varphi}$	$(\varphi^\dagger\varphi)(\bar{q}_p u_r \tilde{\varphi})$
Q_W	$\epsilon^{IJK}W^{I\nu}_\mu W^{J\rho}_\nu W^{K\mu}_\rho$	$Q_{\varphi D}$	$(\varphi^\dagger D^\mu\varphi)^*(\varphi^\dagger D_\mu\varphi)$	$Q_{d\varphi}$	$(\varphi^\dagger\varphi)(\bar{q}_p d_r \varphi)$
$Q_{\tilde{W}}$	$\epsilon^{IJK}\tilde{W}^{I\nu}_\mu W^{J\rho}_\nu W^{K\mu}_\rho$				
$X^2\varphi^2$		$\psi^2 X\varphi$		$\psi^2\varphi^2 D$	
$Q_{\varphi G}$	$\varphi^\dagger\varphi G^A_{\mu\nu}G^{A\mu\nu}$	Q_{eW}	$(\bar{l}_p\sigma^{\mu\nu}e_r)\tau^I\varphi W^I_{\mu\nu}$	$Q^{(1)}_{\varphi l}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_p\gamma^\mu l_r)$
$Q_{\varphi\tilde{G}}$	$\varphi^\dagger\varphi\tilde{G}^A_{\mu\nu}G^{A\mu\nu}$	Q_{eB}	$(\bar{l}_p\sigma^{\mu\nu}e_r)\varphi B_{\mu\nu}$	$Q^{(3)}_{\varphi l}$	$(\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{l}_p\tau^I\gamma^\mu l_r)$
$Q_{\varphi W}$	$\varphi^\dagger\varphi W^I_{\mu\nu}W^{I\mu\nu}$	Q_{uG}	$(\bar{q}_p\sigma^{\mu\nu}\tau^A u_r)\tilde{\varphi}G^A_{\mu\nu}$	$Q_{\varphi e}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{e}_p\gamma^\mu e_r)$
$Q_{\varphi\tilde{W}}$	$\varphi^\dagger\varphi\tilde{W}^I_{\mu\nu}W^{I\mu\nu}$	Q_{uW}	$(\bar{q}_p\sigma^{\mu\nu}u_r)\tau^I\tilde{\varphi}W^I_{\mu\nu}$	$Q^{(1)}_{\varphi q}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{q}_p\gamma^\mu q_r)$
$Q_{\varphi B}$	$\varphi^\dagger\varphi B_{\mu\nu}B^{\mu\nu}$	Q_{uB}	$(\bar{q}_p\sigma^{\mu\nu}u_r)\tilde{\varphi}B_{\mu\nu}$	$Q^{(3)}_{\varphi q}$	$(\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{q}_p\tau^I\gamma^\mu q_r)$
$Q_{\varphi\tilde{B}}$	$\varphi^\dagger\varphi\tilde{B}_{\mu\nu}B^{\mu\nu}$	Q_{dG}	$(\bar{q}_p\sigma^{\mu\nu}\tau^A d_r)\varphi G^A_{\mu\nu}$	$Q_{\varphi u}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{u}_p\gamma^\mu u_r)$
$Q_{\varphi WB}$	$\varphi^\dagger\tau^I\varphi W^I_{\mu\nu}B^{\mu\nu}$	Q_{dW}	$(\bar{q}_p\sigma^{\mu\nu}d_r)\tau^I\varphi W^I_{\mu\nu}$	$Q_{\varphi d}$	$(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{d}_p\gamma^\mu d_r)$
$Q_{\varphi\tilde{W}B}$	$\varphi^\dagger\tau^I\varphi\tilde{W}^I_{\mu\nu}B^{\mu\nu}$	Q_{dB}	$(\bar{q}_p\sigma^{\mu\nu}d_r)\varphi B_{\mu\nu}$	$Q_{\varphi ud}$	$i(\tilde{\varphi}^\dagger\overleftrightarrow{D}_\mu\varphi)(\bar{u}_p\gamma^\mu d_r)$

Table 3.2: Dimension-six operators other than the four-fermion ones, (taken from [3]).

3.2 Deviations from the Standard Model

There are many relations of SMEFT different with that of SM. Therefore, before calculating the total cross-section for process $e^+ + e^- \to \mu^+ + \mu^-$, we have to derive many new and basic equations when we have dimension-six operators in addition.

3.2.1 Higgs mechanism

The Higgs mechanism in SMEFT with dimension-six, in addition, was introduced in [5], this section shall follow that paper with more unambiguous calculations. The Lagrangian terms which are relevant to the Higgs field read

$$\begin{aligned}\mathcal{L}_H = \mathcal{L}_H^{SM} + \mathcal{L}_H^{(6)} &= (D_\mu\varphi)^\dagger(D^\mu\varphi) + m^2(\varphi^\dagger\varphi) - \frac{\lambda}{2}(\varphi^\dagger\varphi)^2 \\ &+ C^\varphi(\varphi^\dagger\varphi)^3 + C^{\varphi\Box}(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi) + C^{\varphi D}(\varphi^\dagger D_\mu\varphi)^*(\varphi^\dagger D^\mu\varphi).\end{aligned} \tag{3.7}$$

$(\bar{L}L)(\bar{L}L)$		$(\bar{R}R)(\bar{R}R)$		$(\bar{L}L)(\bar{R}R)$	
Q_{ll}	$(\bar{l}_p\gamma_\mu l_r)(\bar{l}_s\gamma^\mu l_t)$	Q_{ee}	$(\bar{e}_p\gamma_\mu e_r)(\bar{e}_s\gamma^\mu e_t)$	Q_{le}	$(\bar{l}_p\gamma_\mu l_r)(\bar{e}_s\gamma^\mu e_t)$
$Q_{qq}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{q}_s\gamma^\mu q_t)$	Q_{uu}	$(\bar{u}_p\gamma_\mu u_r)(\bar{u}_s\gamma^\mu u_t)$	Q_{lu}	$(\bar{l}_p\gamma_\mu l_r)(\bar{u}_s\gamma^\mu u_t)$
$Q_{qq}^{(3)}$	$(\bar{q}_p\gamma_\mu \tau^I q_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$	Q_{dd}	$(\bar{d}_p\gamma_\mu d_r)(\bar{d}_s\gamma^\mu d_t)$	Q_{ld}	$(\bar{l}_p\gamma_\mu l_r)(\bar{d}_s\gamma^\mu d_t)$
$Q_{lq}^{(1)}$	$(\bar{l}_p\gamma_\mu l_r)(\bar{q}_s\gamma^\mu q_t)$	Q_{eu}	$(\bar{e}_p\gamma_\mu e_r)(\bar{u}_s\gamma^\mu u_t)$	Q_{qe}	$(\bar{q}_p\gamma_\mu q_r)(\bar{e}_s\gamma^\mu e_t)$
$Q_{lq}^{(3)}$	$(\bar{l}_p\gamma_\mu \tau^I l_r)(\bar{q}_s\gamma^\mu \tau^I q_t)$	Q_{ed}	$(\bar{e}_p\gamma_\mu e_r)(\bar{d}_s\gamma^\mu d_t)$	$Q_{qu}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{u}_s\gamma^\mu u_t)$
		$Q_{ud}^{(1)}$	$(\bar{u}_p\gamma_\mu u_r)(\bar{d}_s\gamma^\mu d_t)$	$Q_{qu}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{u}_s\gamma^\mu T^A u_t)$
		$Q_{ud}^{(8)}$	$(\bar{u}_p\gamma_\mu T^A u_r)(\bar{d}_s\gamma^\mu T^A d_t)$	$Q_{qd}^{(1)}$	$(\bar{q}_p\gamma_\mu q_r)(\bar{d}_s\gamma^\mu d_t)$
				$Q_{qd}^{(8)}$	$(\bar{q}_p\gamma_\mu T^A q_r)(\bar{d}_s\gamma^\mu T^A d_t)$
$(\bar{L}R)(\bar{R}L)$ and $\bar{R}L)(\bar{L}R)$		B-violating			
Q_{ledq}	$(\bar{l}_p^j e_r)(\bar{d}_s q_t^j)$	Q_{duq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(q_s^{\gamma j})^T C l_t^k\right]$		
$Q_{quqd}^{(1)}$	$(\bar{q}_p^j u_r)\varepsilon_{jk}(\bar{q}_s^k d_t)$	Q_{qqu}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jk}\left[(q_p^{\alpha j})^T C q_r^{\beta k}\right]\left[(u_s^\gamma)^T C e_t\right]$		
$Q_{quqd}^{(8)}$	$(\bar{q}_p^j T^A u_r)\varepsilon_{jk}(\bar{q}_s^k T^A d_t)$	Q_{qqq}	$\varepsilon^{\alpha\beta\gamma}\varepsilon_{jn}\varepsilon_{km}(q_p^{\alpha j})^T C q_r^{\beta k}\left]\left[(q_s^{\gamma m})^T C l_t^n\right]\right.$		
$Q_{lequ}^{(1)}$	$(\bar{l}_p^j e_r)\varepsilon_{jk}(\bar{q}_s^k u_t)$	Q_{duu}	$\varepsilon^{\alpha\beta\gamma}\left[(d_p^\alpha)^T C u_r^\beta\right]\left[(u_s^\gamma)^T C e_t\right]$		
Q_{lequ}	$\bar{l}_p^j \sigma_{\mu\nu} e_r)\varepsilon_{jk}(\bar{q}_s^k \sigma^{\mu\nu} u_t)$				

Table 3.3: Four-fermion operators, (taken from [3]).

The Higgs field after Spontaneous Symmetry Breaking (SSB) in Unitary gauge is as follows

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H \end{pmatrix}, \tag{3.8}$$

where v stand for vacuum expectation value and H is the Higgs field. From the Lagrangian above, we can see the Higgs potential is of the form

$$V(\varphi) = m^2(\varphi^\dagger\varphi) - \frac{\lambda}{2}(\varphi^\dagger\varphi)^2 + C^\varphi(\varphi^\dagger\varphi)^3 \tag{3.9}$$

Finding the solutions for the differential equation $\dfrac{\partial V(\varphi)}{\partial\varphi} = 0$, we obtain

$$(\varphi^\dagger\varphi) = \frac{\lambda + \sqrt{\lambda^2 - 12C^\varphi m^2}}{6C^\varphi} \qquad \text{or} \qquad (\varphi^\dagger\varphi) = \frac{\lambda - \sqrt{\lambda^2 - 12C^\varphi m^2}}{6C^\varphi}. \tag{3.10}$$

We only use the second solution since the first one will lead to a divergence when $C^\varphi \to 0$. We will use the approximation $(1+x)^n = 1 + nx + \mathcal{O}(x^2)$ for the small value x. This approximation is useful and will be used many times later since our coefficients $C^{(6)}$ are considered as small values. Thanks to that approximation, we are able to get the vacuum expectation value

$$v = \sqrt{2(\varphi^\dagger\varphi)} = \sqrt{\frac{2m^2}{\lambda}} + \frac{3}{\sqrt{2}}\frac{C^\varphi m^3}{\lambda^{5/2}}. \tag{3.11}$$

The next thing we wish to figure out is the physical fields of Higgs boson and the fields of Goldstone boson. Thus, we must obtain the Lagrangian containing bilinear terms of those scalar fields.

Expanding the Higgs doublet φ around the vacuum, we have:

$$\varphi = \begin{pmatrix} \Phi^+ \\ \frac{1}{\sqrt{2}}(v + H + i\Phi^0) \end{pmatrix}. \tag{3.12}$$

The field $\Phi^\pm$ and Φ^0 are still the Goldstone boson fields in SM, we will derive the new versions of those fields later in this section. Back on our work, we now try to pull out all scalar bilinear terms from Eq.(3.7) for each operator. For the later calculations, we shall expand one operator each time, since the Lagrangian from now on is too long to expand all terms simultaneously. Moreover, all the total derivative in Lagrangian shall be neglected since they do not affect our results.

$$\begin{aligned}(D_\mu\varphi)^\dagger(D^\mu\varphi) \supset (\partial_\mu\varphi)^\dagger(\partial^\mu\varphi) &= \partial_\mu \begin{pmatrix} \Phi^- & \frac{1}{\sqrt{2}}(H - i\Phi^0) \end{pmatrix} \partial^\mu \begin{pmatrix} \Phi^+ \\ \frac{1}{\sqrt{2}}(H + i\Phi^0) \end{pmatrix} \\ &= (\partial_\mu\Phi^-)(\partial^\mu\Phi^+) + \frac{1}{2}(\partial_\mu H\partial^\mu H + \partial_\mu\Phi^0\partial^\mu\Phi^0),\end{aligned} \tag{3.13}$$

$$\begin{aligned}(\varphi^+\varphi) &= \begin{pmatrix} \Phi^- & \frac{1}{\sqrt{2}}(v + H - i\Phi^0) \end{pmatrix} \begin{pmatrix} \Phi^+ \\ \frac{1}{\sqrt{2}}(v + H + i\Phi^0) \end{pmatrix} \\ &= \Phi^-\Phi^+ + \frac{1}{2}(v^2 + H^2 + 2vH + {\Phi^0}^2) \supset \frac{1}{2}H^2,\end{aligned}$$

$$(\varphi^+\varphi)^2 \supset \frac{1}{4}(v^2 + H^2 + 2vH)^2 \supset \frac{3}{2}v^2H^2, \tag{3.14}$$

$$(\varphi^+\varphi)^3 \supset \frac{1}{8}(v^2 + H^2 + 2vH)^3 \supset \frac{15}{8}v^4H^2, \tag{3.15}$$

$$\begin{aligned}(\varphi^+\varphi)\partial_\mu\partial^\mu(\varphi^+\varphi) &= \partial_\mu\left[(\varphi^+\varphi)\partial^\mu(\varphi^+\varphi)\right] - \partial_\mu(\varphi^+\varphi)\partial^\mu(\varphi^+\varphi) \supset -\partial_\mu(\varphi^+\varphi)\partial^\mu(\varphi^+\varphi) \\ &\supset -v^2(\partial_\mu H\partial^\mu H),\end{aligned} \tag{3.16}$$

$$\begin{aligned}&(\varphi^+D_\mu\varphi)^*(\varphi^+D^\mu\varphi) \\ &\supset \left[\Phi^+\partial_\mu\Phi^- + \frac{1}{2}(v + H + i\Phi^0)(\partial_\mu H - i\partial_\mu\Phi^0)\right]\left[\Phi^-\partial_\mu\Phi^+ + \frac{1}{2}(v + H - i\Phi^0)(\partial_\mu H + i\partial_\mu\Phi^0)\right] \\ &\supset \frac{1}{4}v^2\partial_\mu\Phi^0\partial^\mu\Phi^0 + \frac{1}{4}v^2\partial_\mu H\partial^\mu H.\end{aligned} \tag{3.17}$$

Thus, the bilinear terms of the scalar fields read

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\left(1 + \frac{1}{2}C^{\varphi D}v^2 - 2C^{\varphi\Box}v^2\right)(\partial_\mu H)^2 + \left(\frac{1}{2}m^2 - \frac{3}{4}v^2\lambda + \frac{15}{8}C^{\varphi}v^4\right)H^2 \\ &+ \frac{1}{2}\left(1 + \frac{1}{2}C^{\varphi D}v^2\right)(\partial_\mu\Phi^0)^2 + (\partial_\mu\Phi^-)(\partial^\mu\Phi^+).\end{aligned} \tag{3.18}$$

In order to obtain the physical fields, we need to normalize the kinetic terms into the canonical forms. The powerful way to do it is rescaling the fields as

$$h \equiv \sqrt{1 + \frac{1}{2}C^{\varphi D}v^2 - 2C^{\varphi\Box}v^2}H = \left(1 + \frac{1}{4}C^{\varphi D}v^2 - C^{\varphi\Box}v^2\right)H, \tag{3.19}$$

$$G^0 \equiv \sqrt{1 + \frac{1}{2}C^{\varphi D}v^2}\Phi^0 = 1 + \frac{1}{4}C^{\varphi D}v^2\Phi^0, \tag{3.20}$$

$$G^\pm \equiv \Phi^\pm. \tag{3.21}$$

We have introduced the notation h as physical Higgs field, and G^0, $G^\pm$ as Goldstone fields. The mass of Higgs boson, denoted by m_h, obey the gauge symmetry form $\frac{1}{2}(\partial_\mu h)^2 - \frac{1}{2}m_h^2 h^2$. Therefore

$$-\frac{1}{2}m_h^2 = \frac{\frac{1}{2}m^2 - \frac{3}{4}v^2\lambda + \frac{15}{8}C^\varphi v^4}{1 + \frac{1}{2}C^{\varphi D}v^2 - 2C^{\varphi\Box}v^2}. \tag{3.22}$$

Using the the approximation $1/\Lambda^4 = 0$, we will have the Higgs mass in term of m or v

$$m_h^2 = 2m\left[1 - \frac{m^2}{\lambda^2}(3C^\varphi - 4\lambda C^{\varphi\Box} + \lambda C^{\varphi D})\right] \tag{3.23}$$

$$= \lambda v^2 - v^4\left(3C^\varphi - 2\lambda C^{\varphi\Box} + \frac{\lambda}{2}C^{\varphi D}\right). \tag{3.24}$$

3.2.2 Mass of W and Z bosons

In this thesis, we do not have strong interaction in the process $ee \to \mu\mu$, so the Lagrangian of QCD is out of our discussion. The Lagrangian that relevant to free gauge fields reads

$$\begin{aligned}\mathcal{L}_{EW} &= \mathcal{L}_{EW}^{SM} + \mathcal{L}_{EW}^{(6)} = \mathcal{L}_{EW}^{SM} + Q_{\varphi W} + Q_{\varphi B} + Q_{\varphi WB} + Q_{\varphi\Box} + Q_{\varphi D} \\ &= -\frac{1}{4}W_{\mu\nu}^I W^{I\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (D_\mu\varphi)^\dagger(D^\mu\varphi) + C^{\varphi W}(\varphi^\dagger\varphi)W_{\mu\nu}^I W^{I\mu\nu} + C^{\varphi B}(\varphi^\dagger\varphi)B_{\mu\nu}B^{\mu\nu} \\ &\quad + C^{\varphi WB}(\varphi^\dagger\tau^I\varphi)W_{\mu\nu}^I B^{\mu\nu} + C^{\varphi\Box}(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi) + C^{\varphi D}(\varphi^\dagger D^\mu\varphi)^*(\varphi^\dagger D_\mu\varphi).\end{aligned} \tag{3.25}$$

Similar to above subsection, we need to find the bilinear terms of gauge fields from each operator

$$(\varphi^\dagger\varphi)\Box(\varphi^\dagger\varphi) \supset v^2\Box v^2 = 0, \tag{3.26}$$

$$\begin{aligned}(\varphi^\dagger D^\mu\varphi)^*(\varphi^\dagger D_\mu\varphi) &\supset \frac{1}{4}\left[\begin{pmatrix}0 & v\end{pmatrix}\left(igS^I W^{I\mu} + ig'YB^\mu\right)\begin{pmatrix}0\\v\end{pmatrix}\right]^*\left[\begin{pmatrix}0 & v\end{pmatrix}\left(igS^J W^{J\mu} + ig'YB^\mu\right)\begin{pmatrix}0\\v\end{pmatrix}\right] \\ &= \frac{v^4}{16}\left(g^2W_\mu^3W^{3\mu} + g'^2B_\mu B^\mu - 2gg'B^\mu W_\mu^3\right).\end{aligned} \tag{3.27}$$

$$\begin{aligned}(D_\mu\varphi)^\dagger(D^\mu\varphi) &\supset -i\frac{g}{2}\varphi^\dagger\tau^I W_\mu^I i\frac{g}{2}\tau^J W^{J\mu}\varphi - i\frac{g}{2}\varphi^\dagger\tau^I W_\mu^I ig'YB_\mu\varphi \\ &\quad - ig'\varphi^\dagger YB_\mu i\frac{g}{2}\tau^I W^{I\mu}\varphi - ig'\varphi^\dagger YB_\mu ig'YB^\mu\varphi \\ &\supset \frac{g^2}{8}v^2(W_\mu^1W^{1\mu} + W_\mu^2W^{2\mu}) + \frac{v^2}{8}(g^2W_\mu^3W^{3\mu} + g'^2B_\mu B^\mu - 2gg'W_\mu^3B^\mu).\end{aligned}$$

In the calculation above, we have used the identity $\tau^I\tau^J = \delta_{IJ} + i\epsilon_{IJK}\tau^K$ and substitute the value of hypercharge of Higgs field by 1/2. Then, The bilinear part of Lagrangian for gauge fields in Eq.(3.25) becomes

$$\mathcal{L}_{EW} = -\frac{1}{4}\left(1 - 2C^{\varphi W}v^2\right)W^I_{\mu\nu}W^{I\mu\nu} - \frac{1}{4}\left(1 - 2C^{\varphi B}v^2\right)B_{\mu\nu}B^{\mu\nu} - \frac{1}{2}C^{\varphi WB}v^2W^3_{\mu\nu}B^{\mu\nu} + \frac{v^2g^2}{8}(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu}) + \frac{v^2}{8}\left(1 + \frac{1}{2}v^2C^{\varphi D}\right)\left(g^2W^3_\mu W^{3\mu} + g'^2B_\mu B^\mu - 2gg'B^\mu W^3_\mu\right). \tag{3.28}$$

Now, we introduce the new rescaling fields in order to normalize the kinetic terms become canonical forms. The new fields' appearances have differences are that they have overhead bar

$$\bar{W}^I_\mu \equiv \sqrt{1 - 2C^{\varphi W}v^2}W^I_\mu = \left(1 - C^{\varphi W}v^2\right)W^I_\mu, \tag{3.29}$$

$$\bar{B}_\mu \equiv \sqrt{1 - 2C^{\varphi B}v^2}B_\mu = \left(1 - C^{\varphi B}v^2\right)B_\mu. \tag{3.30}$$

There is a caution that the gauge invariant property must be obeyed by those transformations. Thus, it is necessary to rewrite the covariant derivative in the form of

$$\bar{D}_\mu = D_\mu = \partial_\mu + i\bar{g}'\bar{B}_\mu Y + i\bar{g}\bar{W}^I_\mu T^I, \tag{3.31}$$

where $\bar{g}$ and $\bar{g}'$ have the form

$$\bar{g} \equiv \left(1 + C^{\varphi W}v^2\right)g, \qquad \bar{g}' \equiv \left(1 + C^{\varphi B}v^2\right)g', \tag{3.32}$$

as consequences of above rescaling steps. The Lagrangian in Eq.(3.28) which is rewritten in terms of $\bar{B}_\mu$ and $\bar{W}_\mu$ is

$$\mathcal{L}^{bilinear}_{EW} = -\frac{1}{4}\bar{W}^I_{\mu\nu}\bar{W}^{I\mu\nu} - \frac{1}{4}\bar{B}_{\mu\nu}\bar{B}^{\mu\nu} - \frac{1}{2}C^{\varphi WB}v^2\bar{B}^{\mu\nu}\bar{W}^{3\mu\nu} + \frac{v^2\bar{g}^2}{8}(\bar{W}^1_\mu\bar{W}^{1\mu} + \bar{W}^2_\mu\bar{W}^{2\mu}) + \frac{v^2}{8}\left(1 + \frac{1}{2}v^2C^{\varphi D}\right)(g^2\bar{W}^3_\mu\bar{W}^{3\mu} + g'^2\bar{B}_\mu\bar{B}^\mu - 2gg'\bar{W}^3_\mu\bar{B}^\mu). \tag{3.33}$$

Now, we introduce

$$\epsilon \equiv C^{\varphi WB}v^2. \tag{3.34}$$

From Eq. (3.33), we can find the physical field of W boson as

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(\bar{W}^1_\mu \mp i\bar{W}^2_\mu), \tag{3.35}$$

and the corresponding mass

$$m_W = \frac{1}{2}\bar{g}v. \tag{3.36}$$

The other gauge field in mass eigenstate that we need to identify is neutral gauge boson. Note that this time is not the same with charged W boson since we have $-\frac{1}{2}C^{\varphi WB}v^2\bar{B}^{\mu\nu}\bar{W}^{3\mu\nu}$ in the

Lagrangian. Thus, our mission is not only diagonalizing masses but also reducing the kinetic term to canonical form simultaneously. Based on [5], we have the mass matrix of the form

$$\begin{pmatrix} \bar{W}^3_\mu \\ \bar{B}_\mu \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon/2 \\ -\epsilon/2 & 1 \end{pmatrix} \begin{pmatrix} c_{\bar{W}} & s_{\bar{W}} \\ -s_{\bar{W}} & c_{\bar{W}} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \tag{3.37}$$

where the $\bar{\theta}$ is mixing angle

$$\tan\bar{\theta} = \frac{\bar{\mathrm{g}}}{\bar{\mathrm{g}}'} + \frac{\epsilon}{2}\left(1 - \frac{\bar{\mathrm{g}}'^2}{\bar{\mathrm{g}}^2}\right). \tag{3.38}$$

From now on we will use the notations $c_{\bar{W}}$ and $s_{\bar{W}}$ for $\cos\bar{\theta}$, $\sin\bar{\theta}$. After some simple calculations, we obtain

$$c_{\bar{W}} = \frac{\bar{g}}{\sqrt{\bar{g}^2 + \bar{g}'^2}}\left(1 - \frac{\epsilon}{2}\frac{\bar{g}'}{\bar{g}}\frac{\bar{g}^2 - \bar{g}'^2}{\bar{g}^2 + \bar{g}'^2}\right), \qquad s_{\bar{W}} = \frac{\bar{g}'}{\sqrt{\bar{g}^2 + \bar{g}'^2}}\left(1 + \frac{\epsilon}{2}\frac{\bar{g}}{\bar{g}'}\frac{\bar{g}^2 - \bar{g}'^2}{\bar{g}^2 + \bar{g}'^2}\right). \tag{3.39}$$

Substituting (3.37) into the Lagrangian (3.33), we are able to get the masses of the neutral gauge boson

$$m_Z = \frac{1}{2}\sqrt{\bar{g}^2 + \bar{g}'^2}v\left(1 + \frac{\epsilon\bar{g}\bar{g}'}{\bar{g}^2 + \bar{g}'^2}\right)\left(1 + \frac{1}{4}C^{\varphi D}v^2\right), \tag{3.40}$$

$$m_\gamma = 0. \tag{3.41}$$

The mass of gauge bosons in this thesis was derived independently before the publication of the paper [5] in April 2017. For the case of Z boson mass, O. Nachtmann, F. Nagel, M. Pospischil have already introduced the way to diagonalise the mass matrix also transform the kinetic terms into canonical forms [6], but their results were not correct since they used the list of dimension-six operator derived by Buchmuller and Wyler [4] in 1985 which have many redundant operators. But when I used that manner to find the Z boson mass with the updated list of dimension-six operators. It is turn out that my result agrees with that of [5]. But I do not introduce it here since the conventions and notations of that method are complicated so that it can create many confusions.

3.3 Coupling constants of the vertices in $e^- + e^+ \to \mu^- + \mu^+$

3.3.1 Vertex factors of lepton and gauge boson

Since photon and Z boson fields usually appear together. Thus, it is convenient to find the coupling of photon and Z boson with lepton simultaneously. First, the Lagrangian containing the mixed terms of leptons and gauge fields reads

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\bar{W}^I_{\mu\nu}\bar{W}^{I\mu\nu} - \frac{1}{4}\bar{B}^{\mu\nu}\bar{B}_{\mu\nu} + i\sum(\bar{\psi}_L\bar{\not{D}}\psi_L + \bar{\psi}_R\bar{\not{D}}\psi_R) + C^{eW}(\bar{l}_p\sigma^{\mu\nu}e_r)\tau^I\varphi\bar{W}^I_{\mu\nu} \\ &+ C^{eB}(\bar{l}_p\sigma^{\mu\nu}e_r)\varphi\bar{B}_{\mu\nu} + C^{\varphi l1}(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_p\gamma^\mu l_r) + C^{\varphi l3}(\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{l}_p\tau^I\gamma^\mu l_r) + C^{\varphi e}(\varphi^\dagger i\overleftrightarrow{D}_\mu)(\bar{e}_p\gamma^\mu e_r),\end{aligned} \tag{3.42}$$

The Lagrangian is too long to write all terms simultaneously, thus we must find the contribution for each operator:

$$\begin{aligned} i\bar{\psi}_L \bar{\not{D}} \psi_L &\supset i\bar{e}_L \left(i\bar{g}' \bar{B}_\mu Y + i\bar{g} \bar{W}^I_\mu T^I \right) \gamma^\mu \bar{e}_L \\ &\supset i\bar{e}_L \Big\{ i\bar{g}' Y \left[\left((-\epsilon/2) c_{\bar{W}} - s_{\bar{W}} \right) Z_\mu + \left((-\epsilon/2) s_{\bar{W}} + c_{\bar{W}} \right) A_\mu \right] \\ &\qquad + i\bar{g} I^3_e \left[\left(c_{\bar{W}} + (\epsilon/2) s_{\bar{W}} \right) Z_\mu + \left(s_{\bar{W}} - (\epsilon/2) c_{\bar{W}} \right) A_\mu \right] \Big\} \gamma^\mu \bar{e}_L \\ &= i\bar{e}_L \left\{ -\frac{i}{2} c_{\bar{W}} \left(-\frac{\bar{g}'^2}{2}\epsilon - \frac{\bar{g}'^2}{\bar{g}} + \frac{\epsilon}{2}\frac{\bar{g}'^3}{\bar{g}^2} + \bar{g} \right) Z_\mu - \frac{i}{2} c_{\bar{W}} \left(2\bar{g}' - \epsilon \frac{\bar{g}'^2}{\bar{g}} \right) A_\mu \right\} \gamma^\mu \bar{e}_L, \end{aligned} \tag{3.43}$$

$$\begin{aligned} i\bar{\psi}_R \bar{\not{D}} \psi_R &\supset i\bar{e}_R \left(i\bar{g}' \bar{B}_\mu Y \right) \gamma^\mu \bar{e}_R \\ &= i\bar{e}_R i\bar{g}' Y \left[\left((-\epsilon/2) c_{\bar{W}} - s_{\bar{W}} \right) Z_\mu + \left((-\epsilon/2) s_{\bar{W}} + c_{\bar{W}} \right) A_\mu \right] \gamma^\mu \bar{e}_R \\ &= i\bar{e}_R \left\{ -i\bar{g}' c_{\bar{W}} \left[\left(-\epsilon + \frac{\epsilon}{2}\frac{\bar{g}'^2}{\bar{g}^2} - \frac{\bar{g}'}{\bar{g}} \right) Z_\mu + \left(1 - \frac{\epsilon}{2}\frac{\bar{g}'}{\bar{g}} \right) A_\mu \right] \right\} \gamma^\mu \bar{e}_R. \end{aligned}$$

Thus, the contribution of operator $i\sum(\bar{\psi}_L \bar{\not{D}} \psi_L + \bar{\psi}_R \bar{\not{D}} \psi_R)$ in vertex $ee\gamma$ is

$$\begin{aligned} & i\bar{e}_L \left[-\frac{i}{2} c_{\bar{W}} \left(2\bar{g}' - \epsilon \frac{\bar{g}'^2}{\bar{g}} \right) \gamma^\mu \right] e_L A_\mu + i\bar{e}_R \left[-\frac{i}{2} c_{\bar{W}} \left(2\bar{g}' - \epsilon \frac{\bar{g}'^2}{\bar{g}} \right) \gamma^\mu \right] e_R A_\mu \\ &= \frac{i}{2}\bar{e} \left[-i c_{\bar{W}} \left(2\bar{g}' - \epsilon \frac{\bar{g}'^2}{\bar{g}} \right) \gamma^\mu \right] e A_\mu \\ &= \left[\frac{\bar{g}\bar{g}'}{\sqrt{\bar{g}^2 + \bar{g}'^2}} - \frac{\epsilon \bar{g}^2 \bar{g}'^2}{(\bar{g}^2 + \bar{g}'^2)^{3/2}} \right] (\bar{e}\gamma^\mu e) A_\mu, \end{aligned} \tag{3.44}$$

and for eeZ coupling is

$$\begin{aligned} & i\bar{e}_L \left[-\frac{i}{2} c_{\bar{W}} \left(-\frac{\bar{g}'^2}{2}\epsilon - \frac{\bar{g}'^2}{\bar{g}} + \frac{\epsilon}{2}\frac{\bar{g}'^3}{\bar{g}^2} + \bar{g} \right) \gamma^\mu \right] e_L Z_\mu + i\bar{e}_R \left[-i\bar{g}' c_{\bar{W}} \left(-\epsilon + \frac{\epsilon}{2}\frac{\bar{g}'^2}{\bar{g}^2} - \frac{\bar{g}'}{\bar{g}} \right) \gamma^\mu \right] e_R Z_\mu \\ &= \left[-\frac{1}{2\sqrt{\bar{g}^2 + \bar{g}'^2}} \left((\bar{g}'^2 - \bar{g}^2) P_R + 2\bar{g}'^2 P_L \right) + \frac{\epsilon \bar{g}\bar{g}'}{2(\bar{g}^2 + \bar{g}'^2)^{3/2}} \left((\bar{g}'^2 - \bar{g}^2) P_R - 2\bar{g}^2 P_L \right) \right] (\bar{e}\gamma^\mu e) Z_\mu. \end{aligned} \tag{3.45}$$

The contribution of operator Q^{eW} is

$$\begin{aligned} C^{eW} (\bar{l}_p \sigma^{\mu\nu} e_r) \tau^I \varphi \bar{W}^I_{\mu\nu} &\supset C^{eW} \left[\begin{pmatrix} \bar{\nu}_e & \bar{e}_L \end{pmatrix} \sigma^{\mu\nu} e_r \right] \tau^3 \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} W^3_{\mu\nu} \\ &\supset -C^{eW} (\bar{e}_L \sigma^{\mu\nu} e_r) \frac{v}{\sqrt{2}} (\partial_\mu \bar{W}^3_\nu - \partial_\nu \bar{W}^3_\mu) \\ &\supset \sqrt{2} C^{eW} v (\bar{e}_L \sigma^{\mu\nu} e_r) \left[c_{\bar{W}} \partial_\nu Z_\mu + s_{\bar{W}} \partial\nu A_\mu \right]. \end{aligned}$$

Now using the Fourier transformation $Z_\mu \to \int e^{-iqx}\widetilde{Z}_\mu$ where q is the momentum of meditators. Similar for A_μ, we have

$$\begin{aligned} Q^{eW} &\supset \sqrt{2}C^{eW}v(\bar{e}_L\sigma^{\mu\nu}e_r)\left[c_{\bar{W}}(-iq_\nu)Z_\mu + s_{\bar{W}}(-iq_\nu)A_\mu\right] \\ &= -i\sqrt{2}C^{eW}v(\bar{e}q_\nu\sigma^{\mu\nu}P_Re)\left[c_{\bar{W}}Z_\mu + s_{\bar{W}}A_\mu\right]. \end{aligned} \tag{3.46}$$

Note that the operator $Q^{eW\dagger}$ also contain the couplings of fermion and gauge boson

$$Q^{eW\dagger} - i\sqrt{2}C^{eW*}v(\bar{e}q_\nu\sigma^{\mu\nu}P_Le)\left[c_{\bar{W}}Z_\mu + s_{\bar{W}}A_\mu\right] \tag{3.47}$$

The calculation for the operator Q^{eB} is completely the same with Q^{eW} and we can obtain the contribution

$$Q^{eB} \supset -i\sqrt{2}C^{eB}v(\bar{e}q_\nu\sigma^{\mu\nu}P_Re)(s_{\bar{W}}Z_\mu - c_{\bar{W}}A_\mu) \tag{3.48}$$

$$Q^{eB\dagger} \supset -i\sqrt{2}C^{eB*}v(\bar{e}q_\nu\sigma^{\mu\nu}P_Le)(s_{\bar{W}}Z_\mu - c_{\bar{W}}A_\mu). \tag{3.49}$$

For the $Q^{\varphi l1}$ operator

$$\begin{aligned} Q^{\varphi l1} &= C^{\varphi l1}(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_p\gamma^\mu l_r) \\ &= C^{\varphi l1}i\varphi^\dagger(\partial_\mu + i\bar{g}'Y\bar{B}_\mu + i\bar{g}\bar{W}^I_\mu T^I - \overleftarrow{\partial}_\mu + i\bar{g}'Y\bar{B}_\mu + i\bar{g}\bar{W}^I_\mu T^I)\varphi(\bar{l}_p\gamma^\mu l_r) \\ &\supset 2iC^{\varphi l1}\begin{pmatrix}0 & \frac{v}{\sqrt{2}}\end{pmatrix}\left[\frac{i\bar{g}'}{2}(-s_{\bar{W}}Z_\mu + c_{\bar{W}}A_\mu) - \frac{i\bar{g}}{2}(c_{\bar{W}}Z_\mu + s_{\bar{W}}A_\mu)\right]\begin{pmatrix}0 \\ \frac{v}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\bar{\nu}_e & \bar{e}\end{pmatrix}\gamma^\mu\begin{pmatrix}\bar{\nu}_e \\ \bar{e}\end{pmatrix} \\ &= C^{\varphi l1}\frac{v^2}{2}\left[(\bar{g}'s_{\bar{W}} + \bar{g}c_{\bar{W}})Z_\mu + 0.A_\mu\right]\bar{e}_L\gamma^\mu e_L. \end{aligned} \tag{3.50}$$

For the $Q^{\varphi l3}$ operator

$$\begin{aligned} Q^{\varphi l3} &= C^{\varphi l3}i\varphi^\dagger(\tau^I D_\mu - \overleftarrow{D}_\mu\tau^I)\varphi(\bar{l}_p\tau^I\gamma^\mu l_r) \\ &\supset C^{\varphi l3}i\begin{pmatrix}0 & \frac{v}{\sqrt{2}}\end{pmatrix}\left[\tau^3(i\bar{g}'\bar{B}_\mu Y + i\bar{g}\bar{W}^I_\mu T^I) + (i\bar{g}'\bar{B}_\mu Y + i\bar{g}\bar{W}^I_\mu T^I)\tau^3\right]\begin{pmatrix}0 \\ \frac{v}{\sqrt{2}}\end{pmatrix}(-\bar{e}_L\gamma^\mu e_L) \\ &= C^{\varphi l3}\frac{v^2}{2}\left[(\bar{g}'s_{\bar{W}} + \bar{g}c_{\bar{W}})Z_\mu + 0.A_\mu\right](\bar{e}_L\gamma^\mu e_L) \\ &= C^{\varphi l3}\frac{v^2}{2}\sqrt{\bar{g}^2 + \bar{g}'^2}Z_\mu(\bar{e}\gamma^\mu P_Le). \end{aligned} \tag{3.51}$$

For the $Q^{\varphi e}$ operator

$$\begin{aligned} Q^{\varphi e} &= C^{\varphi e}i\varphi^\dagger(D_\mu - \overleftarrow{D}_\mu)\varphi(\bar{e}_p\gamma^\mu e_r) \\ &\supset C^{\varphi e}i\begin{pmatrix}0 & \frac{v}{\sqrt{2}}\end{pmatrix}(2i\bar{g}'Y\bar{B}_\mu + 2i\bar{g}\bar{W}^3_\mu T^3)\begin{pmatrix}0 \\ \frac{v}{\sqrt{2}}\end{pmatrix}(\bar{e}_R\gamma^\mu e_R) \\ &= -C^{\varphi e}\frac{1}{2}v^2\left[(-\bar{g}'s_{\bar{W}} - \bar{g}c_{\bar{W}})Z_\mu + 0.A_\mu\right](\bar{e}\gamma^\mu P_Re) \\ &= C^{\varphi e}\frac{v^2}{2}\sqrt{\bar{g}'^2 + \bar{g}^2}Z_\mu(\bar{e}\gamma^\mu P_Re). \end{aligned} \tag{3.52}$$

3.3.2 Vertex factor of lepton and Goldstone boson

Since our process has neutral mediators, so we will ignore all "charge" Goldstone boson. The Lagrangian containing the mixed terms of lepton and Goldstone boson reads

$$\mathcal{L} = -\,\bar{l}'_L\Gamma_e e'_R\varphi - \varphi^\dagger \bar{e}'_R\Gamma_e^\dagger l'_p + C^{\varphi l1}(\varphi^\dagger i\overleftrightarrow{D}_\mu\varphi)(\bar{l}_p\gamma^\mu l_r) \tag{3.53}$$

$$+ C^{\varphi l3}(\varphi^\dagger i\overleftrightarrow{D}^I_\mu\varphi)(\bar{l}_p\tau^I\gamma^\mu l_r) + C^{\varphi e}(\varphi^\dagger i\overleftrightarrow{D}_\mu)(\bar{e}_p\gamma^\mu e_r). \tag{3.54}$$

First, we need to obtain the coupling in SM. Before doing that, we to rotate the fermion fields by the unitary matrices in order to diagonalize lepton and quark masses

$$\psi'_X = U_{\psi X}\psi_X, \tag{3.55}$$

where ψ stand for ν, e, u, d. X denote for left- and right-handed. The unprimed field is the mass eigenstate fields.

$$\begin{aligned}
-\bar{l}'_L\Gamma_e e'_R\varphi - \varphi^\dagger\bar{e}'_R\Gamma_e^\dagger l'_p &= -\bar{l}_L U_L^\dagger\Gamma_e U_R e_R\varphi - \varphi^\dagger\bar{e}_R(U_L^\dagger\Gamma_e U_R)^\dagger l_L \\
&\supset -\begin{pmatrix}\bar{\nu}_e & \bar{e}_L\end{pmatrix}\Gamma'_e e_R\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}i\Phi^0\end{pmatrix} - \begin{pmatrix}0 & -\frac{1}{\sqrt{2}}i\Phi^0\end{pmatrix}\bar{e}_R\Gamma_e'^\dagger\begin{pmatrix}\nu_e\\ e_L\end{pmatrix}.
\end{aligned}$$

After diagonalizing the lepton masses, we have the identity $\Gamma' = \sqrt{2}\dfrac{m}{v}$, the SM Lagrangian then becomes

$$-i\frac{m_e}{v}\bar{e}(P_R - P_L)e\Phi^0 = -i\frac{m_e}{v}\bar{e}\gamma^5 e\Phi^0 = -i\frac{m_e}{v}\bar{e}\gamma^5 e\left(1 - \frac{1}{4}C^{\varphi D}v^2\right)G^0. \tag{3.56}$$

The contribution of operator $Q^{\varphi l1}$,

$$\begin{aligned}
Q^{\varphi l1} &= C^{\varphi l1} i\varphi^\dagger(D_\mu - \overleftarrow{D}_\mu)\varphi(\bar{l}_p\gamma^\mu l_r) \\
&\supset C^{\varphi l1} i\begin{pmatrix}0 & \frac{1}{\sqrt{2}}(v - i\Phi^0)\end{pmatrix}(\partial_\mu - \overleftarrow{\partial}_\mu)\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(v + i\Phi^0)\end{pmatrix}\begin{pmatrix}\bar{\nu}_e & \bar{e}_L\end{pmatrix}\gamma^\mu\begin{pmatrix}\nu_e\\ e_L\end{pmatrix} \\
&= C^{\varphi l1} i\left[\frac{i}{\sqrt{2}}(\partial_\mu\Phi^0)\frac{1}{\sqrt{2}}(v + i\Phi^0) + \frac{1}{\sqrt{2}}(v - i\Phi^0)\frac{i}{\sqrt{2}}(\partial_\mu\Phi^0)\right]\bar{e}_L\gamma^\mu e_L \\
&= -C^{\varphi l1} v(\partial_\mu\Phi^0)\bar{e}\gamma^\mu P_L e \\
&= iC^{\varphi l1} v\not{q} G^0\bar{e}\gamma^\mu P_L e.
\end{aligned} \tag{3.57}$$

The contribution of operator $Q^{\varphi l3}$,

$$\begin{aligned}
Q^{\varphi l3} &= C^{\varphi l3} i\varphi^\dagger(\tau^I D_\mu - \overleftarrow{D}_\mu\tau^I)\varphi(\bar{l}_p\tau^I\gamma^\mu l_r) \\
&\supset C^{\varphi l3} i\begin{pmatrix}0 & \frac{1}{\sqrt{2}}(v - i\Phi^0)\end{pmatrix}(\tau^3\partial_\mu - \overleftarrow{\partial}_\mu\tau^3)\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(v + i\Phi^0)\end{pmatrix}\begin{pmatrix}\bar{\nu}_e & \bar{e}_L\end{pmatrix}\tau^3\gamma^\mu\begin{pmatrix}\nu_e\\ e_L\end{pmatrix} \\
&= C^{\varphi l3} i\left[-\frac{i}{\sqrt{2}}(\partial_\mu\Phi^0)\frac{1}{\sqrt{2}}(v - i\Phi^0) - \frac{1}{\sqrt{2}}(v + i\Phi^0)\frac{i}{\sqrt{2}}(\partial_\mu\Phi^0)\right](-\bar{e}_L\gamma^\mu e_L) \\
&= -C^{\varphi l3} v(\partial_\mu\Phi^0)\bar{e}\gamma^\mu P_L e \\
&= iC^{\varphi l3} v\not{q} G^0\bar{e}\gamma^\mu P_L e.
\end{aligned} \tag{3.58}$$

For the $Q^{\varphi e}$ operator

$$\begin{aligned}
Q^{\varphi e} &= C^{\varphi e} i\varphi^\dagger (D_\mu - \overleftarrow{D}_\mu)\varphi(\bar{e}_p\gamma^\mu e_r) \\
&\supset C^{\varphi e} i \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}(v - i\Phi^0) \end{pmatrix} (\partial_\mu - \overleftarrow{\partial}_\mu) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + i\Phi^0) \end{pmatrix} (\bar{e}_R\gamma^\mu e_R) \\
&= C^{\varphi e} i \left[\frac{i}{\sqrt{2}}(\partial_\mu \Phi^0)\frac{1}{\sqrt{2}}(v + i\Phi^0) + \frac{1}{\sqrt{2}}(v - i\Phi^0)\frac{i}{\sqrt{2}}(\partial_\mu \Phi^0) \right] \bar{e}_R\gamma^\mu e_R \\
&= -C^{\varphi e} v(\partial_\mu \Phi^0)\bar{e}\gamma^\mu P_R e \\
&= iC^{\varphi e} v \not{p} G^0 \bar{e}\gamma^\mu P_R e.
\end{aligned} \tag{3.59}$$

3.3.3 Four-fermion vertex

There are three four-fermion operators in Table. (3.3) contribute to our process. The corresponding Lagrangian reads

$$\mathcal{L} = C^{ll}(\bar{l}_p\gamma_\mu l_r)(\bar{l}_s\gamma^\mu l_t) + C^{ee}(\bar{e}_p\gamma_\mu e_r)(\bar{e}_s\gamma^\mu e_t) + C^{ee}(\bar{l}_p\gamma_\mu l_r)(\bar{e}_s\gamma^\mu e_t). \tag{3.60}$$

The operators Q_{ee} have four cases for our coupling:

$$(\bar{e}_R\gamma_\mu e_R)(\bar{\mu}_R\gamma_\mu \mu_R), \quad (\bar{\mu}_R\gamma_\mu \mu_R)(\bar{e}_R\gamma_\mu e_R), \quad (\bar{\mu}_R\gamma_\mu e_R)(\bar{e}_R\gamma_\mu \mu_R), \quad (\bar{e}_R\gamma_\mu \mu_R)(\bar{\mu}_R\gamma_\mu e_R). \tag{3.61}$$

Note that for the last two cases we can use the Fierz identity [1] $(\bar{e}_p\gamma_\mu e_r)(\bar{e}_s\gamma_\mu e_t) = (\bar{e}_p\gamma_\mu e_t)(\bar{e}_s\gamma_\mu e_r)$. So in the coupling we have: $4iC^{ee}_{e\mu}(\gamma^\mu P_R)_e(\gamma^\mu P_R)_\mu$
However, we can not apply directly the Fierz transformation for Q_{ll} since this term not only have the Dirac indices but also have the SU(2) indices. Thus we have two cases Q_{ll} for our coupling:

$$(\bar{e}_L\gamma_\mu e_L)(\bar{\mu}_L\gamma_\mu \mu_L), \qquad (\bar{\mu}_L\gamma_\mu \mu_L)(\bar{e}_L\gamma_\mu e_L). \tag{3.62}$$

Another two cases which are not applied Fierz identity directly are

$$\begin{pmatrix} \bar{\nu}_e & \bar{e}_L \end{pmatrix} \gamma_\mu \begin{pmatrix} \bar{\nu}_\mu \\ \bar{\mu}_L \end{pmatrix} \begin{pmatrix} \bar{\nu}_\mu & \bar{\mu}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} \bar{\nu}_e \\ \bar{e}_L \end{pmatrix}, \qquad \begin{pmatrix} \bar{\nu}_\mu & \bar{\mu}_L \end{pmatrix} \gamma_\mu \begin{pmatrix} \bar{\nu}_e \\ \bar{e}_L \end{pmatrix} \begin{pmatrix} \bar{\nu}_e & \bar{e}_L \end{pmatrix} \gamma^\mu \begin{pmatrix} \bar{\nu}_\mu \\ \bar{\mu}_L \end{pmatrix}. \tag{3.63}$$

But after we expand it, we can get four terms without $SU(2)$ indices. Two of them are

$$(\bar{e}_L\gamma_\mu \mu_L)(\bar{\mu}_L\gamma_\mu e_L), \qquad (\bar{\mu}_L\gamma_\mu e_L)(\bar{e}_L\gamma_\mu \mu_L), \tag{3.64}$$

and they are Fierz transformable. So in the coupling we have $4iC^{ll}_{e\mu}(\gamma^\mu P_L)_e(\gamma^\mu P_L)_\mu$.
Similar to operator Q_{ll}, the operator Q_{le} has SU(2) indices as well. We will have two cases:

$$(\bar{e}_L\gamma_\mu e_L)(\bar{\mu}_R\gamma^\mu \mu_R). \qquad (\bar{\mu}_L\gamma_\mu \mu_L)(\bar{e}_R\gamma^\mu e_R). \tag{3.65}$$

[1] For futher reading of Fierz transformation, please find the paper [7]

The other cases after $SU(2)$ expansion are $(\bar{e}_L\gamma_\mu\mu_L)(\bar{\mu}_R\gamma^\mu e_R)$ and $(\bar{\mu}_L\gamma_\mu e_L)(\bar{e}_R\gamma^\mu\mu_R)$. The Fierz transformation for Q_{le} in the first case is:

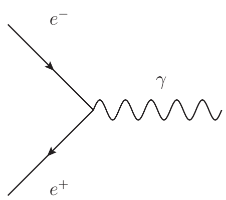
$$\begin{aligned}
&(\bar{e}_L\gamma_\mu\mu_L)(\bar{\mu}_R\gamma^\mu e_R)\\
&= -\frac{1}{4}\left[4(\bar{e}_L e_R)(\bar{\mu}_R\mu_L) - 2(\bar{e}_L\gamma_\mu e_R)(\bar{\mu}_R\gamma^\mu\mu_L) - 2(\bar{e}_L\gamma_\mu\gamma^5 e_R)(\bar{\mu}_R\gamma^\mu\gamma^5\mu_L) - 4(\bar{e}_L\gamma^5 e_R)(\bar{\mu}_R\gamma^5\mu_L)\right]\\
&= -\frac{1}{4}\Big[4\left(\bar{e}P_R e\right)\left(\bar{\mu}P_L\mu\right) - 2\left(\bar{e}P_R\gamma_\mu(P_R-P_L)P_R e\right)\left(\bar{\mu}P_L\gamma^\mu(P_R-P_L)P_L\mu\right)\\
&\qquad\qquad -4\left(\bar{e}P_R(P_R-P_L)P_R e\right)\left(\bar{\mu}P_L(P_R-P_L)P_L\mu\right)\Big]\\
&= -\frac{1}{4}\left[4\left(\bar{e}P_R e\right)\left(\bar{\mu}P_L\mu\right) + 4\left(\bar{e}P_R e\right)\left(\bar{\mu}P_L\mu\right)\right]\\
&= -2\left(\bar{e}P_R e\right)\left(\bar{\mu}P_L\mu\right).
\end{aligned} \tag{3.66}$$

So now we will have $-2i(P_R)_e(P_L)_\mu C^{le}_{e\mu}$ in the coupling. Similar to the case $(\bar{\mu}_L\gamma_\mu e_L)(\bar{e}_R\gamma^\mu\mu_R)$. Thus the coupling corresponding to operator Q^{le} is

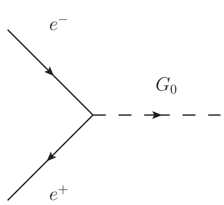
$$iC^{le}_{e\mu}(\gamma^\mu P_L)_e(\gamma^\mu P_R)_\mu + iC^{le}_{\mu e}(\gamma^\mu P_L)_\mu(\gamma^\mu P_R)_e - 2i\left[(P_L)_\mu(P_R)_e C^{le}_{e\mu} + (P_L)_e(P_R)_\mu C^{le}_{\mu e}\right].$$

3.4 Feyman rules

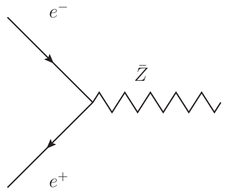
After find all interaction terms for our process, we are now able to write down the Feynman rules for the couplings of our process in SMEFT. The Feynman rules for propagators are not mentioned here since we have rescaled the fields to make the the kinetic terms in SMEFT the same with that of SM. And it lead to the unchanged propagators. It must be understood that all couplings below have incoming momentum.



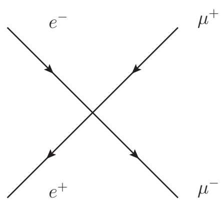
$$\begin{aligned}
&= \frac{i\bar{g}\bar{g}'}{\sqrt{\bar{g}^2+\bar{g}'^2}}\gamma^\mu - \frac{i\bar{g}^2\bar{g}'^2 v^2}{(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\gamma^\mu\\
&\quad + \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\nu\left(C^{eW*}_e\sigma^{\mu\nu}P_L + C^{eW}_e\sigma^{\mu\nu}P_R\right)\\
&\quad - \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\nu\left(C^{eB*}_e\sigma^{\mu\nu}P_L + C^{eB}_e\sigma^{\mu\nu}P_R\right),
\end{aligned} \tag{3.67}$$



$$= \frac{m_e}{v}\gamma^5 - \frac{v}{4}C^{\varphi D}m_e\gamma^5 - v\not{q}\left(P_L C^{\varphi l1}_e + P_L C^{\varphi l1}_e + P_R C^{\varphi e}_e\right), \tag{3.68}$$



$$
\begin{aligned}
=&\frac{i}{\sqrt{\bar{g}^2+\bar{g}'^2}}\left((\bar{g}^2-\bar{g}'^2)\gamma^\mu P_L - 2\bar{g}'^2\gamma^\mu P_R\right)\\
&+\frac{i\bar{g}\bar{g}'v^2}{2(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\left((\bar{g}'^2-\bar{g}^2)\gamma^\mu P_L - 2\bar{g}^2\gamma^\mu P_R\right)\\
&+\frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\nu\left(C_e^{eW*}\sigma^{\mu\nu}P_L + C_e^{eW}\sigma^{\mu\nu}P_R\right)\\
&+\frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\nu\left(C_e^{eB*}\sigma^{\mu\nu}P_L + C_e^{eB}\sigma^{\mu\nu}P_R\right)\\
&+\frac{1}{2}iv^2\sqrt{\bar{g}^2+\bar{g}'^2}\left(C_e^{\varphi l1}\gamma^\mu P_L + C_e^{\varphi l3}\gamma^\mu P_L + C_e^{\varphi e}\gamma^\mu P_R\right),
\end{aligned}
\tag{3.69}
$$



$$
\begin{aligned}
=&4iC^{ll}_{e\mu}(\gamma^\mu P_L)_e(\gamma^\mu P_L)_\mu + 4iC^{ee}_{e\mu}(\gamma^\mu P_R)_e(\gamma^\mu P_R)_\mu\\
&+iC^{le}_{e\mu}(\gamma^\mu P_L)_e(\gamma^\mu P_R)_\mu + iC^{le}_{\mu e}(\gamma^\mu P_L)_\mu(\gamma^\mu P_R)_e\\
&-2i\left[(P_L)_\mu(P_R)_e C^{le}_{e\mu} + (P_L)_e(P_R)_\mu C^{le}_{\mu e}\right].
\end{aligned}
\tag{3.70}
$$

3.5 The independence of Feynman amplitude on gauge fixing parameters

We have apparently discussed the properties of gauge fixing parameters in Chap.1 . In this section, we will check that if the Feynman amplitude of the diagrams in Fig.(3.1) depend on those parameters or not. Therefore, we will only focus on terms in propagators that contain gauge-fixing parameters only. Note that we will not use any approximation of $m_f = 0$, where m_f is the fermion's mass. First, the Feynman for the diagram which mediated by photon reads

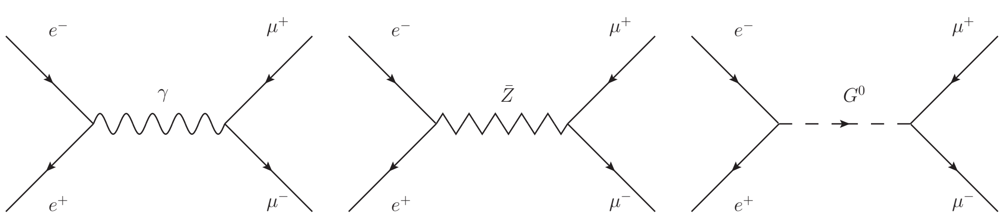


Figure 3.1: Feynman diagrams containing gauge fixing parameters of process $e^+ + e^- \to \mu^+\mu^-$ in SMEFT

$$
\begin{aligned}
\mathcal{M}_\gamma =& \bar{v}_{s'}(p')\left[\frac{i\bar{g}\bar{g}'}{\sqrt{\bar{g}^2+\bar{g}'^2}}\gamma^\mu - \frac{i\bar{g}^2\bar{g}'^2v^2}{(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\gamma^\mu - \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eW*}\sigma^{\mu\sigma}P_L + C_e^{eW}\sigma^{\mu\sigma}P_R\right)\right.\\
&\left. + \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eB*}\sigma^{\mu\sigma}P_L + C_e^{eB}\sigma^{\mu\sigma}P_R\right)\right]u_s(p)\left[\frac{-i}{q^2}\left(g_{\mu\nu}-(1-\xi_\gamma)\frac{q_\mu q_\nu}{q^2}\right)\right]\\
&\bar{u}_r(k)\left[\frac{i\bar{g}\bar{g}'}{\sqrt{\bar{g}^2+\bar{g}'^2}}\gamma^\nu - \frac{i\bar{g}^2\bar{g}'^2v^2}{(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\gamma^\nu + \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eW*}\sigma^{\nu\rho}P_L + C_\mu^{eW}\sigma^{\nu\rho}P_R\right)\right.\\
&\left. - \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eB*}\sigma^{\nu\rho}P_L + C_\mu^{eB}\sigma^{\nu\rho}P_R\right)\right]v_{r'}(k'). \qquad (3.71)
\end{aligned}
$$

We can see that the terms containing gauge-fixing parameter in photon propagator proportion to q_μ. Now look at the first coupling, it is obvious that two terms consist of γ^μ will vanish when we multiply by q_μ

$$
\bar{v}_{s'}(p')(\not{q})u_s(p) = \bar{v}_{s'}(p')(\not{p}' + \not{p})u_s(p) = \bar{v}_{s'}(p')(-m_e + m_e)u_s(p) = 0. \qquad (3.72)
$$

For the rest terms in that coupling, it will be canceled out since $(q_\sigma q_\mu)\sigma^{\sigma\mu} = 0$. Hence, our amplitude does not depend on ξ_γ. For ξ_Z, it is more complicated, the amplitude for Z and Goldstone boson read

$$
\begin{aligned}
&\mathcal{M}_Z =\\
&\bar{v}_{s'}(p')\left[\frac{i}{\sqrt{\bar{g}^2+\bar{g}'^2}}\left((\bar{g}^2-\bar{g}'^2)\gamma^\mu P_L - 2\bar{g}'^2\gamma^\mu P_R\right) + \frac{i\bar{g}\bar{g}'v^2}{2(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\left((\bar{g}'^2-\bar{g}^2)\gamma^\mu P_L - 2\bar{g}^2\gamma^\mu P_R\right)\right.\\
&\quad - \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eW*}\sigma^{\mu\sigma}P_L + C_e^{eW}\sigma^{\mu\sigma}P_R\right) - \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eB*}\sigma^{\mu\sigma}P_L + C_e^{eB}\sigma^{\mu\sigma}P_R\right)\\
&\quad\left. + \frac{1}{2}iv^2\sqrt{\bar{g}^2+\bar{g}'^2}\left(C_e^{\varphi l1}\gamma^\mu P_L + C_e^{\varphi l3}\gamma^\mu P_L + C_e^{\varphi e}\gamma^\mu P_R\right)\right]u_s(p)\\
&\times\left[\frac{-i}{q^2-\bar{m}_Z^2}\left(g_{\mu\nu}-(1-\xi_Z)\frac{q_\mu q_\nu}{q^2-\xi_Z\bar{m}_Z^2}\right)\right]\\
&\times\bar{u}_r(k)\left[\frac{i}{\sqrt{\bar{g}^2+\bar{g}'^2}}\left((\bar{g}^2-\bar{g}'^2)\gamma^\nu P_L - 2\bar{g}'^2\gamma^\nu P_R\right) + \frac{i\bar{g}\bar{g}'v^2}{2(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\left((\bar{g}'^2-\bar{g}^2)\gamma^\nu P_L - 2\bar{g}^2\gamma^\nu P_R\right)\right.\\
&\quad + \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eW*}\sigma^{\nu\rho}P_L + C_\mu^{eW}\sigma^{\nu\rho}P_R\right) + \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eB*}\sigma^{\nu\rho}P_L + C_\mu^{eB}\sigma^{\nu\rho}P_R\right)\\
&\quad\left. + \frac{1}{2}iv^2\sqrt{\bar{g}^2+\bar{g}'^2}\left(C_\mu^{\varphi l1}\gamma^\nu P_L + C_\mu^{\varphi l3}\gamma^\nu P_L + C_\mu^{\varphi e}\gamma^\nu P_R\right)\right]v_{r'}(k'), \qquad (3.73)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_{G_0} =& \bar{v}_{s'}(p')\left[\frac{m_e}{v}\gamma^5 - \frac{v}{4}C^{\varphi D}m_e\gamma^5 + v\not{q}\left(P_L C_e^{\varphi l1} + P_L C_e^{\varphi l1} + P_R C_e^{\varphi e}\right)\right]u_s(p)\left[\frac{i}{q^2-\xi_Z\bar{m}_Z^2}\right]\\
&\bar{u}_r(k)\left[\frac{m_\mu}{v}\gamma^5 - \frac{v}{4}C^{\varphi D}m_\mu\gamma^5 - v\not{q}\left(P_L C_e^{\varphi l1} + P_L C_e^{\varphi l1} + P_R C_e^{\varphi e}\right)\right]v_{r'}(k'). \qquad (3.74)
\end{aligned}
$$

After some calculations, we can write the parts containing ξ_Z of two above amplitudes of the form

$$\mathcal{M}_Z^\xi = \frac{im_e m_\mu(1-\xi_Z)}{(q^2-\xi_Z\bar{m}_Z^2)(q^2-\bar{m}_Z^2)}\left[\bar{v}_{s'}(p')\gamma^5 u_s(p)\right]\left[\bar{u}_r(k)\gamma^5 v_{r'}(k')\right] \times\left[\frac{\bar{g}^2+\bar{g}'^2}{4}+\frac{\bar{g}\bar{g}'\epsilon}{2}+\frac{1}{4}v^2(\bar{g}^2+\bar{g}'^2)(C_e^{\varphi l1}+C_e^{\varphi l3}-C_e^{\varphi e}+C_\mu^{\varphi l1}+C_\mu^{\varphi l3}-C_\mu^{\varphi e})\right], \tag{3.75}$$

$$\mathcal{M}_{G_0}^\xi = \frac{im_e m_\mu}{q^2-\xi_Z\bar{m}_Z^2}\left[\bar{v}_{s'}(p')\gamma^5 u_s(p)\right]\left[\bar{u}_r(k)\gamma^5 v_{r'}(k')\right] \times\left[\frac{1}{v^2}-\frac{C^{\varphi D}}{2}+(C_e^{\varphi l1}+C_e^{\varphi l3}-C_e^{\varphi e}+C_\mu^{\varphi l1}+C_\mu^{\varphi l3}-C_\mu^{\varphi e})\right]. \tag{3.76}$$

Let's now temporarily denote $(C_e^{\varphi l1}+C_e^{\varphi l3}-C_e^{\varphi e}+C_\mu^{\varphi l1}+C_\mu^{\varphi l3}-C_\mu^{\varphi e})$ as A.

$$\mathcal{M}_Z^\xi+\mathcal{M}_{G_0}^\xi \sim \frac{1}{q^2-\xi_Z\bar{m}_Z^2}\left[\frac{1-\xi_Z}{q^2-\bar{m}_Z^2}\left(\frac{\bar{g}^2+\bar{g}'^2}{4}+\frac{\bar{g}\bar{g}'\epsilon}{2}+\frac{1}{4}v^2(\bar{g}^2+\bar{g}'^2)A\right)+\frac{1}{v^2}-\frac{C^{\varphi D}}{2}+A\right]. \tag{3.77}$$

Using the form of m_Z in Eq.(3.40) and the property $C^2_{\text{Wilson}}=0$ of the Wilson coefficients, we have

$$\begin{aligned} m_Z C_{Wilson} &= \frac{1}{2}\sqrt{\bar{g}^2+\bar{g}'^2}v\left(1+\frac{\epsilon\bar{g}\bar{g}'}{\bar{g}^2+\bar{g}'^2}\right)\left(1+\frac{1}{4}C^{\varphi D}v^2\right)C_{Wilson} \\ &= \frac{1}{2}\sqrt{\bar{g}^2+\bar{g}'^2}vC_{Wilson}. \end{aligned} \tag{3.78}$$

Eq(3.77) will be rewritten as

$$\begin{aligned} &\mathcal{M}_Z^\xi+\mathcal{M}_{G_0}^\xi \\ &\sim \frac{1}{q^2-\xi_Z\bar{m}_Z^2}\left\{A\left[\frac{1-\xi_Z}{q^2-\bar{m}_Z^2}m_Z^2+1\right]+\frac{1}{v^2}-\frac{C^{\varphi D}}{2}\right. \\ &\qquad\left.+\frac{1-\xi_Z}{q^2-\bar{m}_Z^2}\frac{1}{v^2}\left[\frac{\bar{g}^2+\bar{g}'^2}{4}v^2+\frac{\bar{g}\bar{g}'\epsilon}{2}v^2+\frac{1}{8}(\bar{g}^2+\bar{g}'^2)v^4C^{\varphi D}-\frac{1}{8}(\bar{g}^2+\bar{g}'^2)v^4C^{\varphi D}\right]\right\} \\ &= \frac{1}{q^2-\xi_Z\bar{m}_Z^2}\left\{A\frac{q^2-\xi_Z m_Z^2}{q^2-\bar{m}_Z^2}+\frac{1}{v^2}-\frac{C^{\varphi D}}{2}+\frac{1-\xi_Z}{q^2-\bar{m}_Z^2}\frac{1}{v^2}\left[m_Z^2-\frac{1}{8}(\bar{g}^2+\bar{g}'^2)v^4C^{\varphi D}\right]\right\} \\ &= \frac{1}{q^2-\xi_Z\bar{m}_Z^2}\left\{A\frac{q^2-\xi_Z m_Z^2}{q^2-\bar{m}_Z^2}+\frac{1}{v^2}\left[1+\frac{1-\xi_Z}{q^2-\bar{m}_Z^2}m_Z^2-\frac{C^{\varphi D}}{2}v^2\left(1+\frac{1-\xi_Z}{q^2-\bar{m}_Z^2}m_Z^2\right)\right]\right\} \\ &= \frac{1}{q^2-\xi_Z\bar{m}_Z^2}\left\{A\frac{q^2-\xi_Z m_Z^2}{q^2-\bar{m}_Z^2}+\frac{1}{v^2}\left[\frac{q^2-\xi_Z m_Z^2}{q^2-\bar{m}_Z^2}-\frac{C^{\varphi D}}{2}v^2\frac{q^2-\xi_Z m_Z^2}{q^2-\bar{m}_Z^2}\right]\right\} \\ &= \frac{1}{q^2-\bar{m}_Z^2}\left\{A+\frac{1}{v^2}\left[1-\frac{C^{\varphi D}}{2}v^2\right]\right\}. \end{aligned} \tag{3.79}$$

That is what we expected when the Feynman amplitude is independent on gauge fixing-parameters.

3.6 Z boson decay width

Another quantity that we are interesting in is the decay width of Z boson in SMEFT. In the previous section, we can see that the Z propagator have the term $q^2 - m_Z^2$ in the denominator and it leads to divergence in the total cross-section. To avoid that, the solution was introduced in Chap. 2 that we used the Breit-Wigner propagator instead. But the Breit-Wigner propagator does contain the decay width of Z boson, that is the first reason for this section. The second is that we already have the accurate value of Z decay width by experiment, thus it is the great quantity for us to constraint the value of Wilson coefficients. We shall use the coupling of Z with fermions in [5], some of them were checked in Sec.(3.3).

We still work on the Center of Mass frame, our process is Z decay to two fermions process $Z \to \bar{f} + f$, where Z, f, $\bar{f}$ have the the momentum $q = (2E, \vec{0})$, $k_1 = (E, \vec{k})$, and $k_2 = (E, -\vec{k})$, respectively. The Feynman amplitude is then of the form

$$\mathcal{M}_f = \bar{u}(k_1) g_f^\mu v(k_2) \varepsilon_{\lambda\mu}(q) \tag{3.80}$$

and the squared amplitude result reads

$$|\mathcal{M}_f|^2 = \frac{1}{3}\mathcal{M}_f \mathcal{M}_f^\dagger. \tag{3.81}$$

We do not have to multiply $\frac{1}{4}$ to squared amplitude since we accept all spin states of the two final particles. The $\frac{1}{3}$ factor stand for the spin states average of Z boson. In the couplings of Z boson with fermions in paper [5], we will ignore all operators which are not relevant to our process $e^+ + e^- \to \mu^+ + \mu^-$ and only take the operator $Q^{\varphi WB}$ into account. Then will obtain the results for each fermion

$$|\mathcal{M}_\nu|^2 = \frac{\bar{g}^2 + \bar{g}'^2}{6} m_Z^2 + \frac{\bar{g}\bar{g}' v^2}{3} m_Z^2 \mathrm{ReC}^{\varphi \mathrm{WB}}, \tag{3.82}$$

$$\begin{aligned} |\mathcal{M}_e|^2 = {} & \frac{1}{(\bar{g}^2 + \bar{g}'^2)} \left(\frac{(\bar{g}'^2 - \bar{g}^2)^2}{6} + \frac{2}{3}\bar{g}'^4 \right) m_Z^2 - \frac{\bar{g}\bar{g}'(\bar{g}'^2 - \bar{g}^2)^2 v^2}{3(\bar{g}^2 + \bar{g}'^2)^2} m_Z^2 \mathrm{ReC}^{\varphi \mathrm{WB}} \\ & + \frac{4\bar{g}^3 \bar{g}'^3 v^2}{3(\bar{g}^2 + \bar{g}'^2)^2} m_Z^2 \mathrm{ReC}^{\varphi \mathrm{WB}}, \end{aligned} \tag{3.83}$$

$$\begin{aligned} |\mathcal{M}_u|^2 = {} & \frac{1}{\bar{g}^2 + \bar{g}'^2} \left[\frac{\bar{g}'^4}{54}(17m_Z^2 + 7m_f^2) - \frac{\bar{g}^2 \bar{g}'^2}{9}(m_Z^2 + 11m_f^2) + \frac{\bar{g}^4}{6}(m_Z^2 - m_f^2) \right] \\ & - \frac{\bar{g}\bar{g}'^5 v^2}{9(\bar{g}^2 + \bar{g}'^2)^2}((m_Z^2 + 11m_f^2)\mathrm{ReC}^{\varphi \mathrm{WB}} + \frac{\mathrm{\bar{g}^3 \bar{g}'^3 v^2}}{27(\mathrm{\bar{g}^2 + \bar{g}'^2})^2}(26\mathrm{m_Z^2} - 2\mathrm{m_f^2})\mathrm{ReC}^{\varphi \mathrm{WB}} \\ & - \frac{\bar{g}^5 \bar{g}' v^2}{9(\bar{g}^2 + \bar{g}'^2)^2}(m_Z^2 + 11m_f^2)\mathrm{ReC}^{\varphi \mathrm{WB}}, \end{aligned} \tag{3.84}$$

$$|\mathcal{M}_d|^2 = \frac{1}{\bar{g}^2+\bar{g}'^2}\left[\frac{\bar{g}'^4}{54}(5m_Z^2-17m_f^2)+\frac{\bar{g}^2\bar{g}'^2}{9}(m_Z^2-7m_f^2)+\frac{\bar{g}^4}{6}(m_Z^2-m_f^2)\right]$$
$$+\frac{\bar{g}\bar{g}'^5v^2}{9(\bar{g}^2+\bar{g}'^2)^2}(m_Z^2-7m_f^2)\mathrm{ReC}^{\varphi\mathrm{WB}}+\frac{\bar{\mathrm{g}}^3\bar{\mathrm{g}}'^3\mathrm{v}^2}{27(\bar{\mathrm{g}}^2+\bar{\mathrm{g}}'^2)^2}(14\mathrm{m}_\mathrm{Z}^2-26\mathrm{m}_\mathrm{f}^2)\mathrm{ReC}^{\varphi\mathrm{WB}}$$
$$+\frac{\bar{g}^5\bar{g}'v^2}{9(\bar{g}^2+\bar{g}'^2)^2}(m_Z^2-7m_f^2)\mathrm{ReC}^{\varphi\mathrm{WB}}. \tag{3.85}$$

We have used the relations $|\vec{k}|^2 = E^2 - m_f^2$ and $q^2 = m_Z^2 = 4E^2$. Applying the the differential decay width in the paper [2]

$$\frac{d\Gamma_f}{d\Omega} = \frac{1}{64\pi^2 m_Z}\sqrt{1-\frac{4m_f^2}{m_Z^2}}|\mathcal{M}_f|^2, \tag{3.86}$$

and ingerating over θ and ϕ, we have

$$\Gamma_f = \frac{1}{16\pi m_Z}\sqrt{1-\frac{4m_f^2}{m_Z^2}}|\mathcal{M}_f|^2. \tag{3.87}$$

Because of the term $\sqrt{1-\frac{4m_f^2}{m_Z^2}}$, so Z boson can not decay to particles with have $2m_f > m_Z$. Then we will calculate the total decay width of Z boson with the absence of top quark

$$\Gamma = 3\Gamma_\nu + \Gamma_e + \Gamma_\mu + \Gamma_\tau + 3\Gamma_u + 3\Gamma_d + 3\Gamma_c + 3\Gamma_s + 3\Gamma_b \tag{3.88}$$

The result of Z decay width in SMEFT is a function of $C^{\varphi WB}$, it's too long and there is no need to write it down, but when I set $C^{\varphi WB} = 0$, the result is 2.44402 GeV, which is a bit far from the experimental value 2.4952 GeV. Therefore, I decided to use the result of Nghia's thesis[8], he calculated the next-to-leading order (NLO) with QCD corrections to the decay $Z \to b\bar{b}$, his result reads

$$\Gamma_{QCD} = \Gamma_0\left(1+\frac{\alpha_s}{\pi}\right), \tag{3.89}$$

where Γ_{QCD}, Γ_0 are the decay width of Z boson at NLO and tree level of SM, respectively. while α_s stand for the strong interaction coupling constant. Eq.(3.89) can apply for all $Z \to q\bar{q}$ channels so I have used the NLOQCD factor for all quarks. I also checked both cases of massless and massive fermions, and it turned out that the mass of fermion has weak effect on our result. So now to easily compare with Nghia's thesis, in which he has used the approximation $m_b = 0$, I will set $m_f = 0$ for all particles. My result for the decay width of Z to $b\bar{b}$ at NLOQCD based on Eq.(3.89) is 0.385748 GeV which agree with that of Nghia and the paper [9]
Moreover, I also use the NLOQCD decay width for SMEFT, since QCD correction factor is the same for the SM and SMEFT when only operator $Q^{\varphi WB}$ is kept. Fig.(3.2) shows the values of Z boson decay width with respect to $\bar{C}^{\varphi WB}$ in many models, the way we chose the value of $\bar{C}^{\varphi WB}$ will be discuss more in Sec.(3.9). The input parameters, which I chose from Particle Data Group [10], are

$$m_Z = 91.1876(\mathrm{GeV}), \qquad m_W = 80.385(\mathrm{GeV}),$$
$$G_F = 1.1663787.10^{-5}(\mathrm{GeV}^{-2}), \qquad \alpha_s = \alpha_s(m_Z) = 0.1181.$$

Note again that I have used the approximation $m_f = 0$ for all fermions. Since our result in SMEFT is a function of Wilson coefficients, so we can change the value of those coefficients to get the result close to the experimental value, this case is changing $C^{\varphi WB}$.

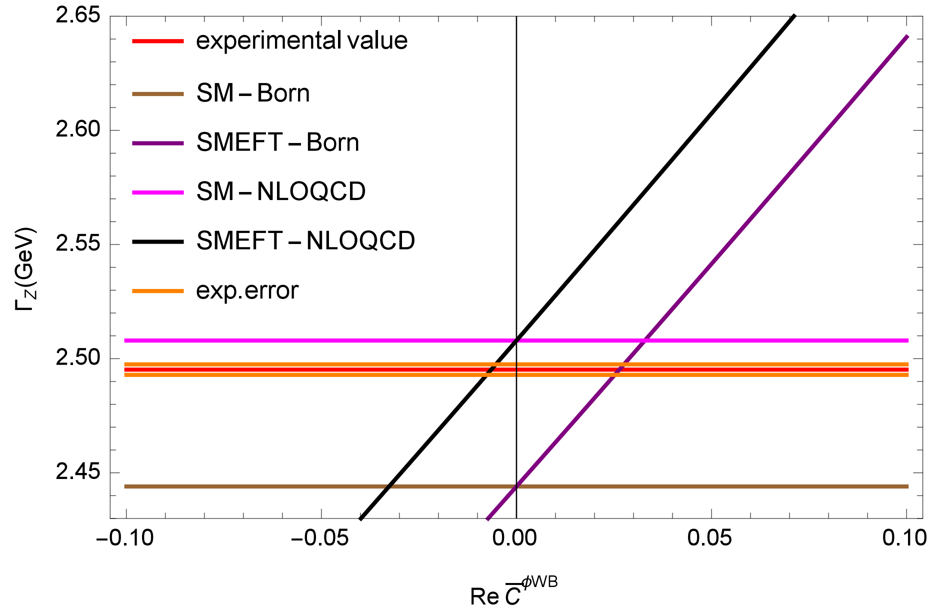


Figure 3.2: The values of Z decay width

3.7 Feynman amplitude

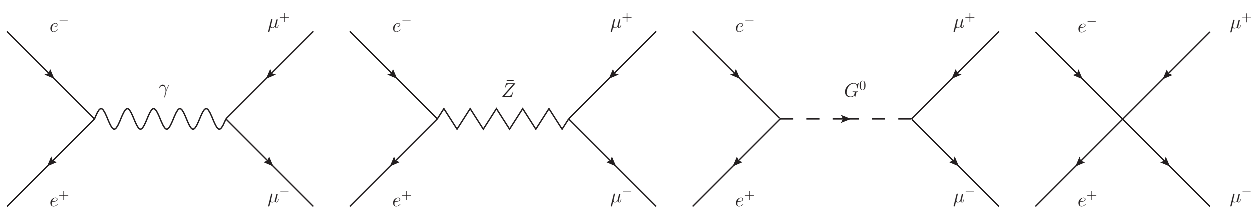


Figure 3.3: Feynman diagrams of process $e^+ + e^- \to \mu^+\mu^-$ in SMEFT

The notations of spins and momentums is the same with previous chapters. But for easy following, we now mention again that we have denoted the four momenta and spin indices of e^-, e^+, μ^-, μ^+ to be $(p, s), (p', s'), (k, r), (k', r')$, respectively. $q = p + p' = k + k'$ is the momentum of the mediators. The Feynman amplitudes for diagrams in Fig.(3.3) reads

$$\mathcal{M}_\gamma = \bar{v}_{s'}(p')\left[\frac{i\bar{g}\bar{g}'}{\sqrt{\bar{g}^2+\bar{g}'^2}}\gamma^\mu - \frac{i\bar{g}^2\bar{g}'^2v^2}{(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\gamma^\mu - \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eW*}\sigma^{\mu\sigma}P_L + C_e^{eW}\sigma^{\mu\sigma}P_R\right)\right.$$
$$\left. + \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eB*}\sigma^{\mu\sigma}P_L + C_e^{eB}\sigma^{\mu\sigma}P_R\right)\right] u_s(p)\left[\frac{-i}{q^2}\left(g_{\mu\nu} - (1-\xi_\gamma)\frac{q_\mu q_\nu}{q^2}\right)\right]$$
$$\bar{u}_r(k)\left[\frac{i\bar{g}\bar{g}'}{\sqrt{\bar{g}^2+\bar{g}'^2}}\gamma^\nu - \frac{i\bar{g}^2\bar{g}'^2v^2}{(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\gamma^\nu + \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eW*}\sigma^{\nu\rho}P_L + C_\mu^{eW}\sigma^{\nu\rho}P_R\right)\right.$$
$$\left. - \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eB*}\sigma^{\nu\rho}P_L + C_\mu^{eB}\sigma^{\nu\rho}P_R\right)\right] v_{r'}(k') \tag{3.90}$$

$$\mathcal{M}_Z =$$
$$\bar{v}_{s'}(p')\left[\frac{i}{\sqrt{\bar{g}^2+\bar{g}'^2}}\left((\bar{g}^2-\bar{g}'^2)\gamma^\mu P_L - 2\bar{g}'^2\gamma^\mu P_R\right) + \frac{i\bar{g}\bar{g}'v^2}{2(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\left((\bar{g}'^2-\bar{g}^2)\gamma^\mu P_L - 2\bar{g}^2\gamma^\mu P_R\right)\right.$$
$$- \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eW*}\sigma^{\mu\sigma}P_L + C_e^{eW}\sigma^{\mu\sigma}P_R\right) - \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\sigma\left(C_e^{eB*}\sigma^{\mu\sigma}P_L + C_e^{eB}\sigma^{\mu\sigma}P_R\right)$$
$$\left. + \frac{1}{2}iv^2\sqrt{\bar{g}^2+\bar{g}'^2}\left(C_e^{\varphi l1}\gamma^\mu P_L + C_e^{\varphi l3}\gamma^\mu P_L + C_e^{\varphi e}\gamma^\mu P_R\right)\right] u_s(p)$$
$$\times\left[\frac{-ig_{\mu\nu}}{q^2 - \bar{m}_Z^2 + i\bar{\Gamma}_Z m_z}\right]$$
$$\times\bar{u}_r(k)\left[\frac{i}{\sqrt{\bar{g}^2+\bar{g}'^2}}\left((\bar{g}^2-\bar{g}'^2)\gamma^\nu P_L - 2\bar{g}'^2\gamma^\nu P_R\right) + \frac{i\bar{g}\bar{g}'v^2}{2(\bar{g}^2+\bar{g}'^2)^{3/2}}C^{\varphi WB}\left((\bar{g}'^2-\bar{g}^2)\gamma^\nu P_L - 2\bar{g}^2\gamma^\nu P_R\right)\right.$$
$$+ \frac{\sqrt{2}\bar{g}v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eW*}\sigma^{\nu\rho}P_L + C_\mu^{eW}\sigma^{\nu\rho}P_R\right) + \frac{\sqrt{2}\bar{g}'v}{\sqrt{\bar{g}^2+\bar{g}'^2}}q_\rho\left(C_\mu^{eB*}\sigma^{\nu\rho}P_L + C_\mu^{eB}\sigma^{\nu\rho}P_R\right)$$
$$\left. + \frac{1}{2}iv^2\sqrt{\bar{g}^2+\bar{g}'^2}\left(C_\mu^{\varphi l1}\gamma^\nu P_L + C_\mu^{\varphi l3}\gamma^\nu P_L + C_\mu^{\varphi e}\gamma^\nu P_R\right)\right] v_{r'}(k') \tag{3.91}$$

$$\mathcal{M}_{G_0} = \bar{v}_{s'}(p')\left[\frac{m_e}{v}\gamma^5 - \frac{v}{4}C^{\varphi D}m_e\gamma^5 + v\not{q}\left(P_L C_e^{\varphi l1} + P_L C_e^{\varphi l1} + P_R C_e^{\varphi e}\right)\right] u_s(p)\left[\frac{i}{q^2 - \xi_Z\bar{m}_Z^2}\right]$$
$$\bar{u}_r(k)\left[\frac{m_\mu}{v}\gamma^5 - \frac{v}{4}C^{\varphi D}m_\mu\gamma^5 - v\not{q}\left(P_L C_e^{\varphi l1} + P_L C_e^{\varphi l1} + P_R C_e^{\varphi e}\right)\right] v_{r'}(k') \tag{3.92}$$

$$\mathcal{M}_4 = 4iC_{e\mu}^{ll}\bar{v}_{s'}(p')(\gamma_\mu P_L)u_s(p)\bar{u}_r(k)(\gamma^\mu P_L)v_{r'}(k') + 4iC_{e\mu}^{ee}\bar{v}_{s'}(p')(\gamma_\mu P_R)u_s(p)\bar{u}_r(k)(\gamma^\mu P_R)v_{r'}(k')$$
$$+ iC_{e\mu}^{le}\bar{v}_{s'}(p')(\gamma_\mu P_L)u_s(p)\bar{u}_r(k)(\gamma^\mu P_R)v_{r'}(k') + iC_{\mu e}^{le}\bar{v}_{s'}(p')(\gamma_\mu P_R)u_s(p)\bar{u}_r(k)(\gamma^\mu P_L)v_{r'}(k')$$
$$- 2iC_{e\mu}^{le}\bar{v}_{s'}(p')P_R u_s(p)\bar{u}_r(k)P_L v_{r'}(k') - 2iC_{\mu e}^{le}\bar{v}_{s'}(p')P_L u_s(p)\bar{u}_r(k)P_R v_{r'}(k') \tag{3.93}$$

Notice that we used the Breit-Wigner propergator of Z boson to avoid the divergence in our results.

3.8 Result of squared-amplitude

I have used FORM[2] to calculate this squared-amplitude. In this result, we obtain some great things that we expected are the independence on gauge-fixing parameters, the squared-amplitude is the real number athough the initial set of Wilson coefficients is complex numbers. And the most important result is that our distributions later are completely the same with SM when we set all Wilson coefficients equal to zero.
With the approximation $m_e = m_\mu = 0$, we will obtain $\mathcal{M}_{G_0} = 0$, thus

$$|\mathcal{M}_{G_0}|^2 = \mathcal{M}_\gamma \mathcal{M}_{G_0}^\dagger = \mathcal{M}_Z \mathcal{M}_{G_0}^\dagger = 0 \tag{3.94}$$

The other results are

$$\frac{1}{4}|\mathcal{M}_\gamma|^2 = \left[\frac{16E^4\bar{g}^4\bar{g}'^4}{q^4(\bar{g}^2+\bar{g}'^2)^2} - \frac{64v^2E^4\bar{g}^5\bar{g}'^5}{q^4(\bar{g}^2+\bar{g}'^2)^3}\mathrm{ReC}^{\varphi\mathrm{WB}}\right](1+\cos^2\theta). \tag{3.95}$$

$$\frac{1}{4}|\mathcal{M}_Z|^2 = \frac{v^2E^4}{(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2 m_Z^2}\left[C_{Z1}(1+\cos^2\theta) + \mathrm{C_{Z2}cos}\theta\right], \tag{3.96}$$

in which we have

$$\begin{aligned}
C_{Z1}(\sqrt{s}) =& \frac{1}{v^2}\left[\frac{4\bar{g}'^8}{(\bar{g}^2+\bar{g}'^2)^2} + \frac{2\bar{g}'^4(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}^2+\bar{g}'^2)^2} + \frac{(\bar{g}'^2-\bar{g}^2)^4}{4(\bar{g}^2+\bar{g}'^2)^2}\right] \\
&- \frac{4\bar{g}'^6}{(\bar{g}^2+\bar{g}'^2)}\mathrm{Re}(\mathrm{C_e^{\varphi e}} + \mathrm{C}_\mu^{\varphi\mathrm{e}}) - \frac{2\bar{\mathrm{g}}'^4(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)}{(\bar{\mathrm{g}}^2+\bar{\mathrm{g}}'^2)}\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{l3}} + \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C_e^{\varphi l3}} + \mathrm{C_e^{\varphi l1}}) \\
&- \frac{\bar{g}'^2(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)}\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{e}} + \mathrm{C_e^{\varphi e}}) - \frac{(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)^3}{2(\bar{\mathrm{g}}'^2+\bar{\mathrm{g}}^2)}\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{l3}} + \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C_e^{\varphi l3}} + \mathrm{C_e^{\varphi l1}}) \\
&+ \left(\frac{16\bar{g}^3\bar{g}'^7}{(\bar{g}'^2+\bar{g}^2)^3} - \frac{4\bar{g}\bar{g}'^5(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)^3} + \frac{4\bar{g}^3\bar{g}'^3(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)^3} - \frac{\bar{g}\bar{g}'(\bar{g}'^2-\bar{g}^2)^4}{(\bar{g}'^2+\bar{g}^2)^3}\right)\mathrm{ReC}^{\varphi\mathrm{WB}},
\end{aligned} \tag{3.97}$$

$$\begin{aligned}
C_{Z2}(\sqrt{s}) =& \frac{1}{v^2}\left[\frac{8\bar{g}'^8}{(\bar{g}^2+\bar{g}'^2)^2} - \frac{4\bar{g}'^4(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}^2+\bar{g}'^2)^2} + \frac{(\bar{g}'^2-\bar{g}^2)^4}{2(\bar{g}^2+\bar{g}'^2)^2}\right] \\
&- \frac{8\bar{g}'^6}{(\bar{g}^2+\bar{g}'^2)}\mathrm{Re}(\mathrm{C_e^{\varphi e}} + \mathrm{C}_\mu^{\varphi\mathrm{e}}) + \frac{4\bar{\mathrm{g}}'^4(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)}{(\bar{\mathrm{g}}^2+\bar{\mathrm{g}}'^2)}\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{l3}} + \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C_e^{\varphi l3}} + \mathrm{C_e^{\varphi l1}}) \\
&+ \frac{2\bar{g}'^2(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)}\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{e}} + \mathrm{C_e^{\varphi e}}) - \frac{(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)^3}{(\bar{\mathrm{g}}'^2+\bar{\mathrm{g}}^2)}\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{l3}} + \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C_e^{\varphi l3}} + \mathrm{C_e^{\varphi l1}}) \\
&+ \left(\frac{32\bar{g}^3\bar{g}'^7}{(\bar{g}'^2+\bar{g}^2)^3} + \frac{8\bar{g}\bar{g}'^5(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)^3} - \frac{8\bar{g}^3\bar{g}'^3(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)^3} - \frac{2\bar{g}\bar{g}'(\bar{g}'^2-\bar{g}^2)^4}{(\bar{g}'^2+\bar{g}^2)^3}\right)\mathrm{ReC}^{\varphi\mathrm{WB}}.
\end{aligned} \tag{3.98}$$

[2]FORM is a symbolic manipulation system which is used by many HEP physicists. For more information, please find [11] [12] [13] [14] or read here https://www.nikhef.nl/ form/license/license.html

$$\frac{1}{4}(\mathcal{M}_\gamma \mathcal{M}_Z^\dagger + \mathcal{M}_Z \mathcal{M}_\gamma^\dagger) = \frac{v^2 E^4}{q^2[(q^2-\bar{m}_Z^2)^2 + \bar{\Gamma}_Z^2 m_Z^2]} \left(C_{\gamma Z1}(1+\cos^2\theta) + \mathrm{C}_{\gamma Z2}\cos\theta \right), \tag{3.99}$$

where

$$\begin{aligned} C_{\gamma Z1}(\sqrt{s}) =& \frac{2(q^2-\bar{m}_Z^2)}{v^2}\left[\frac{4\bar{g}^2\bar{g}'^6}{(\bar{g}'^2+\bar{g}^2)^2} + \frac{4\bar{g}^2\bar{g}'^4(\bar{g}'^2-\bar{g}^2)}{(\bar{g}'^2+\bar{g}^2)^2} + \frac{\bar{g}^2\bar{g}'^2(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)^2}\right] \\ &- \frac{\bar{g}^2\bar{g}'^2(3\bar{g}'^2-\bar{g}^2)}{(\bar{g}'^2+\bar{g}^2)}\Big[(q^2-\bar{m}_Z^2)2\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{e}} + \mathrm{C}_\mu^{\varphi\mathrm{l3}} + \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{e}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l3}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l1}}) \\ &\qquad + 2\bar{\Gamma}_Z\bar{m}_Z\mathrm{Im}(\mathrm{C}_\mu^{\varphi\mathrm{e}} + \mathrm{C}_\mu^{\varphi\mathrm{l3}} + \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{e}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l3}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l1}})\Big] \\ &+ \frac{8\bar{g}^3\bar{g}'^3}{(\bar{g}'^2+\bar{g}^2)^2}\left(3\bar{g}'^2-\bar{g}^2\right)\left[\bar{\Gamma}_Z\bar{m}_Z\mathrm{Im}\mathrm{C}^{\varphi\mathrm{WB}} - \frac{2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)}{(\bar{\mathrm{g}}'^2+\bar{\mathrm{g}}^2)}(\mathrm{q}^2-\bar{\mathrm{m}}_\mathrm{Z}^2)\mathrm{Re}\mathrm{C}^{\varphi\mathrm{WB}}\right], \end{aligned} \tag{3.100}$$

$$\begin{aligned} C_{\gamma Z2}(\sqrt{s}) =& \frac{2(q^2-\bar{m}_Z^2)}{v^2}\left[\frac{8\bar{g}^2\bar{g}'^6}{(\bar{g}'^2+\bar{g}^2)^2} - \frac{8\bar{g}^2\bar{g}'^4(\bar{g}'^2-\bar{g}^2)}{(\bar{g}'^2+\bar{g}^2)^2} + \frac{2\bar{g}^2\bar{g}'^2(\bar{g}'^2-\bar{g}^2)^2}{(\bar{g}'^2+\bar{g}^2)^2}\right] \\ &- \frac{2\bar{g}^2\bar{g}'^2(\bar{g}'^2+\bar{g}^2)}{(\bar{g}'^2+\bar{g}^2)}\Big[(q^2-\bar{m}_Z^2)2\mathrm{Re}(\mathrm{C}_\mu^{\varphi\mathrm{e}} - \mathrm{C}_\mu^{\varphi\mathrm{l3}} - \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{e}} - \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l3}} - \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l1}}) \\ &\qquad + 2\bar{\Gamma}_Z\bar{m}_Z\mathrm{Im}(\mathrm{C}_\mu^{\varphi\mathrm{e}} - \mathrm{C}_\mu^{\varphi\mathrm{l3}} - \mathrm{C}_\mu^{\varphi\mathrm{l1}} + \mathrm{C}_\mathrm{e}^{\varphi\mathrm{e}} - \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l3}} - \mathrm{C}_\mathrm{e}^{\varphi\mathrm{l1}})\Big] \\ &+ \frac{8\bar{g}^3\bar{g}'^3}{(\bar{g}'^2+\bar{g}^2)}\bar{\Gamma}_Z\bar{m}_Z 2\mathrm{Im}\mathrm{C}^{\varphi\mathrm{WB}}. \end{aligned} \tag{3.101}$$

$$\frac{1}{4}(\mathcal{M}_4 \mathcal{M}_\gamma^\dagger + \mathcal{M}_\gamma \mathcal{M}_4^\dagger) = \frac{E^4\bar{g}^2\bar{g}'^2}{q^2(\bar{g}'^2+\bar{g}^2)}\left[C_{4\gamma1}(1+\cos^2\theta) + \mathrm{C}_{4\gamma2}\cos\theta\right], \tag{3.102}$$

where

$$C_{4\gamma1}(\sqrt{s}) = 8\mathrm{Re}(\mathrm{C}_{\mu\mathrm{e}}^{\mathrm{le}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{le}}) + 32\mathrm{Re}(\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ee}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ll}}), \tag{3.103}$$

$$C_{4\gamma2}(\sqrt{s}) = -16\mathrm{Re}(\mathrm{C}_{\mu\mathrm{e}}^{\mathrm{le}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{le}}) + 64\mathrm{Re}(\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ee}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ll}}). \tag{3.104}$$

$$\frac{1}{4}(\mathcal{M}_4 \mathcal{M}_Z^\dagger + \mathcal{M}_Z \mathcal{M}_4^\dagger) = \frac{E^4}{[(q^2-\bar{m}_Z^2)^2 + \bar{\Gamma}_Z^2\bar{m}_Z^2]}\left[C_{4Z1}(1+\cos^2\theta) + \mathrm{C}_{4\mathrm{Z}2}\cos\theta\right], \tag{3.105}$$

where

$$\begin{aligned} C_{4Z1}(\sqrt{s}) =& \frac{4}{(\bar{g}'^2+\bar{g}^2)}\Big\{(q^2-\bar{m}_Z^2)\left[8\bar{g}'^4\mathrm{Re}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ee}} + \bar{\mathrm{g}}'^2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)\mathrm{Re}(\mathrm{C}_{\mu\mathrm{e}}^{\mathrm{le}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{le}}) + 2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)^2\mathrm{Re}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ll}}\right] \\ &- \bar{\Gamma}_Z\bar{m}_Z\left[8\bar{g}'^4\mathrm{Im}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ee}} + \bar{\mathrm{g}}'^2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)\mathrm{Im}(\mathrm{C}_{\mu\mathrm{e}}^{\mathrm{le}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{le}}) + 2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)^2\mathrm{Im}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ll}}\right]\Big\}, \end{aligned} \tag{3.106}$$

$$\begin{aligned} C_{4Z2}(\sqrt{s}) =& \frac{4}{(\bar{g}'^2+\bar{g}^2)}\Big\{(q^2-\bar{m}_Z^2)\left[16\bar{g}'^4\mathrm{Re}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ee}} - 2\bar{\mathrm{g}}'^2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)\mathrm{Re}(\mathrm{C}_{\mu\mathrm{e}}^{\mathrm{le}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{le}}) + 4(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)^2\mathrm{Re}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ll}}\right] \\ &- \bar{\Gamma}_Z\bar{m}_Z\left[16\bar{g}'^4\mathrm{Im}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ee}} - 2\bar{\mathrm{g}}'^2(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)\mathrm{Im}(\mathrm{C}_{\mu\mathrm{e}}^{\mathrm{le}} + \mathrm{C}_{\mathrm{e}\mu}^{\mathrm{le}}) + 4(\bar{\mathrm{g}}'^2-\bar{\mathrm{g}}^2)^2\mathrm{Im}\mathrm{C}_{\mathrm{e}\mu}^{\mathrm{ll}}\right]\Big\}. \end{aligned} \tag{3.107}$$

Now we can write the squared-amplitude in a compact form

$$|\mathcal{M}_0|^2 = A(\sqrt{s})(1+\cos^2\theta) + \mathrm{B}(\sqrt{\mathrm{s}})\mathrm{cos}\theta, \tag{3.108}$$

in which we have

$$\begin{aligned} A(\sqrt{s}) = &\left[\frac{16E^4\bar{g}^4\bar{g}'^4}{q^4(\bar{g}^2+\bar{g}'^2)^2} - \frac{64v^2E^4\bar{g}^5\bar{g}'^5}{q^4(\bar{g}^2+\bar{g}'^2)^3}\mathrm{ReC}^{\varphi\mathrm{WB}}\right] + \frac{v^2E^4}{(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2 m_Z^2}C_{Z1}(\sqrt{s}) \\ &+ \frac{v^2E^4}{q^2[(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2 m_Z^2]}C_{\gamma Z1}(\sqrt{s}) + \frac{E^4\bar{g}^2\bar{g}'^2}{q^2(\bar{g}'^2+\bar{g}^2)}C_{4\gamma 1}(\sqrt{s}) + \frac{E^4}{[(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2\bar{m}_Z^2]}C_{4Z1}(\sqrt{s}), \end{aligned} \tag{3.109}$$

$$\begin{aligned} B(\sqrt{s}) = &\frac{v^2E^4}{(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2 m_Z^2}C_{Z2}(\sqrt{s}) + \frac{v^2E^4}{q^2[(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2 m_Z^2]}C_{\gamma Z2}(\sqrt{s}) \\ &+ \frac{E^4\bar{g}^2\bar{g}'^2}{q^2(\bar{g}'^2+\bar{g}^2)}C_{4\gamma 2}(\sqrt{s}) + \frac{E^4}{[(q^2-\bar{m}_Z^2)^2+\bar{\Gamma}_Z^2\bar{m}_Z^2]}C_{4Z2}(\sqrt{s}). \end{aligned} \tag{3.110}$$

3.9 Some distributions for the process $e^+ + e^- \to \mu^+ + \mu^-$

Before going to this section, some assumptions are needed for us. First, it is convenient to set all Wilson coefficients that we used for field rescaling equal to zero except for $C^{\varphi WB}$ since the decay width and the mass of Z boson are depend on that coefficient. The next step is classifying all coefficients in groups, if you pay attention to the squared amplitude, you can see that there are some operators always appear with each other as a sum. Therefore they have the same contribution to our process. These groups are $[C^{\varphi WB}]$. $[C^{\varphi e}_e, C^{\varphi e}_\mu]$, $[C^{\varphi l1}_e, C^{\varphi l3}_e, C^{\varphi l1}_\mu, C^{\varphi l3}_\mu]$, $[C^{le}_{e\mu}, C^{le}_{\mu e}]$, $[C^{ee}_{e\mu}]$, $[C^{ll}_{e\mu}]$. Since the coefficients in the same group have the same contribution, so we only take one representative operator for each group into account. Now we have six coefficients to consider, but it is hard to put many lines in one plot, so I will again separate them into two group

- **The Higgs group**: The coefficients which their corresponding operators contain Higgs fields: $C^{\varphi WB}$, $C^{\varphi e}$ and $C^{\varphi l1}$.
- **The four-fermion group**: The coefficients which were contained in four-fermion operators: $C^{le}_{e\mu}$, $C^{ee}_{e\mu}$ and $C^{ll}_{e\mu}$.

The important notations that we will use now are the dimensionless coefficients which are denoted by

$$\bar{C}_{\mathrm{Wilson}} = \frac{v^2}{\Lambda^2}\mathrm{Re}(\mathrm{C_{Wilson}}), \qquad \mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = \frac{v^2}{\Lambda^2}\mathrm{Im}(\mathrm{C_{Wilson}}), \tag{3.111}$$

and their possible values are from -0.2 to 0.2, note that in this thesis we only consider the real part of Wilson coefficients. In most of the cases, I will choose the value of 0.1 for my calculations. To see how different of SMEFT from SM, we introduce the ratio $\frac{SMEFT}{SM}$. As we expect, it must equal to 1 when all Wilson coefficients are set to be zero. The input parameters for all below graphics are

$$m_Z = 91.1876(\mathrm{GeV}), \qquad m_W = 80.385(\mathrm{GeV}), \qquad G_F = 1.1663787.10^{-5}(\mathrm{GeV}^{-2}). \tag{3.112}$$

which were chosen from Particle Data Group [10]. Note again that we use the approximation $m_e = m_\mu = 0$.

3.9.1 Total cross-section

From Eq.(1.12), the cross-section of our process is of the form

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\mathbf{k}|}{|\mathbf{p}|}|\mathcal{M}_0|^2 = \frac{1}{64\pi^2 s}\left[A(\sqrt{s})(1+\cos^2\theta) + \mathrm{B}(\sqrt{\mathrm{s}})\cos\theta\right]. \tag{3.113}$$

The total cross-section then reads

$$\sigma = \int \frac{d\sigma}{d\Omega} d\cos\theta d\phi = \frac{1}{12\pi \mathrm{s}}\mathrm{A}(\sqrt{\mathrm{s}}). \tag{3.114}$$

The result of total cross-section are indicated in Fig.(3.4). Notice that in all below plots, for each line we will set zero value for all Wilson coefficients except for the coefficient which is denoted for that line.

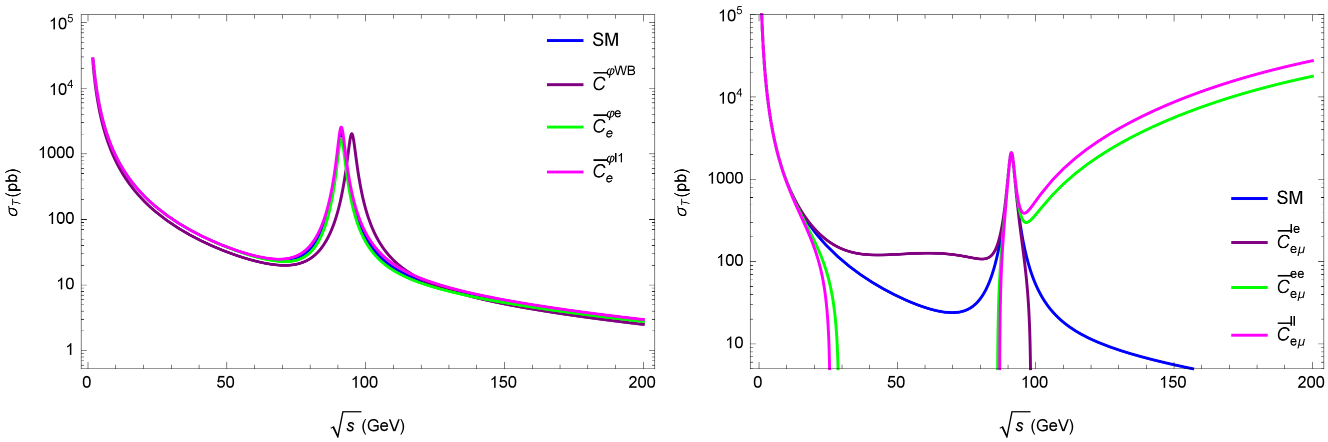


Figure 3.4: Total cross-section of muon in SMEFT when $\bar{C}_{\text{Wilson}} = 0.1$ each time

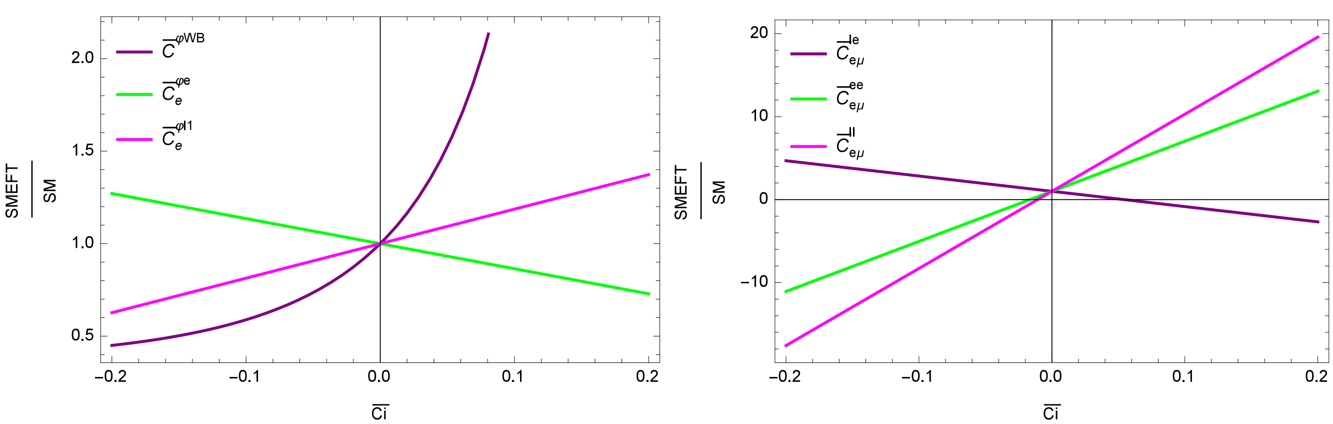


Figure 3.5: The SMEFT/SM ratio of total cross-section when $\sqrt{s} = 100$GeV

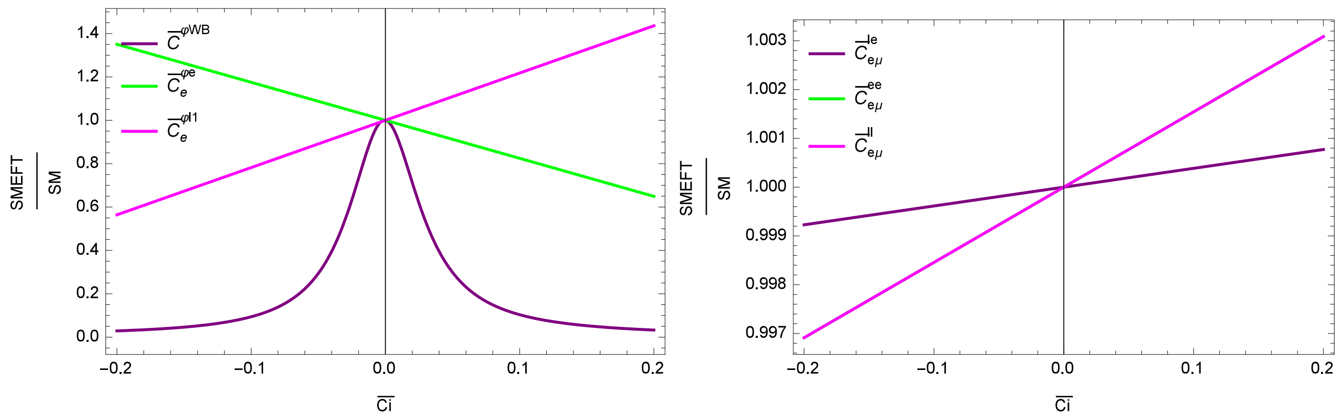


Figure 3.6: The SMEFT/SM ratio of total cross-section for special case $\sqrt{s} = m_Z$ GeV

In Fig.(3.5), we can see that the dependence of SMEFT on Wilson coefficients is linear, but the special case is for $C^{\varphi WB}$, since we not only have it in the coupling but also in the mass and decay width of Z. Therefore our result is not linear depend on $C^{\varphi WB}$. Comparing Fig.(3.5) and Fig.(3.6), we see that for the case $\sqrt{s} = 100$ GeV, the coefficients in four-fermion group have stronger effect than that of Higgs group. While in the special case $\sqrt{s} = m_Z$, the contributions of coefficients in four-fermion group are negligible. Note that in $\sqrt{s} = m_Z$ case, $C^{ee}_{e\mu}$ and $C^{ll}_{e\mu}$ have the same contribution so that you can not distinguish them in the graphic.

3.9.2 Angular distribution

The angular distribution reads

$$\frac{d\sigma}{d\theta} = \frac{1}{32\pi s}\left[A(\sqrt{s})(1+\cos^2\theta) + \mathrm{B}(\sqrt{\mathrm{s}})\cos\theta\right]\sin\theta. \tag{3.115}$$

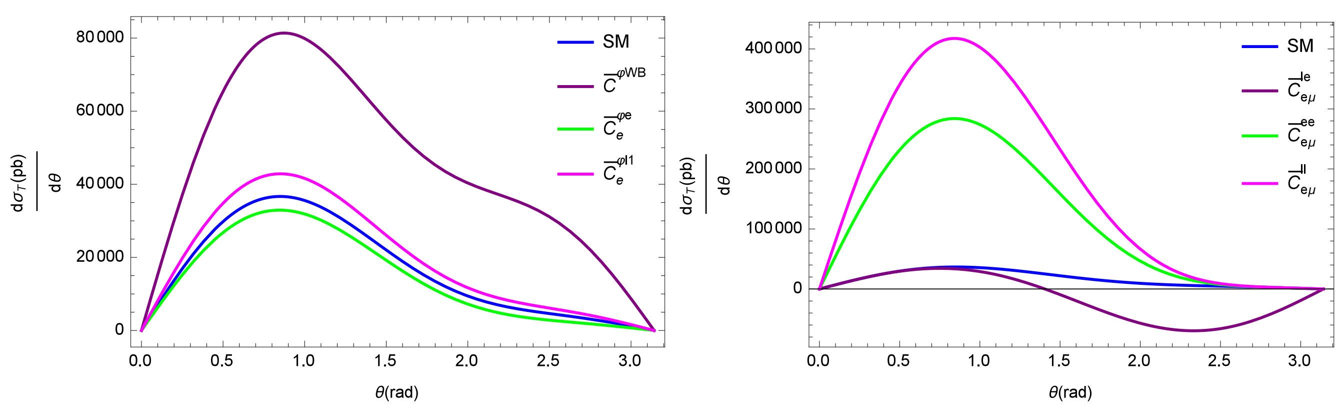


Figure 3.7: The angular distribution for $\bar{C}_{\text{Wilson}} = 0.1$ when $\sqrt{s} = 100$(GeV)

3.9.3 Fordward-backward asymmetry

the fordward-backward asymmetry

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{8}\frac{B(\sqrt{s})}{A(\sqrt{s})}. \tag{3.116}$$

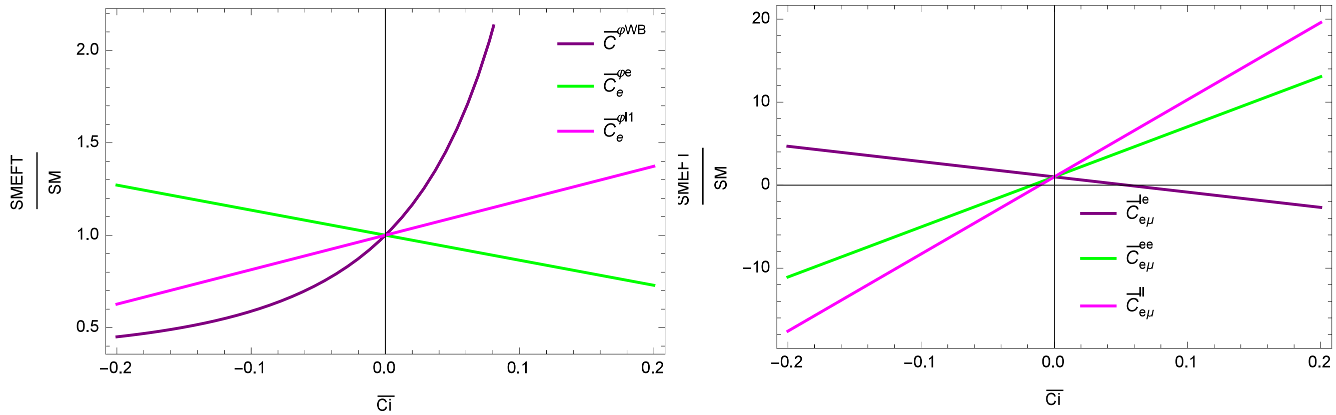


Figure 3.8: The SMEFT/SM ratio of angular distribution when $\sqrt{s} = 100$GeV and θ from $\pi/4$ to $3\pi/4$

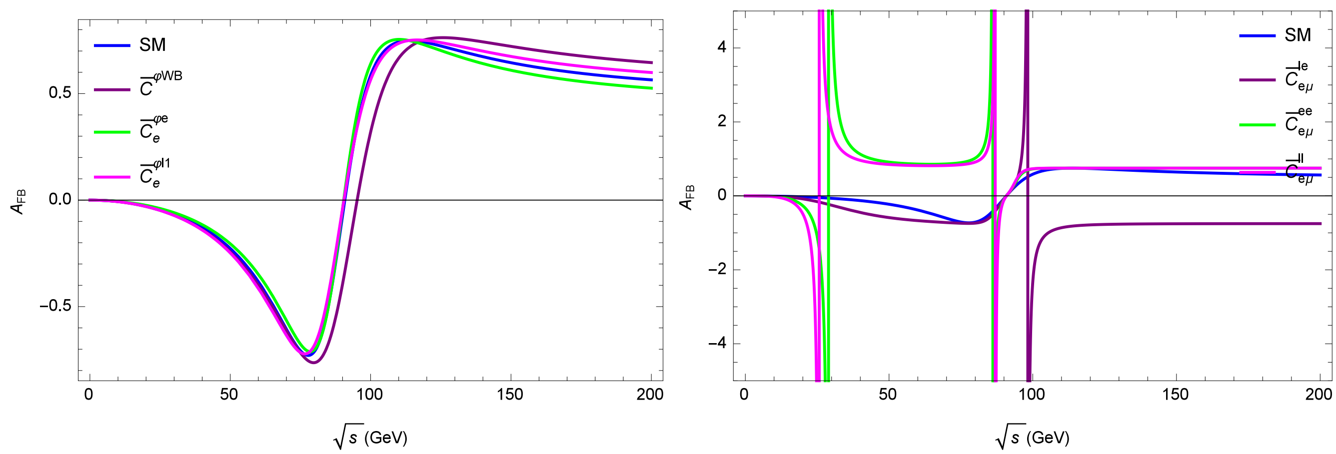


Figure 3.9: The fordward-backward asymmertry in SMEFT when $\bar{C}_{\text{Wilson}} = 0.1$

3.9.4 Transverse momentum distribution

The transverse momentum distribution is given by

$$\frac{d\sigma}{dk_t} = \frac{A(\sqrt{s})}{16\pi s} \frac{(2E^2 - k_t^2)k_t}{E^3\sqrt{E^2 - k_t^2}}. \tag{3.117}$$

3.9.5 Longitudinal momentum distribution

The longitudinal momentum distribution is of the form

$$\frac{d\sigma}{dk_l} = \frac{1}{32\pi s E}\left[A(\sqrt{s})\left(1 + \frac{k_l^2}{E^2}\right) + B(\sqrt{s})\frac{k_l}{E}\right]. \tag{3.118}$$

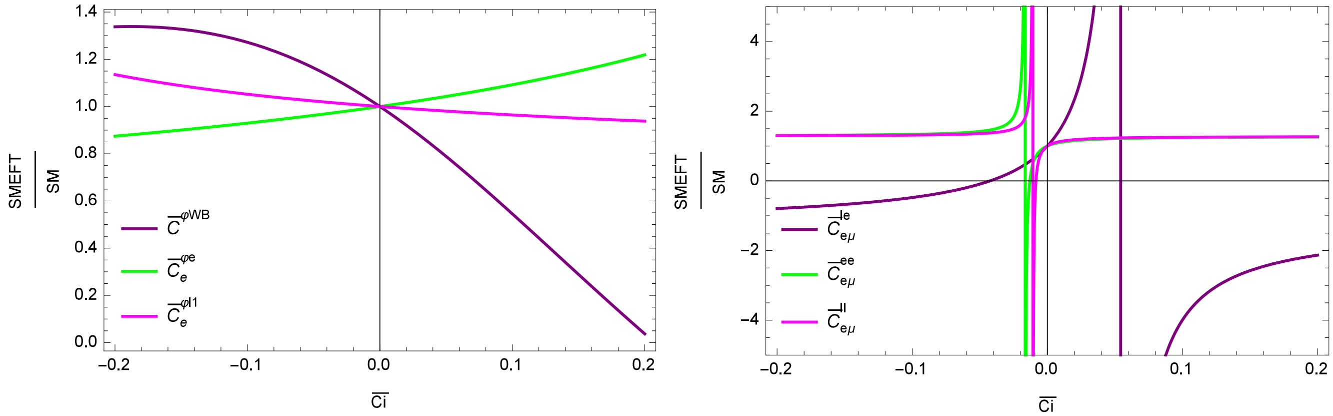


Figure 3.10: The SMEFT/SM ratio of fordward-backward asymmertry when $\sqrt{s} = 100$GeV

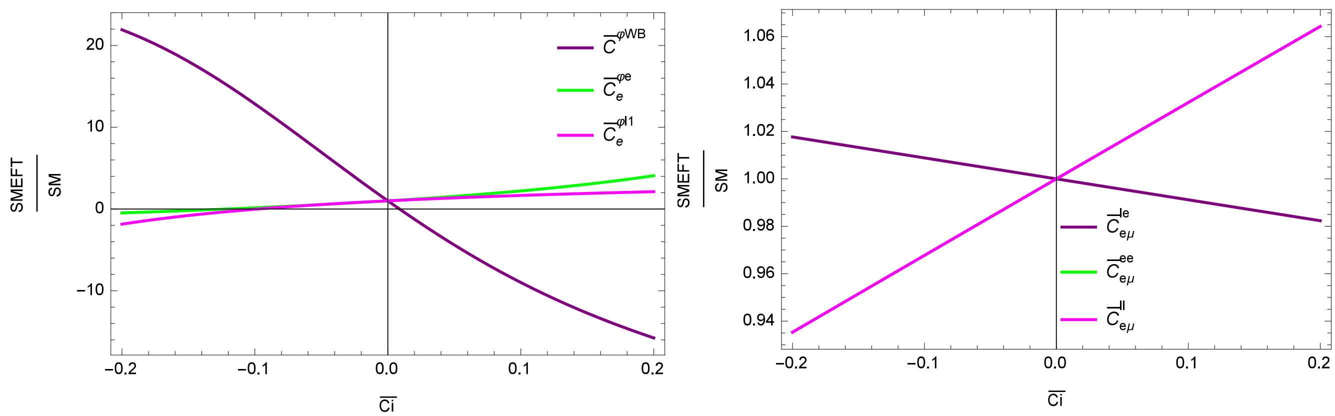


Figure 3.11: The SMEFT/SM ratio of fordward-backward asymmertry when $\sqrt{s} = m_Z$ GeV

3.9.6 Rapidity distribution

The rapidity is defined as

$$y = \frac{1}{2}\ln\left(\frac{\mathrm{E} + \mathrm{k_l}}{\mathrm{E} - \mathrm{k_l}}\right). \tag{3.119}$$

From the original definition, we can write down the another form of it

$$y = arc\tanh\frac{\mathrm{k_l}}{\mathrm{E}} \tag{3.120}$$

Thus, the Jacobian then read

$$k_l = E\mathrm{tanhy} \Rightarrow \left|\frac{\mathrm{dk_l}}{\mathrm{dy}}\Big|_{\mathrm{k_l=Etanhy}}\right| = \mathrm{E}(1 - \mathrm{tanh^2y}) \tag{3.121}$$

Thus the rapidity distribution of muon is:

$$\begin{aligned}\frac{d\sigma}{dy} &= \frac{d\sigma}{dk_l}\Big|_{k_l=E\mathrm{tanhy}} \left|\frac{dk_l}{dy}\Big|_{k_l=E\mathrm{tanhy}}\right| \\ &= \frac{1}{32\pi s}\left[A(\sqrt{s})\left(1 + \mathrm{tanhy}^2\right) + B(\sqrt{s})\mathrm{tanhy}\right](1 - \mathrm{tanh^2y}).\end{aligned} \tag{3.122}$$

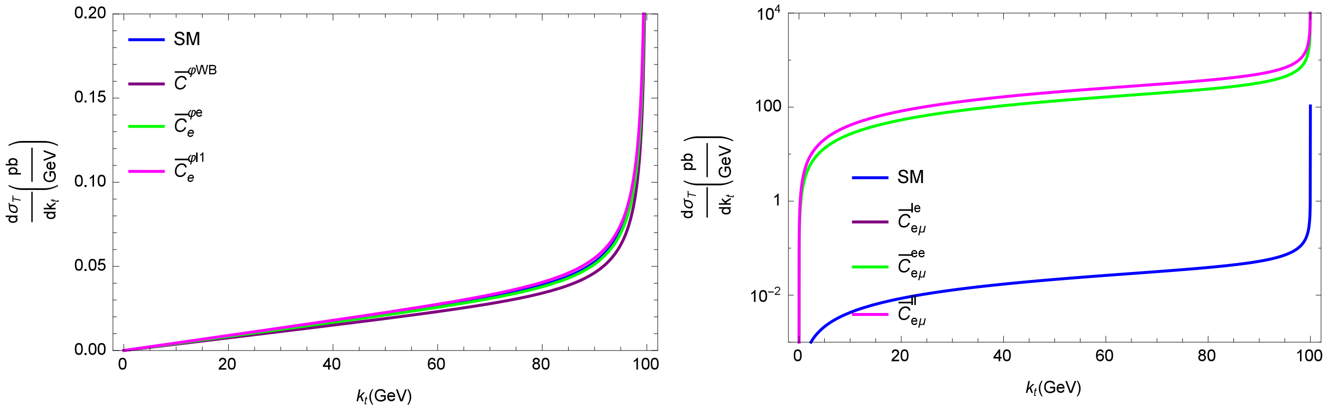


Figure 3.12: The transverse momentum distribution in SMEFT when $\bar{C}_{\text{Wilson}} = 0.1$ and $\sqrt{s} = 200\text{GeV}$

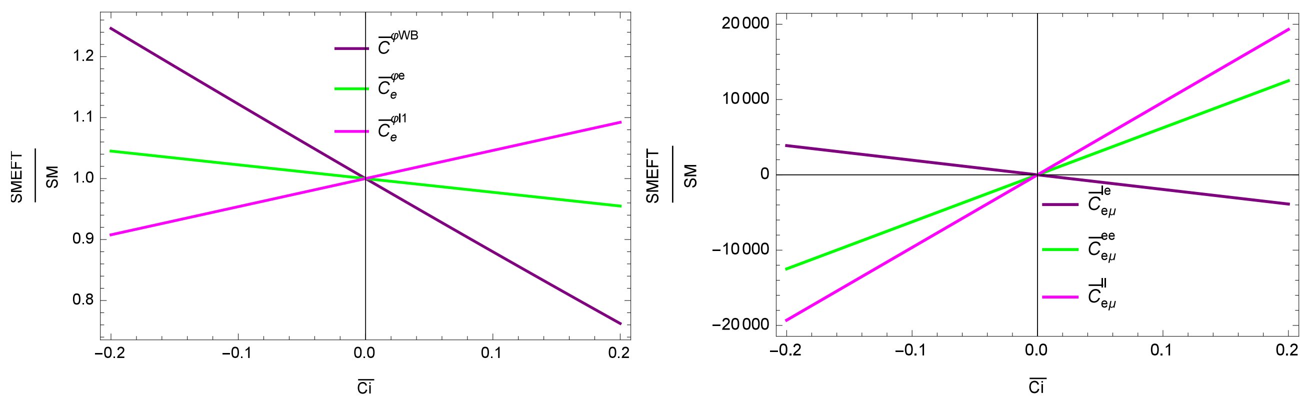


Figure 3.13: The SMEFT/SM ratio of transverse momentum distribution when $\sqrt{s} = 200\text{GeV}$ and k_t from 90 to 100(GeV)

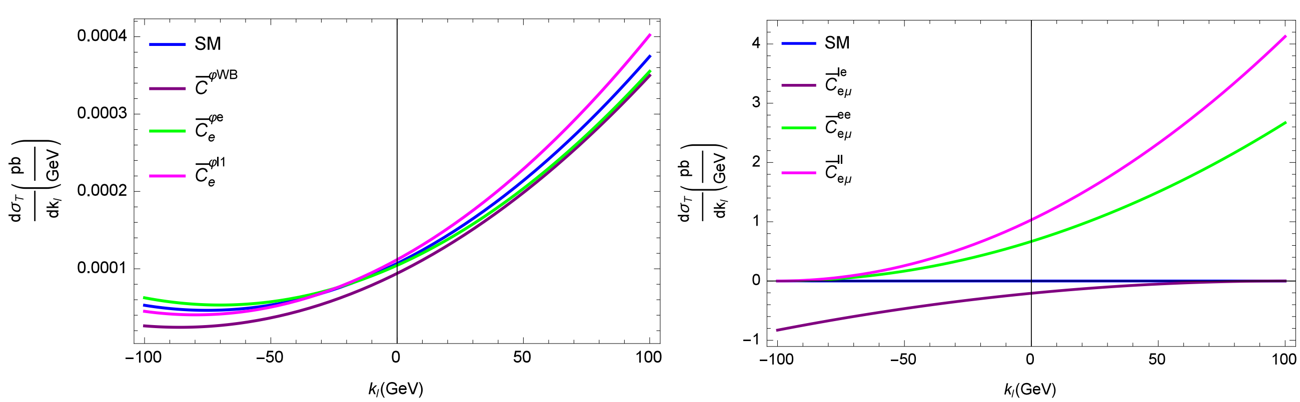


Figure 3.14: The longitudinal momentum distribution in SMEFT when $\bar{C}_{\text{Wilson}} = 0.1$ and $\sqrt{s} = 200\text{GeV}$

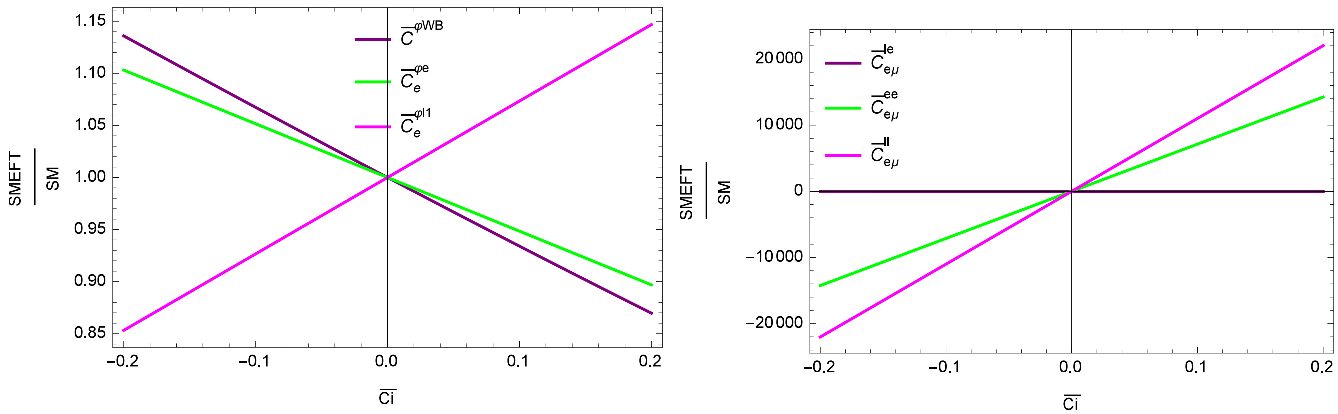


Figure 3.15: The SMEFT/SM ratio of longitudinal momentum distribution when $\sqrt{s} = 200\text{GeV}$ and k_l from 90 to 100 (GeV)

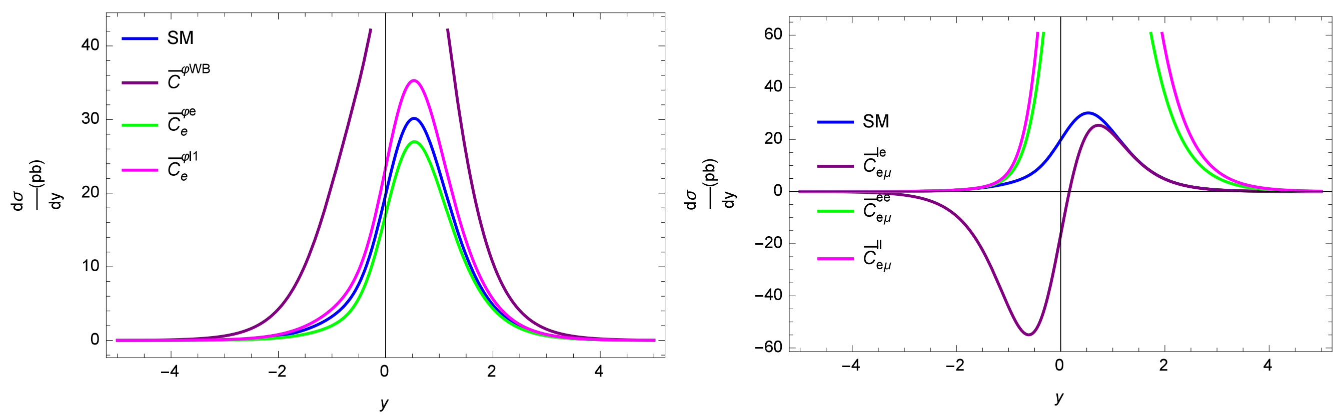


Figure 3.16: The rapidity distribution in SMEFT when $\bar{C}_{\text{Wilson}} = 0.1$ and $\sqrt{s} = 100\text{GeV}$

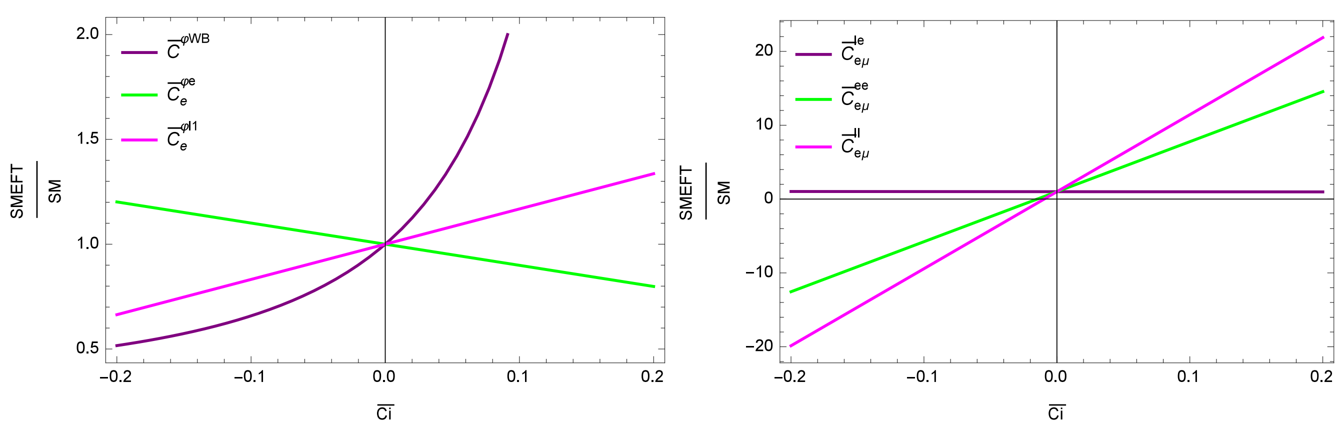


Figure 3.17: The SMEFT/SM ratio of rapidity distribution when $\sqrt{s} = 100\text{GeV}$ and y from 1 to 2

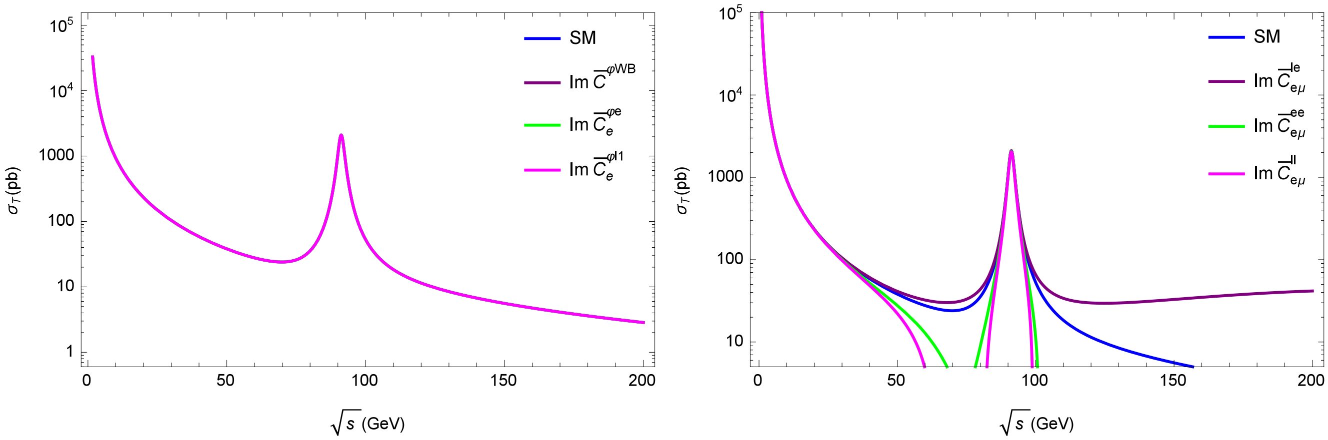


Figure 3.18: Total cross-section of muon in SMEFT when $\mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = 0.1$ each time

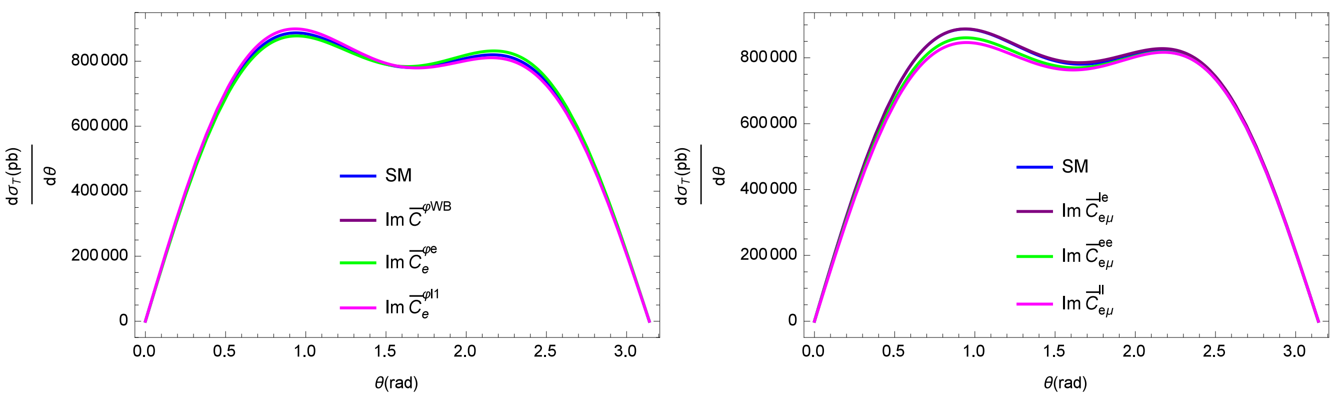


Figure 3.19: The angular distribution for $\mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = 0.1$ when $\sqrt{s} = m_Z(\mathrm{GeV})$

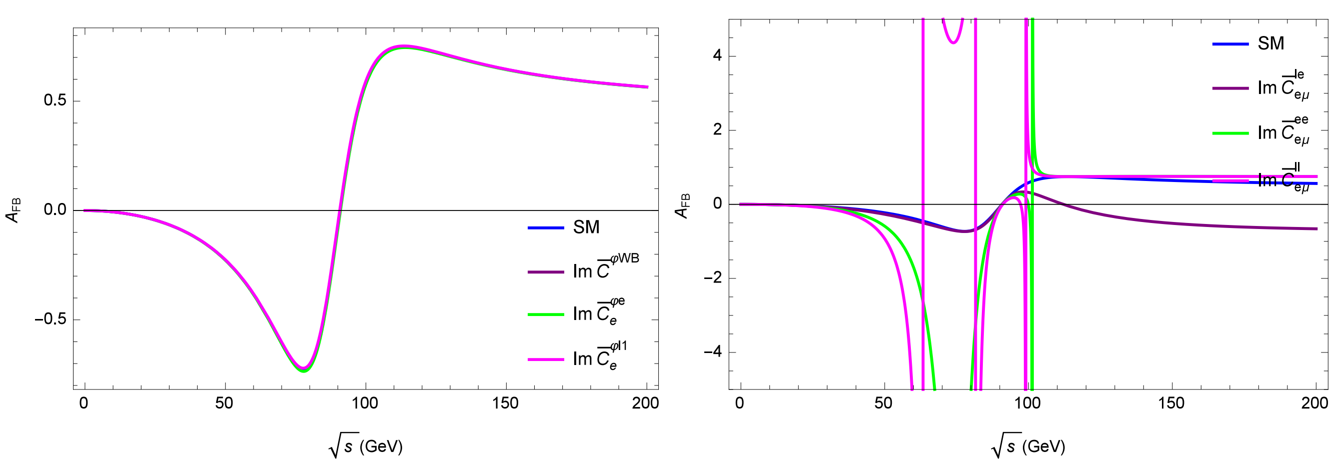


Figure 3.20: The fordward-backward asymmertry in SMEFT when $\mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = 0.1$

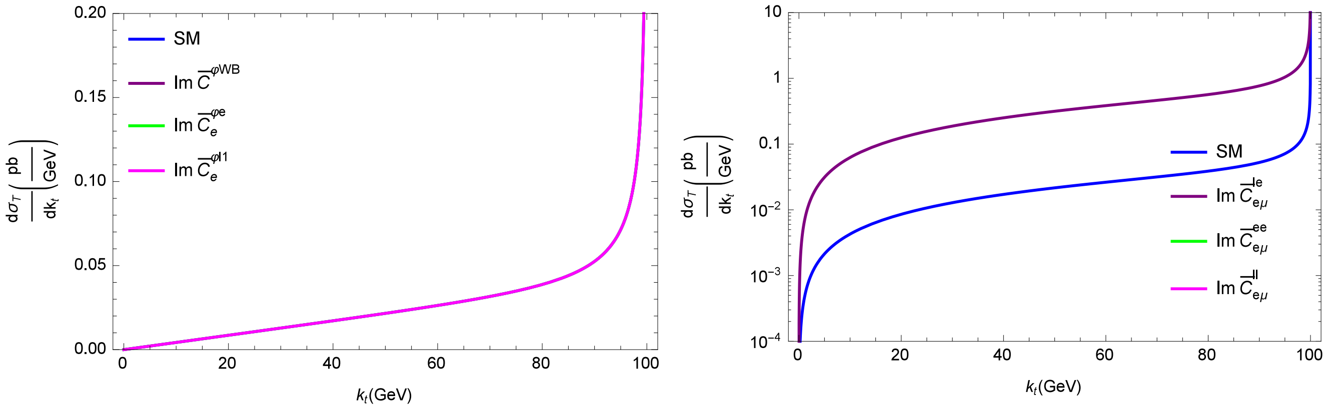


Figure 3.21: The transverse momentum distribution in SMEFT when $\mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = 0.1$ and $\sqrt{s} = 200\mathrm{GeV}$

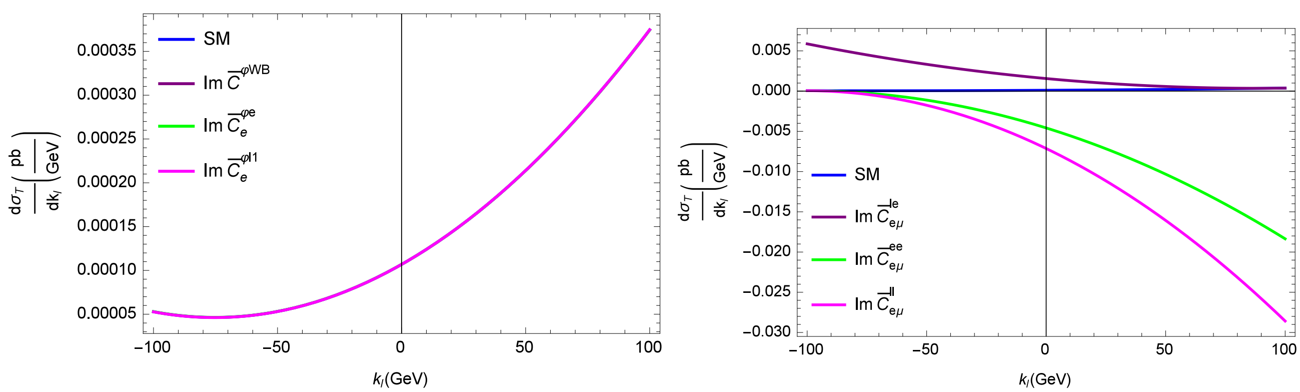


Figure 3.22: The longitudinal momentum distribution in SMEFT when $\mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = 0.1$ and $\sqrt{s} = 200\mathrm{GeV}$

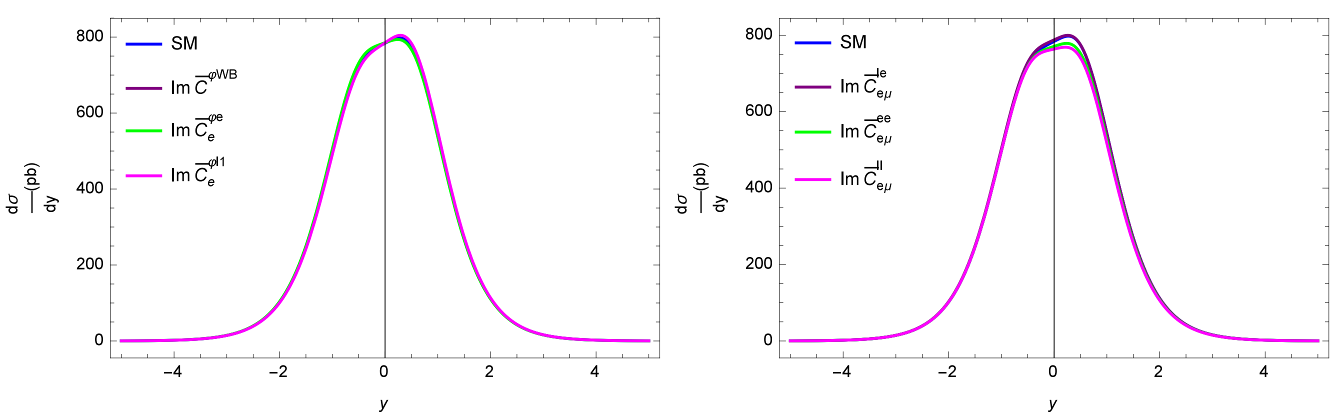


Figure 3.23: The rapidity distribution in SMEFT when $\mathrm{Im}\bar{\mathrm{C}}_{\mathrm{Wilson}} = 0.1$ and $\sqrt{s} = m_Z(\mathrm{GeV})$

4

Application for the spin-density matrix elements of Z boson

4.1 Relations of the Z boson decay angular distributions with its spin-density matrix elements

A density matrix is a matrix that describes a quantum system in a mixed state, a statistical ensemble of several quantum states. Density matrix is contradicted with a single state vector which describes a quantum system in a pure state. We call pure state when a quantum system has one state vector $|\psi\rangle$ only. Nevertheless, it is possible for a system to be in a statistical ensemble of different state vectors as well. This time we call the system is in a mixed state. For instance, we have a mixed state of a system when the system have two distinguishable states are $|\psi_1\rangle$ and $|\psi_2\rangle$, with the probability for each state is 50%. That means there is 50% probability that the state vector is $|\psi_1\rangle$ and the rest chance that the state vector is $|\psi_2\rangle$. Hence it turns out that density matrix is the powerful tool to describe the mixed state. Note that a mixed state is not the same with a quantum superposition. The probabilities in a mixed state are classical probabilities, dissimilar to the quantum probabilites in a quantum superposition.

Being a spin-1 particle, Z boson's spin state is described in a form of a 3×3 density matrix with 8 observables. We have the number 8 rather than another number since the density matrix of Z boson has 8 degrees of freedom. Now the main purpose of this section is re-producing the relations in [15] between the Z boson decay angular distributions and the spin-density matrix elements of the Z boson (it is noted that Z bosons produced at e^+e^- or pp collision are polarized). The method to find those relations was firstly introduced by J.A. Aguilar-Saavedra and J. Bernabeu in 2016 [16] when they used that method for the W boson case.
In addition to the form of 3×3 matrix, the density matrix has the other features which are Hermitian and unit trace. The general form of it reads

$$\rho = \frac{1}{3} + \frac{1}{2}\sum_{m=-1}^{1}\langle S_m\rangle^* S_m + \sum_{m=-2}^{2}\langle T_m\rangle^* T_m, \tag{4.1}$$

where S_m is the three spin operators which are given in the spherical basis by

$$S_{\pm 1} = \mp\frac{1}{\sqrt{2}}(S_x \pm iS_y), \qquad S_0 = S_z, \tag{4.2}$$

and T_m are five irreducible tensors establishing from S_m as

$$T_{\pm 2} = S_{\pm 1}^2, \qquad T_{\pm 1} = \frac{1}{2}[S_{\pm 1}S_0 + S_0 S_{\pm 1}], \qquad T_0 = \frac{1}{\sqrt{6}}[S_{+1}S_{-1} + S_{-1}S_{+1} + 2S_0^2]. \tag{4.3}$$

We have used the notation $S_{x,y,z}$ for the spin operators which have the below forms for the spin-1 particles

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{1}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.4}$$

Substituting eq.(4.4) into (4.2) and (4.3), we can obtain

$$S_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$T_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T_0 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \qquad T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.5}$$

For later convenience, we will define the operators

$$A_1 = \frac{1}{2}(T_1 - T_{-1}), \quad A_2 = \frac{1}{2i}(T_1 + T_{-1}), \quad B_1 = \frac{1}{2}(T_2 + T_{-2}), \quad B_2 = \frac{1}{2i}(T_2 - T_{-2}). \tag{4.6}$$

From the explicit form of the above operators, we are now able to write each density matrix elements of Z boson in terms of expected values of observables

$$\begin{aligned}
\rho_{\pm 1 \pm 1} &= \frac{1}{3} \pm \frac{1}{2}\langle S_3 \rangle + \frac{1}{\sqrt{6}}\langle T_0 \rangle, \\
\rho_{\pm 1 0} &= \frac{1}{2\sqrt{2}}[\langle S_1 \rangle \mp i\langle S_2 \rangle] \mp \frac{1}{\sqrt{2}}[\langle A_1 \rangle \mp i\langle A_2 \rangle], \\
\rho_{00} &= \frac{1}{3} - \frac{2}{\sqrt{6}}\langle T_0 \rangle, \\
\rho_{1-1} &= \langle B_1 \rangle - i\langle B_2 \rangle,
\end{aligned} \tag{4.7}$$

and the other elements have the forms $\rho_{a,b} = \rho_{b,a}^*$. As we will see later, the angular distribution of Z decay totally depends on those density matrix elements. To figure out that, we first need to write the amplitude for the decay of Z boson by using the helicity formalism of Jacob and Wick [17]

$$\mathcal{M}_{m,\lambda_1\lambda_2} = a_{\lambda_1\lambda_2} D_{m\lambda}^{1*}(\phi, \theta, 0), \tag{4.8}$$

where m is the third spin component and $\lambda_1\lambda_2$ stand for the helicities of two fermions which are the production of Z decay. $\lambda = \lambda_1 - \lambda_2 = \pm 1$. We use the Z boson rest frame and $\phi,\ \theta$ are the polar and azimuthal angles of fermion respectively. The notation $D^s_{m'm}(\alpha,\beta,\gamma)$ is the Wigner D-matrix [18] which is of the form

$$D^s_{m'm}(\alpha,\beta,\gamma) \equiv \langle sm'|R(\alpha,\beta,\gamma)|sm\rangle = e^{-im'\alpha}e^{-im\gamma}d^s_{m'm}(\beta), \tag{4.9}$$

where $R(\alpha,\beta,\gamma) = e^{-i\alpha S_x}e^{-i\beta S_y}e^{-i\gamma S_z}$ is a generic rotation in 3-dimensional space which is built by compounding operators using Euler angles, and $d^s_{m'm}(\beta) = \langle sm'|e^{-i\beta S_y}|sm\rangle$ is the Wigner's small d-matrix. The amplitude in Eq.(4.8) then becomes

$$\mathcal{M}_{m\lambda_1\lambda_2} = a_{\lambda_1\lambda_2}e^{im\phi}d^1_{m\lambda}(\theta). \tag{4.10}$$

The squared amplitude is easily found

$$\begin{aligned}|\mathcal{M}|^2 &= \sum_{m',m}\rho_{mm'}\mathcal{M}^\dagger_{m'\lambda_1\lambda_2}\mathcal{M}_{m\lambda_1\lambda_2}\\ &= \sum_{m',m}\rho_{mm'}a^*_{\lambda_1\lambda_2}a_{\lambda_1\lambda_2}e^{i(m-m')\phi}d^1_{m\lambda}(\theta)d^1_{m'\lambda}(\theta).\end{aligned} \tag{4.11}$$

The decay width of Z boson in two final particles reads [2]

$$\begin{aligned}\frac{d\Gamma}{d\Omega} &= \frac{1}{64\pi m_Z^2}\sqrt{1-\frac{4m^2}{m_Z^2}}|\mathcal{M}|^2\\ &= C\sum_{m,m'm\lambda_1\lambda_2}\rho_{m,m'}|a_{\lambda_1\lambda_2}|^2e^{i(m-m')\phi}d^1_{m\lambda}(\theta)d^1_{m'\lambda}(\theta)\end{aligned} \tag{4.12}$$

Note that $a_{1/2\ -1/2}$ is different with $a_{-1/2\ 1/2}$ and their values are proportional to the right- and left-handed couplings of Z boson and charged lepton, respectively. It is convenient to denote them as $a_1 = a^2_{1/2\ -1/2}$ and $a_2 = a^2_{-1/2\ 1/2}$ for short, then we have

$$\frac{a^2_{1/2\ -1/2}}{a^2_{-1/2\ 1/2}} = \frac{a_1}{a_2} = \frac{g_R^{l^2}}{g_L^{l^2}}. \tag{4.13}$$

For later apparent calculations, it is necessary to introduce some values of the Wigner's small d-matrix elements for spin-1 particles

$$d^1_{1,1}(\theta) = \frac{1+\cos\theta}{2}, \qquad d^1_{1,0}(\theta) = -\frac{\sin\theta}{\sqrt{2}}, \qquad d^1_{1,-1}(\theta) = \frac{1-\cos\theta}{2}, \qquad d^1_{0,0}(\theta) = \cos\theta. \tag{4.14}$$

The other values of d-matrix elements can derive by using the feature below

$$d^j_{m',m}(\theta) = (-1)^{m-m'}d^j_{m,m'}(\theta) = d^j_{-m,-m'}(\theta). \tag{4.15}$$

The sum in Eq.(4.12) have the below result, note that the mediate calculation steps is too long and there is no need to explicitly write it down

$$\begin{aligned}\frac{d\Gamma}{d\Omega} = C\frac{a_1+a_2}{2}\Bigg\{&\left(\frac{1}{3}+\frac{T_0}{\sqrt{6}}\right)(1+\cos^2\theta) + \left(\frac{1}{3}-\frac{2}{\sqrt{6}}\mathrm{T}_0\right)\sin^2\theta + \langle \mathrm{B}_1\rangle\cos2\phi\sin^2\theta + \langle \mathrm{B}_2\rangle\sin2\phi\sin^2\theta\\ &-\langle A_1\rangle\cos\phi\sin2\theta - \langle \mathrm{A}_2\rangle\sin\phi\sin2\theta + \frac{\mathrm{a}_1-\mathrm{a}_2}{\mathrm{a}_1+\mathrm{a}_2}(\langle \mathrm{S}_1\rangle\cos\phi\sin\theta + \langle \mathrm{S}_2\rangle\sin\phi\sin\theta + \langle \mathrm{S}_3\rangle\cos\theta)\Bigg\}.\end{aligned} \tag{4.16}$$

Integrating the terms in {} bracket, we will get the value of $8\pi/3$. Let's introduce the factor

$$\eta = \frac{a_1 - a_2}{a_1 + a_2} = \frac{g_R^{l2} - g_L^{l2}}{g_R^{l2} + g_L^{l2}} \simeq -0.14. \tag{4.17}$$

We are now able to write Eq.(4.12) in the normalised distribution form

$$\begin{aligned}\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi} = \frac{3}{8\pi}\Bigg\{&\frac{1}{2}(1+\cos^2\theta) + \left(\frac{1}{6} - \frac{1}{\sqrt{6}}\langle \mathrm{T}_0\rangle\right)(1-3\cos^2\theta) + \langle \mathrm{B}_1\rangle \cos2\phi \sin^2\theta + \langle \mathrm{B}_2\rangle \sin2\phi \sin^2\theta \\ &-\langle A_1\rangle \cos\phi \sin2\theta - \langle \mathrm{A}_2\rangle \sin\phi \sin2\theta + \eta(\langle \mathrm{S}_1\rangle \cos\phi \sin\theta + \langle \mathrm{S}_2\rangle \sin\phi \sin\theta + \langle \mathrm{S}_3\rangle \cos\theta)\Bigg\}.\end{aligned} \tag{4.18}$$

We have changed it into the normalized form so that we can compare with the form which is usually used by experimentalist. Because of the term $e^{i(m-m')\phi}$ in squared amplitude, the off-diagonal elements will become zero when we integrate over the azimuthal angle. The polar angle distribution, therefore, is given by

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} = \frac{3}{8\pi}\int \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi} d\phi = \frac{1}{2} + \frac{3\langle T_0\rangle}{4\sqrt{6}}(3\cos^2\theta - 1) + \frac{3}{4}\eta\langle \mathrm{S}_3\rangle \cos\theta. \tag{4.19}$$

The forward-backward (FB) asymmetry

$$\begin{aligned}A_{FB} &= \frac{1}{\Gamma}\left[\Gamma(\cos\theta > 0) - \Gamma(\cos\theta < 0)\right] \\ &= \int_0^1 \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} d\cos\theta - \int_{-1}^0 \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta \\ &= \left(\frac{1}{2} + \frac{3}{8}\eta\langle S_3\rangle\right) - \left(\frac{1}{2} - \frac{3}{8}\eta\langle S_3\rangle\right) \\ &= \frac{3}{4}\eta\langle S_3\rangle.\end{aligned} \tag{4.20}$$

The edge-central asymmetry

$$\begin{aligned}A_{EC} &= \frac{1}{\Gamma}\left[\Gamma\left(|\cos\theta| > \frac{1}{2}\right) - \Gamma\left(|\cos\theta| < \frac{1}{2}\right)\right] \\ &= \int_{-1}^{-1/2} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta} d\cos\theta + \int_{1/2}^1 \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta - \int_{-1/2}^{1/2} \frac{1}{\Gamma}\frac{\mathrm{d}\Gamma}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta \\ &= \left(\frac{1}{2} + \frac{9\langle T_0\rangle}{16\sqrt{6}}\right) - \left(\frac{1}{2} - \frac{9\langle T_0\rangle}{16\sqrt{6}}\right) \\ &= \frac{3}{8}\sqrt{\frac{3}{2}}\langle T_0\rangle.\end{aligned} \tag{4.21}$$

All the diagonal density matrix elements have been found already. In order to measure the off-diagonal one, we need to avoid the vanishment in the azimuthal integral. The issue is that all off-diagonal elements contain $\cos\phi$ or $\sin\phi$. While $\cos\phi$ has opposite value between forward- and backward-part of the trigonometric circle, the value of $\sin\phi$ is opposite between upper- and lower-part of the circle. That is the reason for the cancellation of those terms when we integrate over

ϕ. Thus if we can make our function has the same sign in two opposite part of the trigonometric circle, our integration will remain off-diagonal elements. The solution for this is using function $f_1(\phi) = \text{sign}(\cos\phi)$ or $f_2(\phi) = \text{sign}(\sin\phi)$. The corresponding results of the multiplication $f_1(\phi)$, $f_2(\phi)$ are denoted as $\delta_1\Gamma$ and $\delta_2\Gamma$

$$\begin{aligned}
\frac{1}{\Gamma}\frac{d(\delta_1\Gamma)}{d\cos\theta} &= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}\text{sign}(\cos\phi)\mathrm{d}\phi \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}d\phi \\
&= \frac{3}{2\pi}\eta\langle S_1\rangle\sin\theta - \frac{3}{2\pi}\langle \mathrm{A}_1\rangle\sin2\theta. \qquad (4.22)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\Gamma}\frac{d(\delta_2\Gamma)}{d\cos\theta} &= \int_{0}^{2\pi} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}\text{sign}(\sin\phi)\mathrm{d}\phi \\
&= \int_{0}^{\pi} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}d\phi - \int_{\pi}^{2\pi} \frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}d\phi \\
&= \frac{3}{2\pi}\eta\langle S_2\rangle\sin\theta - \frac{3}{2\pi}\langle \mathrm{A}_2\rangle\sin2\theta. \qquad (4.23)
\end{aligned}$$

Because of the sign(sinϕ) and sign(sinϕ) function, when we integrate (4.22) and (4.23) over the polar angle, we will obtain

$$A^x_{FB} = \frac{\delta_1\Gamma}{\Gamma} = \int_{-1}^{1}\frac{1}{\Gamma}\frac{d\delta_1\Gamma}{d\cos\theta}d\cos\theta = \frac{1}{\Gamma}\big[\Gamma(\cos\phi > 0) - \Gamma(\cos\phi < 0)\big] = \frac{3}{4}\eta\langle \mathrm{S}_1\rangle, \qquad (4.24)$$

$$A^y_{FB} = \frac{\delta_2\Gamma}{\Gamma} = \int_{-1}^{1}\frac{1}{\Gamma}\frac{d\delta_2\Gamma}{d\cos\theta}d\cos\theta = \frac{1}{\Gamma}\big[\Gamma(\sin\phi > 0) - \Gamma(\sin\phi < 0)\big] = \frac{3}{4}\eta\langle \mathrm{S}_2\rangle. \qquad (4.25)$$

We have used the notation A^x_{FB} and A^y_{FB} since they look like the forward-backward asymmetry with respect to x and y axes, respectively. So we have just gained two more observables. The rest observables in (4.22) and (4.23) can be found by calculating the Forward-Backward asymmetries A^1_{FB} and A^2_{FB} of $\delta_1\Gamma$, $\delta_2\Gamma$ distributions.

$$\begin{aligned}
A^1_{FB} &= \int_{0}^{1}\frac{1}{\Gamma}\frac{d(\delta_1\Gamma)}{d\cos\theta}\mathrm{d}\cos\theta - \int_{-1}^{0}\frac{1}{\Gamma}\frac{\mathrm{d}(\delta_1\Gamma)}{\mathrm{d}\cos\theta}\mathrm{d}\cos\theta \\
&= \frac{1}{\Gamma}\big[\Gamma(\cos\theta > 0, \cos\phi > 0) - \Gamma(\cos\theta > 0, \cos\phi < 0)\big] \\
&\quad - \frac{1}{\Gamma}\big[\Gamma(\cos\theta < 0.\cos\phi > 0) - \Gamma(\cos\theta < 0, \cos\phi < 0)\big] \\
&= \frac{1}{\Gamma}\big[\Gamma(\cos\theta\cos\phi > 0) - \Gamma(\cos\theta\cos\phi < 0)\big] \\
&= \left(\frac{3}{8}\langle S_1\rangle - \frac{1}{\pi}\langle A_1\rangle\right) - \left(\frac{3}{8}\langle S_1\rangle + \frac{1}{\pi}\langle A_1\rangle\right) \\
&= -\frac{2}{\pi}\langle A_1\rangle. \qquad (4.26)
\end{aligned}$$

In the same manner, we can get

$$A^2_{FB} = \frac{1}{\Gamma}\big[\Gamma(\cos\theta\sin\phi > 0) - \Gamma(\cos\theta\sin\phi < 0)\big] = -\frac{2}{\pi}\langle A_2\rangle. \qquad (4.27)$$

Although we have the sign function in addition, we still not be able to avoid the cancelltion of the terms containing $\langle B_1\rangle$ and $\langle B_2\rangle$. Therefore, we must change a bit our sign functions to $f_3(\phi) = \text{signcos}2\phi$, $f_4(\phi) = \text{signsin}2\phi$. Then, the $\delta_3\Gamma$ distribution reads

$$\begin{aligned}\frac{1}{\Gamma}\frac{d(\delta_3\Gamma)}{d\cos\theta} &= \int_0^{2\pi}\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta \mathrm{d}\phi}\text{sign}(\cos 2\phi)\mathrm{d}\phi \\ &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta\mathrm{d}\phi}d\phi + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}}\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta\mathrm{d}\phi}d\phi - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta\mathrm{d}\phi}d\phi - \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}}\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta\mathrm{d}\phi}d\phi \\ &= \frac{3}{2\pi}\langle B_1\rangle\sin^2\theta. \end{aligned} \tag{4.28}$$

After the total integration over the polar angular, it yields

$$A_\phi^1 = \int_{-1}^{1}\frac{1}{\Gamma}\frac{d(\delta_3\Gamma)}{d\cos\theta}d\cos\theta = \frac{1}{\Gamma}\big[\Gamma(\cos 2\phi > 0) - \Gamma(\cos 2\phi < 0)\big] = \frac{2}{\pi}\langle \mathrm{B}_1\rangle. \tag{4.29}$$

Similar to the case of $f_4(\phi) = \text{signsin}2\phi$, we have

$$A_\phi^2 = \frac{1}{\Gamma}\big[\Gamma(\sin 2\phi > 0) - \Gamma(\sin 2\phi < 0)\big] = \frac{2}{\pi}\langle B_2\rangle. \tag{4.30}$$

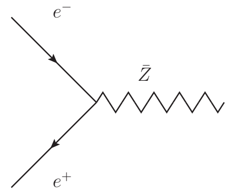
4.2 Z boson spin-density matrix elements using spinor helicity amplitudes method

In ordinary way, when we calculate amplitude of scattering process, we usually write down the amplitude using Feynman rules. After that, we square the amplitude and sum over the spins of external states. Then using the trace theorems to obtain the squared-amplitude in a more compact form without γ matrix. This basic technique is widely used in many textbooks, but it can create many issues at a deeper level since the Feynman amplitude for the process with many diagrams is much more complicated to square. For the spinor helicity amplitude method, it was first introduced by J.D. Bjorken and M. Chen [19] in 1966. In this method, we must choose a specific polarization states of the external particles, then write all spinors or gauge boson polarization vectors in explicit form. Next, find the individual helicity amplitude corresponding with each polarized external state and sum over the squares of polarized amplitudes. It is so different with the squared-amplitude method when we squared each helicity amplitude and then sum over all possible squared helicity-amplitudes later. We are able to do that because the helicity-amplitudes for each external state are independent and not interfere with each other. [1]

4.2.1 Squared amplitude of Z boson squared amplitude method

Our goal is not using the squared amplitude method, but we need to use it since it is necessary for us to compare the results of two methods together. First, we consider the Z boson that was created by ee collider. Hence, our Feynman diagram is

[1]For further discussion on that, please see [20]



With the coressponding momentum of Z boson is as previous chapters: $q = p + p'$. The Feynman amplitude reads

$$\mathcal{M} = \bar{v}_{s'}(p')\frac{i}{2}(g_V\gamma^\mu - g_A\gamma^\mu\gamma^5)u_s(p)\varepsilon^*_{\lambda\mu}, \tag{4.31}$$

where $\varepsilon_{\lambda\mu}(q)$ is the Z boson polarization vector, the squared-amplitude is then given by

$$|\mathcal{M}|^2 = \frac{1}{4}\sum_{s,s',\lambda}\left[\bar{u}_s(p)\left(-\frac{i}{2}\right)\gamma^\mu(g_V - g_A\gamma^5)v_{s'}(p')\varepsilon_{\lambda\mu}(q)\right]\left[\bar{v}_{s'}(p')\frac{i}{2}\gamma^\nu(g_V - g_A\gamma^5)u_s(p)\varepsilon^*_{\lambda\nu}(q)\right]. \tag{4.32}$$

Now, using the trace theorems like the former chapters, also the below identities

$$\sum_{\lambda=1}^{3}\varepsilon^\mu_\lambda(q)\varepsilon^{\nu*}_\lambda(q) = -g^{\mu\nu} + \frac{q^\mu q^\nu}{m_Z^2}, \qquad \text{and} \qquad \varepsilon^\mu_\lambda(q)\varepsilon^*_{\lambda'\mu}(q) = -\delta_{\lambda\lambda'}, \tag{4.33}$$

we are able to obtain

$$|\mathcal{M}|^2 = \frac{1}{4}\left\{(g_A^2 + g_V^2)\left[\frac{2(p'.q)(p.q)}{m_Z^2} + (p.p')\right] + 2g_V g_A i\epsilon^{\rho\mu\sigma\nu}p_\rho p'_\sigma q_\mu q_\nu\right\}. \tag{4.34}$$

Now if we work on the CM frame like chapter 1, and choose the case of on-shell Z boson, i.e. $q^2 = m_Z^2 = 4E^2$, the squared amplitude becomes

$$|\mathcal{M}|^2 = (g_V^2 + g_A^2)E^2. \tag{4.35}$$

4.2.2 Squared amplitude of Z boson using helicity amplitude method

As I have introduced above, first, we must choose a basis for the polarization states of the external particles, note that the momentum notation is unchanged from Chap. 1, where $q^2 = m_Z^2$

$$p^\mu = (E, \vec{p}) = (E, 0, 0, -E), \qquad p'^\mu = (E, -\vec{p}) = (E, 0, 0, E), \qquad q^\mu = (2E, \vec{0}). \tag{4.36}$$

and the Z boson polarization vectors reads

$$\varepsilon^\mu_0 = (0, 0, 0, 1), \qquad \varepsilon^\mu_+ = -\frac{1}{\sqrt{2}}(0, 1, i, 0), \qquad \varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0). \tag{4.37}$$

Before we construct the Dirac spinor, we need to consider some definitions. The Weyl spinors are defined as

$$\chi_+(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p_z)}}\begin{pmatrix}|\vec{p}| + p_z\\ p_x + ip_y\end{pmatrix}, \qquad \chi_-(p) = \frac{1}{\sqrt{2|\vec{p}|(|\vec{p}| + p_z)}}\begin{pmatrix}-p_x + ip_y\\ |\vec{p}| + p_z\end{pmatrix}, \tag{4.38}$$

which satisfy the equation

$$\chi^\dagger_\lambda(p)\chi_{\lambda'}(p) = \delta_{\lambda\lambda'} \tag{4.39}$$

$$H\chi_\pm(p) = \pm\chi_\pm(p), \tag{4.40}$$

where $H = \dfrac{\vec{\sigma}\vec{p}}{|\vec{p}|}$ is the helicity operator. Then the Dirac spinors can be constructed from the Weyl spinor as follows

$$u_\lambda(p) = \begin{pmatrix} \sqrt{p_0 + \lambda|\vec{p}|}\chi_\lambda(p) \\ \sqrt{p_0 - \lambda|\vec{p}|}\chi_\lambda(p) \end{pmatrix}, \qquad v_\lambda(p) = \begin{pmatrix} \lambda\sqrt{p_0 - \lambda|\vec{p}|}\chi_{-\lambda}(p) \\ -\lambda\sqrt{p_0 + \lambda|\vec{p}|}\chi_{-\lambda}(p) \end{pmatrix}. \tag{4.41}$$

First, we need to find the explicit form of Weyl spinor. Using Eq.(4.40), we have

$$\frac{\vec{\sigma}\vec{p}}{|\vec{p}|}\chi_+(p) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\chi_+(p) = \chi_+(p). \tag{4.42}$$

Thus the spinor above must have the form $\chi_+(p) = \begin{pmatrix} 0 \\ a \end{pmatrix}$ with a is an arbitrary number. But our Weyl spinor have to obey the normalize condition (4.39), therefore we can obtain $\chi_+(p)$ spinor and the other three Weyl spinors using the same trick

$$\chi_+(p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \chi_-(p) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_+(p') = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_-(p') = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.43}$$

From (4.41), the expression of Dirac spinors is

$$u_+(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_-(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_+(p') = -\sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_-(p') = -\sqrt{2E}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.44}$$

With all explicit forms of spinors and polarization vectors, we are now able to compute the helicity amplitude

$$A(\lambda, s', s) = \bar{v}_{s'}(p')\frac{i}{2}(g_V\gamma^\mu - g_A\gamma^\mu\gamma^5)u_s(p)\varepsilon^*_{\lambda\mu}. \tag{4.45}$$

The results turn out with no surprise when almost all terms vanish except for two cases

$$A(1, 1, -1) = -iE\sqrt{2}(g_V - g_A), \qquad A(-1, -1, 1) = -iE\sqrt{2}(g_V + g_A). \tag{4.46}$$

The squared amplitude then becomes

$$\begin{aligned}\frac{1}{4}\sum_{s',s,\lambda} A^*(\lambda, s', s)A(\lambda, s', s) &= \frac{1}{4}\left[A^*(1,1,-1)A(1,1,-1) + A^*(-1,-1,1)A(-1,-1,1)\right] \\ &= E^2(g_V^2 + g_A^2)\end{aligned} \tag{4.47}$$

which agrees with the squared amplitude in the previous subsection.

4.2.3 Z boson spin-density matrix elements

The next step is calculating Z boson density matrix. The relation between density matrix elements and helicity amplitudes is indicated by the formula

$$\rho_{ij} = \sum_{s',s} aA^*(i,s',s)A(j,s',s) \tag{4.48}$$

where a is a constant. For the cases of Z boson, we have two non-zero elements since there are only two helicity amplitudes of Z remain in above subsection. Thus, the density matrix elements of Z boson reads

$$\rho_{11} = a2E^2(g_V - g_A)^2, \qquad \rho_{-1-1} = a2E^2(g_V + g_A)^2 \tag{4.49}$$

and all other elements are canceled out. Notice that the density matrix have the feature of unit trace, therefore

$$\rho_{11} + \rho_{-1-1} = a2E^2(2g_V^2 + 2g_A^2) = 1 \Rightarrow a = \frac{1}{4E^2(g_V^2 + g_A^2)}. \tag{4.50}$$

Using Eq.(4.7), we rewrite the density matrix elements as

$$\rho_{11} = \frac{1}{3} + \frac{1}{2}\langle S_3\rangle + \frac{1}{\sqrt{6}}\langle T_0\rangle = \frac{1}{2} - \frac{g_V g_A}{g_V^2 + g_A^2}, \tag{4.51}$$

$$\rho_{-1-1} = \frac{1}{3} - \frac{1}{2}\langle S_3\rangle + \frac{1}{\sqrt{6}}\langle T_0\rangle = \frac{1}{2} + \frac{g_V g_A}{g_V^2 + g_A^2}. \tag{4.52}$$

The observables values are now obtained

$$\langle S_3\rangle = -\frac{2g_V g_A}{g_V^2 + g_A^2}, \qquad \langle T_0\rangle = \frac{1}{\sqrt{6}}. \tag{4.53}$$

while all other observables vanish. Note that if we change the basis states of the external particles, such as

$$p^\mu = (E,\vec{p}) = (E,0,0,E), \qquad p'^\mu = (E,-\vec{p}) = (E,0,0,-E), \tag{4.54}$$

then our results are quite different when $\langle S_3\rangle$ value has the opposite sign. This is because when we change the basis, our spinors form will change which leads to different helicity amplitudes.

4.3 Z boson spin-density matrix elements using squared amplitude method

As I have mentioned, our purpose to write the normalised distribution of Z decay in Eq.(4.18) is that we can compare with the cross-section distribution in normalised form. From [21] [22] [23], the normalised cross-section distribution using by experimentalist in the CollinsSoper (CS) reference frame [24] is parameterised in terms of eight coefficients V_{0-7}

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta d\phi} = \frac{3}{16\pi}\Big\{(1+\cos^2\theta) + \frac{1}{2}\mathrm{V}_0(1-3\cos^2\theta) + \mathrm{V}_1\cos\phi\sin2\theta + \frac{1}{2}\mathrm{V}_2\cos2\phi\sin^2\theta + \mathrm{V}_3\cos\phi\sin\theta$$
$$+V_4\cos\theta + \mathrm{V}_5\sin2\phi\sin^2\theta + \mathrm{V}_6\sin\phi\sin2\theta + \mathrm{V}_7\sin\phi\sin\theta\Big\}. \tag{4.55}$$

Comparing with Eq.(4.18), we have these relations

$$V_0 = \frac{2}{3} - 2\sqrt{\frac{2}{3}}\langle T_0\rangle, \qquad V_1 = -2\langle A_1\rangle, \qquad V_2 = 4\langle B_1\rangle, \qquad V_3 = 2\eta\langle S_1\rangle,$$
$$V_4 = 2\eta\langle S_3\rangle, \qquad V_5 = 2\langle B_2\rangle, \qquad V_6 = -2\langle A_2\rangle, \qquad V_7 = 2\eta\langle S_2\rangle. \tag{4.56}$$

Now our mission is finding the normalized distribution of the cross-section using the amplitude-squared method, then compare with the relation produced above to obtain density matrix elements of Z boson.
First, we have already found the squared amplitude of process $e^+ + e^- \to Z \to \mu^+ + \mu^-$ with the only meditator is Z boson in Eq.(2.31)

$$|\mathcal{M}_Z|^2 = \frac{e^4}{E^2}|\chi_0|^2\left[(g_V^2+g_A^2)^2\left(E^2+|\mathbf{k}|^2\cos^2\theta\right)+(g_V^4-g_A^4)m_\mu^2+8g_A^2g_V^2E|\mathbf{k}|\cos\theta\right] \tag{4.57}$$

Using the approximation $m_e = m_\mu = 0$, we can find the total cross section as

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2 s}|\chi_0|^2\left[(g_V^2+g_A^2)^2\left(1+\cos^2\theta\right)+8g_A^2g_V^2\cos\theta\right],$$
$$\Rightarrow \sigma = \frac{e^4}{12\pi s}|\chi_0|^2(g_V^2+g_A^2)^2. \tag{4.58}$$

The nomalised distribution of cross-section is then reads

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta d\phi} = \frac{3}{16\pi}\left[(1+\cos^2\theta)+\frac{8\mathrm{g_V^2g_A^2}}{(\mathrm{g_V^2+g_A^2})^2}\cos\theta\right]. \tag{4.59}$$

Comparing with Eq.(4.55) and using the relations in Eq.(4.56), we have

$$V_4 = 2\eta\langle S_3\rangle = \frac{8g_V^2g_A^2}{(g_V^2+g_A^2)^2}, \qquad V_0 = \frac{2}{3} - 2\sqrt{\frac{2}{3}}\langle T_0\rangle = 0 \tag{4.60}$$

where $\eta = \frac{g_R^2 - g_L^2}{g_R^2+g_L^2} = -\frac{2g_Vg_A}{g_V^2+g_A^2}$. we can obtain

$$\langle S_3\rangle = -\frac{2g_Vg_A}{g_V^2+g_A^2}, \qquad \langle T_0\rangle = \frac{1}{\sqrt{6}} \tag{4.61}$$

while all other observables equal to zero which is totally the same with the results in helicity method.

4.4 The so-called "spin-density matrix" of Z boson and photon in SM

Using the above method, If we can find the normalized angular distribution for the process $e^+ + e^- \to \mu^+ + \mu^-$ which mediated by Z boson and photon, we will able to derive the so-called density matrix elements. We use the word "so-called" since it is not actually the density matrix while it has two mediators. Based on the result in Eq.(2.39) of Chapter 2

$$\frac{d\sigma}{d\Omega} = \frac{e^4|\mathbf{k}|}{16\pi^2Es^2}\left[G_1(s)|\mathbf{k}|^2\cos^2\theta + \mathrm{G_2(s)E^2} + 4\mathrm{G_3(s)E}|\mathbf{k}|\cos\theta\right]. \tag{4.62}$$

We will find the total cross-section of our process with two mediators using approximation $m_\mu = 0$ as

$$\Rightarrow \sigma = \frac{e^4}{32\pi}\left[\frac{2}{3}G_1(s) + 2G_2(s)\right]. \tag{4.63}$$

Note that the approximation $m_\mu = 0$ leads to $G_1(s) = G_2(s)$. The nomalised distribution of cross-section is then reads

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{3}{16\pi}\left[(1+\cos^2\theta) + \frac{4\mathrm{G}_3(\mathrm{s})}{\mathrm{G}_1(\mathrm{s})}\cos\theta\right]. \tag{4.64}$$

Using the same trick, Now we compare with Eq.(4.55) and use the relations in Eq.(4.56), we have

$$V_4 = 2\eta\langle S_3\rangle = \frac{4G_3(s)}{G_1(s)}, \qquad V_0 = \frac{2}{3} - 2\sqrt{\frac{2}{3}}\langle T_0\rangle = 0. \tag{4.65}$$

Then two observables values read

$$\langle T_0\rangle = \frac{1}{\sqrt{6}}, \qquad \langle S_3\rangle = \frac{2G_3(s)}{\eta G_1(s)}. \tag{4.66}$$

Note that in both Z and Z-photon density matrix, there are only two non-zero observable. While $\langle T_0\rangle$ in both cases have the same value and it is a constant. Thus, let's focus on the $\langle S_3\rangle$ value, it is just a number in Z density matrix while it becomes a function on the threshold energy $\sqrt{s}$ in the so-called Z-photon density matrix. The values of $\langle S_3\rangle$ are indicated in Fig.(4.2), we can see that the intersection is at $\sqrt{s} = m_Z$, so the so-called density matrix of Z and photon will becomes the density matrix of Z boson when the threshold energy equal to mass of Z boson.

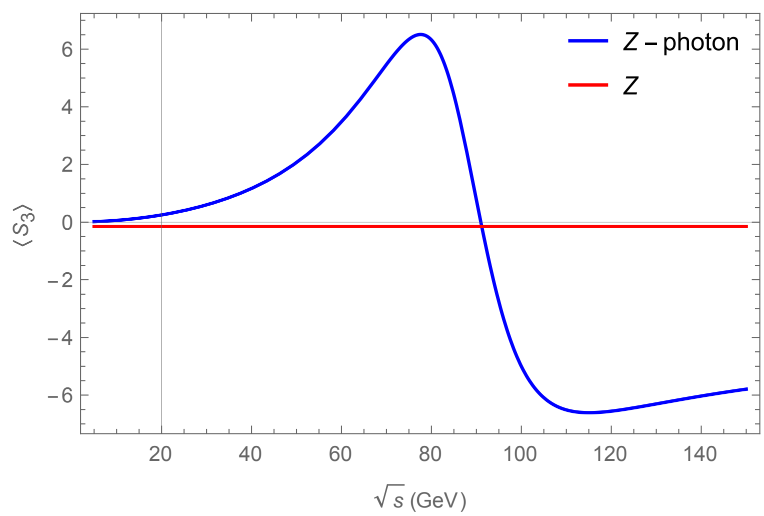


Figure 4.1: The values of $\langle S_3\rangle$ in two density matries

4.5 The so-called "spin-density matrix" of Z boson and photon in SMEFT

Using the result in Chap.3, we have the cross-section of the process $e^+ + e^- \to \mu^+ + \mu^-$ in SMEFT as follows

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left[A(\sqrt{s})(1+\cos^2\theta) + \mathrm{B}(\sqrt{\mathrm{s}})\cos\theta\right], \tag{4.67}$$

and the total cross-section reads

$$\sigma = \frac{1}{12\pi s}A(\sqrt{s}). \tag{4.68}$$

Now, we are able to find the normalized distribution of cross-section as

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega} = \frac{3}{16\pi}\left[(1+\cos^2\theta) + \frac{\mathrm{B}(\sqrt{\mathrm{s}})}{\mathrm{A}(\sqrt{\mathrm{s}})}\cos\theta\right]. \tag{4.69}$$

Once again, we compare with Eq.(4.55) and use the relations in Eq.(4.56)

$$V_4 = 2\eta\langle S_3\rangle = \frac{B(\sqrt{s})}{A(\sqrt{s})}, \tag{4.70}$$

$$V_0 = \frac{2}{3} - 2\sqrt{\frac{2}{3}}\langle T_0\rangle = 0. \tag{4.71}$$

Then two observables values then read

$$\langle T_0\rangle = \frac{1}{\sqrt{6}}, \qquad \langle S_3\rangle = \frac{B(\sqrt{s})}{2\eta A(\sqrt{s})}. \tag{4.72}$$

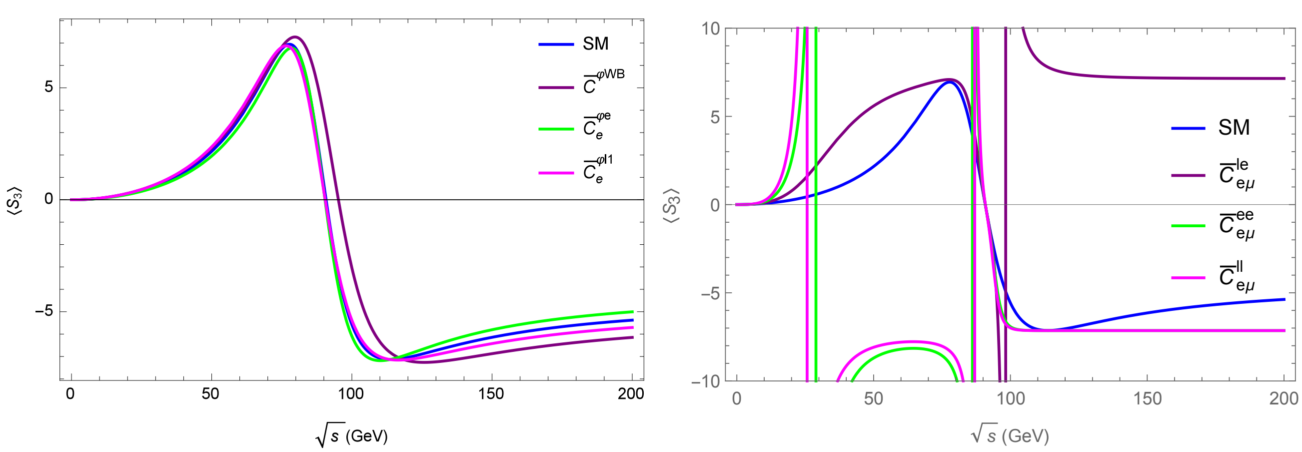


Figure 4.2: The values of $\langle S_3\rangle$ in SMEFT when the $\bar{C}_{\mathrm{Wilson}} = 0.1$

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