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INTRODUCTION

In this paper I will represent what I learnt last two weeks and what I'am going to learn in next time. However, since my presentation is about neutrino oscillation so I will represent in detail this one in the chapter 2.

In chapter one, I have learnt about the theory of neutrino physics. In which there are the theory of weak interaction as a basic background, the properties of neutrino in SM and the seesaw mechanism.

Chapter two is about neutrino oscillation. Since T2K experiment concerns with a search for neutrino oscillation from muon neutrinos to electron neutrinos, I'll represent here detailed calculation of oscillation probability of that case and involving dicussion about CP violation.

In chapter three, I intend to learn about neutrino oscillation experiments, particularly focus on T2K.

CHAPTER 1
THE THEORY OF NEUTRINO PHYSICS

+ How parity is conserved in QED and QCD.

+ Wu 1957 experiment about nuclear β decay and the parity violation in weak interaction leads to V-A structure of the weak interaction.

+ Chiral structure leads to the existance of *left-handed particles* and *right-handed antiparticles* in weak interaction.

+ Neutrino properties in the SM such as form a $SU(2)_L$ doublet with lepton, form left Weyl spinors.

+ Due to neutral current interaction at LEP leads to an argument that there must be three families of neutrino in SM.

+ Dirac and Majoran neutrinos.

+ Seesaw mechanism.

CHAPTER 2

NEUTRINO OSCILLATION

0.1 Mass and flavor eigenstates

The flavor eigenstates are related to the mass eigenstates by the 3×3 unitary PMNS matrix.

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{1}$$

PMNS matrix is unitary so $U^{-1} = U^\dagger \equiv (U^*)^T$. And hence, the mass eigenstates also can be performed via flavor eigenstates as

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} U^*_{e1} & U^*_{e2} & U^*_{e3} \\ U^*_{\mu1} & U^*_{\mu2} & U^*_{\mu3} \\ U^*_{\tau1} & U^*_{\tau2} & U^*_{\tau3} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{2}$$

The unitary condition $UU^\dagger = I$ implies:

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{pmatrix} \begin{pmatrix} U^*_{e1} & U^*_{\mu1} & U^*_{\tau1} \\ U^*_{e2} & U^*_{\mu2} & U^*_{\tau2} \\ U^*_{e3} & U^*_{\mu3} & U^*_{\tau3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

In compact form

$$\sum_i^3 U_{\alpha,i} U^*_{\beta,i} = \sigma_{\alpha\beta} \tag{4}$$

0.2 Wave function

From (1), we derive wave function at time $t = 0$

$$|\Psi(0)\rangle = |\nu_\mu\rangle = U^*_{\mu1}|\nu_1\rangle + U^*_{\mu2}|\nu_2\rangle + U^*_{\mu3}|\nu_3\rangle \tag{5}$$

Time-dependent wave function

$$|\Psi(\vec{x},t)\rangle = U^*_{\mu 1}|\nu_1\rangle e^{-i\phi_1} + U^*_{\mu 2}|\nu_2\rangle e^{-i\phi_2} + U^*_{\mu 3}|\nu_3\rangle e^{-i\phi_3} \tag{6}$$

In compact form

$$|\nu_\alpha\rangle = \sum_i^3 U^*_{\alpha,i}|\nu_i\rangle e^{-i\phi_i} \tag{7}$$

Where $\phi_i = p_i.x_i = E_i t - \vec{p}_i.\vec{x}_i$
From (2) we have

$$|\nu_1\rangle = U_{e1}|\nu_e\rangle + U_{\mu 1}|\nu_\mu\rangle + U_{\tau 1}|\nu_\tau\rangle$$

$$|\nu_2\rangle = U_{e2}|\nu_e\rangle + U_{\mu 2}|\nu_\mu\rangle + U_{\tau 2}|\nu_\tau\rangle$$

$$|\nu_3\rangle = U_{e3}|\nu_e\rangle + U_{\mu 3}|\nu_\mu\rangle + U_{\tau 3}|\nu_\tau\rangle$$

The equation (6) can be rewritten as

$$\begin{aligned}
|\Psi(\vec{x},t)\rangle &= U^*_{\mu 1}(U_{e1}|\nu_e\rangle + U_{\mu 1}|\nu_\mu\rangle + U_{\tau 1}|\nu_\tau\rangle)e^{-i\phi_1} \\
&+ U^*_{\mu 2}(U_{e2}|\nu_e\rangle + U_{\mu 2}|\nu_\mu\rangle + U_{\tau 2}|\nu_\tau\rangle)e^{-i\phi_2} \\
&+ U^*_{\mu 3}(U_{e3}|\nu_e\rangle + U_{\mu 3}|\nu_\mu\rangle + U_{\tau 3}|\nu_\tau\rangle)e^{-i\phi_3} \\
&= (U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3})|\nu_e\rangle \\
&+ (U^*_{\mu 1}U_{\mu 1}e^{-i\phi_1} + U^*_{\mu 2}U_{\mu 2}e^{-i\phi_2} + U^*_{\mu 3}U_{\mu 3}e^{-i\phi_3})|\nu_\mu\rangle \\
&+ (U^*_{\mu 1}U_{\tau 1}e^{-i\phi_1} + U^*_{\mu 2}U_{\tau 2}e^{-i\phi_2} + U^*_{\mu 3}U_{\tau 3}e^{-i\phi_3})|\nu_\tau\rangle \\
&= c_e|\nu_e\rangle + c_\mu|\nu_\mu\rangle + c_\tau|\nu_\tau\rangle
\end{aligned} \tag{8}$$

In compact form

$$|\Psi(\vec{x},t)\rangle = \sum_i U^*_{\alpha i}U_{\beta i}e^{-i\phi_i}|\nu_\beta\rangle \tag{9}$$

0.3 Oscillation probability

The oscillation probability from muon neutrino to electron neutrino is

$$\begin{aligned}
P(\nu_\mu \to \nu_e) &= |\langle\nu_e|\Psi(\vec{x},t)\rangle|^2 = c_e c_e^* \\
&= |U^*_{\mu 1}U_{e1}e^{-i\phi_1} + U^*_{\mu 2}U_{e2}e^{-i\phi_2} + U^*_{\mu 3}U_{e3}e^{-i\phi_3}|^2
\end{aligned} \tag{10}$$

In compact form

$$P(\nu_\alpha \to \nu_\beta) = |\sum_i U^*_{\alpha i}U_{\beta i}e^{-i\phi_i}|^2 \tag{11}$$

If $\phi_1 = \phi_2 = \phi_3 (\approx \frac{m^2}{2E})$, from unitary condition (4) we have $P(\nu_\alpha \to \nu_\beta) = \sigma_{\alpha\beta}$. This means that the oscillations occur if the neutrinos have mass and the masses are not the same.

Using the identity properties of complex number:

$$|z_1 + z_2 + z_3|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2Re[z_1 z_2^* + z_1 z_3^* + z_2 z_3^*] \tag{12}$$

Then equation (10) becomes

$$\begin{aligned} P(\nu_\mu \to \nu_e) &= |U_{\mu 1}^* U_{e1} e^{-i\phi_1} + U_{\mu 2}^* U_{e2} e^{-i\phi_2} + U_{\mu 3}^* U_{e3} e^{-i\phi_3}|^2 \\ &= |U_{\mu 1}^* U_{e1}|^2 + |U_{\mu 2}^* U_{e2}|^2 + |U_{\mu 3}^* U_{e3}|^2 \\ &+ 2Re[U_{\mu 1}^* U_{e1} U_{\mu_2} U_{e2}^* e^{i(\phi_2 - \phi_1)}] \\ &+ 2Re[U_{\mu 1}^* U_{e1} U_{\mu_3} U_{e3}^* e^{i(\phi_3 - \phi_1)}] \\ &+ 2Re[U_{\mu 2}^* U_{e2} U_{\mu_3} U_{e3}^* e^{i(\phi_3 - \phi_2)}] \end{aligned} \tag{13}$$

In compact form

$$P(\nu_\alpha \to \nu_\beta) = \sum_i |U_{\alpha i}^* U_{\beta i}|^2 + 2 \sum_{j>i} Re[U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta_j}^* e^{i(\phi_j - \phi_i)}] \tag{14}$$

From the unitary condition (3) we have

$$\begin{aligned} & |U_{\mu 1}^* U_{e1} + U_{\mu 2}^* U_{e2} + U_{\mu 3}^* U_{e3}|^2 = 0 \\ \Rightarrow\ & |U_{\mu 1}^* U_{e1}|^2 + |U_{\mu 2}^* U_{e2}|^2 + |U_{\mu 3}^* U_{e3}|^2 \\ +\ & 2Re[U_{\mu 1}^* U_{e1} U_{\mu_2} U_{e2}^* + U_{\mu 1}^* U_{e1} U_{\mu_3} U_{e3}^* + U_{\mu 2}^* U_{e2} U_{\mu_3} U_{e3}^*] \\ =\ & 0 \end{aligned} \tag{15}$$

In compact form

$$\sum_i |U_{\alpha i}^* U_{\beta i}|^2 + 2 \sum_{j>i} Re[U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta_j}^*] = \sigma_{\alpha\beta} \tag{16}$$

It is followed from (13) and (15):

$$\begin{aligned} P(\nu_\mu \to \nu_e) &= 2Re\left[U_{\mu 1}^* U_{e1} U_{\mu_2} U_{e2}^* \left(e^{i(\phi_2 - \phi_1)} - 1\right)\right] \\ &+ 2Re\left[U_{\mu 1}^* U_{e1} U_{\mu_3} U_{e3}^* \left(e^{i(\phi_3 - \phi_1)} - 1\right)\right] \\ &+ 2Re\left[U_{\mu 2}^* U_{e2} U_{\mu_3} U_{e3}^* \left(e^{i(\phi_3 - \phi_2)} - 1\right)\right] \end{aligned} \tag{17}$$

In compact form

$$P(\nu_\alpha \to \nu_\beta) = \sigma_{\alpha\beta} + 2 \sum_{j>i} Re[U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta_j}^* (e^{i(\phi_j - \phi_i)}) - 1)] \tag{18}$$

We have

$$
\begin{aligned}
& Re[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta_j}(e^{i(\phi_j-\phi_i)})-1)] \\
= & Re[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta_j}(\cos(\phi_j-\phi_i)-1+i\sin(\phi_j-\phi_i))] \\
= & Re\left\{(Re[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta_j}]+iIm[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta_j}])(-2\sin^2(\frac{\phi_j-\phi_i}{2})+i\sin(\phi_j-\phi_i))\right\} \\
= & -2Re[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta_j}]\sin^2(\frac{\phi_j-\phi_i}{2})-Im[U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta_j}]\sin(\phi_j-\phi_i) \qquad (19)
\end{aligned}
$$

From (19), we can write the oscillation pobability in a normal form

$$
\begin{aligned}
& P(\nu_\mu\to\nu_e)= \\
- & 4Re\left[U^*_{\mu 1}U_{e1}U_{\mu_2}U^*_{e2}\right]\sin^2(\frac{\phi_2-\phi_1}{2})-2Im\left[U^*_{\mu 1}U_{e1}U_{\mu_2}U^*_{e2}\right]\sin(\phi_2-\phi_1) \\
- & 4Re\left[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e3}\right]\sin^2(\frac{\phi_3-\phi_1}{2})-2Im\left[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e3}\right]\sin(\phi_3-\phi_1) \\
- & 4Re\left[U^*_{\mu 2}U_{e2}U_{\mu_3}U^*_{e3}\right]\sin^2(\frac{\phi_3-\phi_2}{2})-2Im\left[U^*_{\mu 2}U_{e2}U_{\mu_3}U^*_{e3}\right]\sin(\phi_3-\phi_2) \qquad (20)
\end{aligned}
$$

In compact form

$$
\begin{aligned}
P(\nu_\alpha\to\nu_\beta)=\sigma_{\alpha\beta} & - 4\sum_{j>i}Re\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin^2(\frac{\phi_j-\phi_i}{2}) \\
& - 2\sum_{j>i}Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin(\phi_j-\phi_i) \qquad (21)
\end{aligned}
$$

If the neutrino interacts at a time T at a distance L along its direction of flight, the difference in phase of the three mass eigenstates are written as

$$\phi_j-\phi_i=p_j.x_j-p_i.x_i=(E_j-E_i)T-(p_j-p_i)L$$

With assuming that $p_j=p_i=p$ for neutrinos of the same source, then

$$
\begin{aligned}
\phi_j-\phi_i &= (E_j-E_i)T\approx\left[p_j(1+\frac{m_j^2}{2p_j^2})-p_i(1+\frac{m_i^2}{2p_i^2})\right]T \\
&= \frac{m_j^2-m_i^2}{2p}T=\frac{\Delta m_{ji}^2L}{2E} \qquad (22)
\end{aligned}
$$

In the above calculation, we used the approximation $T\approx L$ and $p\approx E$ for $v_\nu\approx c$ and $m_\nu\ll E_\nu$

We finally get the most common form of the oscillation probability:

$$
\begin{aligned}
P(\nu_\alpha\to\nu_\beta)=\sigma_{\alpha\beta} & - 4\sum_{j>i}Re\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin^2(\frac{\Delta m_{ji}^2}{4E}L) \\
& - 2\sum_{j>i}Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin(\frac{\Delta m_{ji}^2}{2E}L) \qquad (23)
\end{aligned}
$$

For antineutrinos, we just take the complex conjugate of the product matrix and get

$$\begin{aligned} P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = \sigma_{\alpha\beta} &- 4\sum_{j>i} Re\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin^2(\frac{\Delta m^2_{ji}}{4E}L) \\ &+ 2\sum_{j>i} Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin(\frac{\Delta m^2_{ji}}{2E}L) \end{aligned} \tag{24}$$

The probabilities (23) and (24) are called *transition probabilities*, and the *survival probability* for a flavor is

$$P(\nu_\alpha \to \nu_\alpha) = P(\bar{\nu}_\alpha \to \bar{\nu}_\alpha) = 1 - 4\sum_{j>i}\left[|U_{\alpha i}|^2|U_{\alpha_j}|^2\right]\sin^2(\frac{\Delta m^2_{ji}}{4E}L) \tag{25}$$

By using the natural unit conversion for $1eV^{-1}$of length $= 1.97 \times 10^{-7}m$, for practical purpose we can express the phase in (23), (24) and (25) as

$$\begin{aligned} \frac{\Delta m^2_{ji}[eV^2]L[eV]}{4E[eV]} &= \frac{\Delta m^2_{ji}[eV^2]L[m]}{4 \times 1.97 \times 10^{-7}E[eV]} \\ = 1.269\frac{\Delta m^2_{ji}[eV^2]L[m]}{E[MeV]} &= 1.269\frac{\Delta m^2_{ji}[eV^2]L[km]}{E[GeV]} \end{aligned} \tag{26}$$

0.4 CP violation

From (23) and (24), the difference between the neutrino and antineutrino oscillation probability indicates CP violation

$$\begin{aligned} \text{CP violation} &= P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \\ &= 4\sum_{j>i} Im\left[U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}\right]\sin(\frac{\Delta m^2_{ij}}{2E}L) \end{aligned} \tag{27}$$

If CP is violated, $U^*_{\alpha i}U_{\beta i}U_{\alpha_j}U^*_{\beta_j}$ has to contain an imaginary component.

For $\alpha = \mu$ and $\beta = e$, then

$$\begin{aligned}
\text{CP violation} &= P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e) \\
&= 4\sum_{j>i} Im\left[U^*_{\mu i} U_{ei} U_{\mu_j} U^*_{e_j}\right] \sin(\frac{\Delta m^2_{ij}}{2E} L) \qquad (28) \\
&= 4Im\left[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2}\right] \sin(\frac{\Delta m^2_{12}}{2E} L) \\
&+ 4Im\left[U^*_{\mu 1} U_{e1} U_{\mu_3} U^*_{e_3}\right] \sin(\frac{\Delta m^2_{13}}{2E} L) \\
&+ 4Im\left[U^*_{\mu 2} U_{e2} U_{\mu_3} U^*_{e_3}\right] \sin(\frac{\Delta m^2_{23}}{2E} L) \qquad (29)
\end{aligned}$$

From the unitary condition (3) we have

$$U_{\mu 1} U^*_{e1} + U_{\mu_2} U^*_{e_2} + U_{\mu 3} U^*_{e3} = 0 \qquad (30)$$

Multiply two sides of the equation (30) with $U^*_{\mu 1} U_{e1}$ and $U^*_{\mu 2} U_{e2}$ respectively and then add them up, we have

$$\begin{aligned}
& U^*_{\mu 1} U_{e1} U_{\mu 1} U^*_{e1} + U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2} + U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3} \\
+ \quad & U^*_{\mu 2} U_{e2} U_{\mu 1} U^*_{e1} + U^*_{\mu 2} U_{e2} U_{\mu_2} U^*_{e_2} + U^*_{\mu 2} U_{e2} U_{\mu 3} U^*_{e3} = 0 \\
\Leftrightarrow \quad & 0 = |U_{\mu 1}|^2 |U_{e1}|^2 + |U_{\mu 2}|^2 |U_{e2}|^2 \\
+ \quad & Re[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2}] + Re[U^*_{\mu 2} U_{e2} U_{\mu 1} U^*_{e1}] + Re[U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3}] + Re[U^*_{\mu 2} U_{e2} U_{\mu 3} U^*_{e3}] \\
+ \quad & i\left\{Im[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2}] + Im[U^*_{\mu 2} U_{e2} U_{\mu 1} U^*_{e1}] + Im[U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3}] + Im[U^*_{\mu 2} U_{e2} U_{\mu 3} U^*_{e3}]\right\} \\
\Rightarrow \quad & Im[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2}] + Im[U^*_{\mu 2} U_{e2} U_{\mu 1} U^*_{e1}] + Im[U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3}] + Im[U^*_{\mu 2} U_{e2} U_{\mu 3} U^*_{e3}] = 0
\end{aligned} \qquad (31)$$

Note that $[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2}]^* = U^*_{\mu 2} U_{e2} U_{\mu 1} U^*_{e1} \Rightarrow Im[U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2}] = -Im[U^*_{\mu 2} U_{e2} U_{\mu 1} U^*_{e1}]$ Therefore, from (31) we get

$$Im[U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3}] = -Im[U^*_{\mu 2} U_{e2} U_{\mu 3} U^*_{e3}] \qquad (32)$$

Multiply two sides of the equation (30) with $U^*_{\mu 1} U_{e1}$ and $U^*_{\mu 3} U_{e3}$ respectively and then add them up, we have

$$\begin{aligned}
& U^*_{\mu 1} U_{e1} U_{\mu 1} U^*_{e1} + U^*_{\mu 1} U_{e1} U_{\mu_2} U^*_{e_2} + U^*_{\mu 1} U_{e1} U_{\mu 3} U^*_{e3} \\
+ \quad & U^*_{\mu 3} U_{e3} U_{\mu 1} U^*_{e1} + U^*_{\mu 3} U_{e3} U_{\mu_2} U^*_{e_2} + U^*_{\mu 3} U_{e3} U_{\mu 3} U^*_{e3} = 0 \\
\Leftrightarrow \quad & 0 = |U_{\mu 1}|^2 |U_{e1}|^2 + |U_{\mu 3}|^2 |U_{e3}|^2 \\
+ \quad & Re[U^*_{\mu 1} U_{e1} U_{\mu_3} U^*_{e_3}] + Re[U^*_{\mu 3} U_{e3} U_{\mu 1} U^*_{e1}] + Re[U^*_{\mu 1} U_{e1} U_{\mu 2} U^*_{e2}] + Re[U^*_{\mu 3} U_{e3} U_{\mu 2} U^*_{e2}] \\
+ \quad & i\left\{Im[U^*_{\mu 1} U_{e1} U_{\mu_3} U^*_{e_3}] + Im[U^*_{\mu 3} U_{e3} U_{\mu 1} U^*_{e1}] + Im[U^*_{\mu 1} U_{e1} U_{\mu 2} U^*_{e2}] + Im[U^*_{\mu 3} U_{e3} U_{\mu 2} U^*_{e2}]\right\} \\
\Rightarrow \quad & Im[U^*_{\mu 1} U_{e1} U_{\mu_3} U^*_{e_3}] + Im[U^*_{\mu 3} U_{e3} U_{\mu 1} U^*_{e1}] + Im[U^*_{\mu 1} U_{e1} U_{\mu 2} U^*_{e2}] + Im[U^*_{\mu 3} U_{e3} U_{\mu 2} U^*_{e2}] = 0
\end{aligned} \qquad (33)$$

Note that $[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e_3}]^* = U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1} \Rightarrow Im[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e_3}] = -Im[U^*_{\mu 3}U_{e3}U_{\mu 1}U^*_{e1}]$ and $[U^*_{\mu 3}U_{e3}U_{\mu 2}U^*_{e2}]^* = U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3} \Rightarrow Im[U^*_{\mu 3}U_{e3}U_{\mu 2}U^*_{e2}] = -[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}]$

Therefore, from (33) we get

$$Im[U^*_{\mu 1}U_{e1}U_{\mu 2}U^*_{e2}] = Im[U^*_{\mu 2}U_{e2}U_{\mu 3}U^*_{e3}] \tag{34}$$

By using (32) and (34), we can rewrite (29) as

$$\begin{aligned} \text{CP violation} &= P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e) \\ &= 4Im\left[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e_3}\right](\sin\Delta_{13} - \sin\Delta_{12} - \sin\Delta_{23}) \end{aligned} \tag{35}$$

Where $\Delta_{13} = \frac{\Delta m^2_{13}}{2E}L$, $\Delta_{12} = \frac{\Delta m^2_{12}}{2E}L$ and $\Delta_{23} = \frac{\Delta m^2_{23}}{2E}L = \Delta_{13} - \Delta_{23}$

By a simple trigonometry calculation, we have

$$\begin{aligned} \sin\Delta_{13} - \sin\Delta_{12} - \sin(\Delta_{13} - \Delta_{12}) &= -4\sin\frac{\Delta_{12}}{2}\sin\frac{\Delta_{13}}{2}\sin\frac{\Delta_{23}}{2} \\ &= 4\sin\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2}\sin\frac{\Delta_{32}}{2} \end{aligned}$$

Then we can rewrite (36) as

$$\begin{aligned} \text{CP violation} &= P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e) \\ &= 16Im\left[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e_3}\right]\left(\sin\frac{\Delta_{21}}{2}\sin\frac{\Delta_{31}}{2}\sin\frac{\Delta_{32}}{2}\right) \\ &= 16Im\left[U^*_{\mu 1}U_{e1}U_{\mu_3}U^*_{e_3}\right]\sin\left(\frac{\Delta m^2_{21}L}{4E}\right)\sin\left(\frac{\Delta m^2_{31}L}{4E}\right)\sin\left(\frac{\Delta m^2_{32}L}{4E}\right) \end{aligned} \tag{36}$$

0.5 The PMNS matrix

The unitarity of PMNS matrix $UU^\dagger = I$ yields nine independent parameters.

If the PMNS matrix were real, it could be described by three rotation angles θ_{12}, θ_{13} and θ_{23} via orthogonal rotation matrix R

$$R = R_1 \times R_2 \times R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}\begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{37}$$

Where $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$.

Since PMNS matrix is unitary and not real, it must contain six more additional degrees of freedom in term of complex phase $e^{i\delta}$. Five among these six phases can be absorbed into the definition of the particles and leaves only one single phase δ.

The PMNS matrix then depends on three mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$ and a single complex phase δ and can be expressed as

$$
\begin{aligned}
U_{PMNS} &= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (38) \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \qquad (39)
\end{aligned}
$$

As mentioned before, CP is violated if $U^*_{\alpha i} U_{\beta i} U_{\alpha_j} U^*_{\beta_j}$ contains an imaginary component. Therefore, δ is also called the CP-violating phase.

0.6 Neutrino masses and neutrino mass hierarchy

CHAPTER 3

NEUTRINO OSCILLATION EXPERIMENTS

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