# Neutrino oscillation in vacuum

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# Contents

# The neutrino oscillation probability

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j} |\nu_j\rangle \qquad (\alpha = e, \mu, \tau) \tag{1}$$

where $U$ is the neutrino mixing matrix, $|\nu_j\rangle$ is the massive neutrino state, having mass $m_j \neq 0$.

The massive neutrino states $|\nu_j\rangle$ are eigenstates of the Hamiltonian with eigenvalues $E_j$

$$\mathcal{H}|\nu_j\rangle = E_j|\nu_j\rangle \tag{2}$$

$$E_j = \sqrt{\overrightarrow{p}^2 + m_j^2} \tag{3}$$

We consider the Schrödinger equation

$$i\frac{d}{dt}|\nu_j(t)\rangle = \mathcal{H}|\nu_j(t)\rangle \tag{4}$$

$$\Rightarrow |\nu_j(t)\rangle = e^{-iE_jt}|\nu_j\rangle \tag{5}$$

From (1) and (5), we have

$$|\nu_\alpha(t)\rangle = \sum_j U^*_{\alpha j} e^{-iE_j t} |\nu_j\rangle \quad (6)$$

such that

$$|\nu_\alpha(t=0)\rangle = |\nu_\alpha\rangle \quad (7)$$

Hence

$$|\nu_j\rangle = \sum_\alpha U_{\alpha j} |\nu_\alpha\rangle \quad (8)$$

Substituting (8) into (6), we get

$$|\nu_\alpha(t)\rangle = \sum_{\beta=e,\mu,\tau} \left( \sum_j U^*_{\alpha j} e^{-iE_j t} U_{\beta j} \right) |\nu_\beta\rangle \quad (9)$$

The amplitude of $\nu_\alpha \to \nu_\beta$ transitions is

$$A_{\nu_\alpha \to \nu_\beta}(t) \equiv \langle \nu_\beta | \nu_\alpha(t) \rangle = \sum_j U^*_{\alpha j} U_{\beta j} e^{-iE_j t} \tag{10}$$

Then we obtain the transition probability

$$P_{\nu_\alpha \to \nu_\beta}(t) = \left| A_{\nu_\alpha \to \nu_\beta}(t) \right|^2 = \sum_{j,k} U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} e^{-i(E_j - E_k)t} \tag{11}$$

For ultrarelativistic neutrino, we have

$$E_j \approx E + \frac{m_j^2}{2E} \tag{12}$$

$$E_j - E_k \approx \frac{\Delta m_{jk}^2}{2E} \tag{13}$$

$$E \approx \left| \overrightarrow{p} \right| \tag{14}$$

The transition probability is approximated by

$$P_{\nu_\alpha \to \nu_\beta}(t) = \sum_{j,k} U_{\alpha k} U_{\beta k}^* U_{\alpha j}^* U_{\beta j} \exp\left( -i \frac{\Delta m_{jk}^2 t}{2E} \right) \tag{15}$$

The propagation time $t$ can be approximated by the distance $L$ between the source and the detector, $t = L$. Therefore

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_{j,k} U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} \exp\left(-i \frac{\Delta m^2_{jk} L}{2E}\right) \quad (16)$$

The probability is rewritten in the form

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_{j} |U_{\alpha j}|^2 \, |U_{\alpha j}|^2 + 2 \sum_{j>k} \mathfrak{Re}\left[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} \exp\left(-i \frac{\Delta m^2_{jk} L}{2E}\right)\right] \quad (17)$$

Hence

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sum_j |U_{\alpha j}|^2 |U_{\alpha j}|^2 + 2\sum_{j>k} \mathfrak{Re}\left[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \cos\left(\frac{\Delta m^2_{jk} L}{2E}\right) + 2\sum_{j>k} \mathfrak{Im}\left[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \sin\left(\frac{\Delta m^2_{jk} L}{2E}\right) \tag{18}$$

We have

$$\sum_j |U_{\alpha j}|^2 |U_{\alpha j}|^2 = \delta_{\alpha\beta} - 2\sum_{j>k} \mathfrak{Re}\left[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \tag{19}$$

Therefore

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \delta_{\alpha\beta} - 4 \sum_{j>k} \mathfrak{Re}\left[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \sin^2\left(\frac{\Delta m^2_{jk} L}{2E}\right) + 2 \sum_{j>k} \mathfrak{Im}\left[U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j}\right] \sin\left(\frac{\Delta m^2_{jk} L}{2E}\right) \tag{20}$$

Let us consider the antineutrinos. The antineutrino oscillation, $\overline{\nu}_\alpha \to \overline{\nu}_\beta$, probability

$$P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta}(L, E) = \sum_{j,k} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \exp\left( -i \frac{\Delta m^2_{jk} L}{2E} \right) \quad (21)$$

We have

$$P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta}(L, E) = \delta_{\alpha\beta} - 4 \sum_{j>k} \mathfrak{Re}\left[ U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} \right] \sin^2\left( \frac{\Delta m^2_{jk} L}{2E} \right)$$
$$- 2 \sum_{j>k} \mathfrak{Im}\left[ U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} \right] \sin\left( \frac{\Delta m^2_{jk} L}{2E} \right) \quad (22)$$

# CP violation

The condition of CP invariance is:

$$P_{\nu_\alpha \to \nu_\beta} = P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} \tag{23}$$

Then the CP violation in neutrino oscillation experiments can be measured only in transition between different flavors and could reveal by measuring the asymmetry:

$$A^{CP}_{\alpha\beta} = P_{\nu_\alpha \to \nu_\beta} - P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta} \tag{24}$$

From (20) and (22), we obtain

$$A^{CP}_{\alpha\beta} = 4 \sum_{j>k} \mathfrak{Im} \left[ U_{\alpha k} U^*_{\beta k} U^*_{\alpha j} U_{\beta j} \right] \sin\left( \frac{\Delta m^2_{jk} L}{2E} \right) \tag{25}$$