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Belle II school lectures

- Yoshihide Sakai (KEK, Tsukuba): "CP violation in B decays".
- Nguyen Van Hanh (VNUA, Hanoi): "Practical statistics for particle physics analyses".
- Nguyen Thi Hong Van (IOP, VAST, Hanoi and IFIRSE, Quy Nhon): **Brief course on ROOT and its application to the data analysis in high energy physics.**
- Ha Huy Bang (VNU, Hanoi): "Standard model and CP violation".
- Nguyen Anh Tuan (VNU, Hanoi): Introduction to Python for scientific computing.
- Takanori Hara (KEK, Tsukuba): "Belle II detector overview".
- Dimitri Liventsev (KEK, Tsukuba and Virginia tech, Blacksburg): Search for New Physics particles at Belle II.
- Shohei Nishida (KEK, Tsukuba): "Particle identification (from Belle to Belle II)".
- Yukio Onishi (KEK, Tsukuba): "SuperKEKB".
- Dong Van Thanh (SOKENDAI and KEK, Tsukuba): Calibration and alignment for Belle II central drift chamber.
- Karim Trabelsi (KEK, Tsukuba): Rare B decays.
- Ikuo Ueda (KEK, Tsukuba): Computing in HEP (Belle II as illustration)".
- Shoji Uno (KEK, Tsukuba): "Tracking devices at Belle II".
- Changzheng Yuan (IHEP, Beijing): "Hadron spectroscopy at Belle, BESIII and Belle II".

Belle II and B physics

KEKB/Belle → SuperKEKB/Belle II

KEK B-factory accelerator
(e^+e^- asymmetric energy collider)

Luminosity Frontier
World highest Lum.

Main goal
Study of CPV in B decays

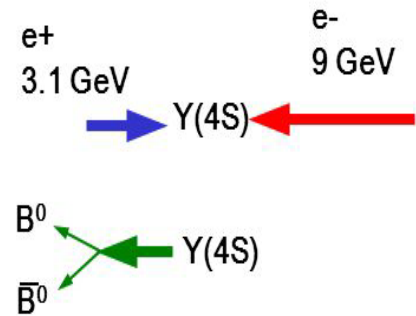
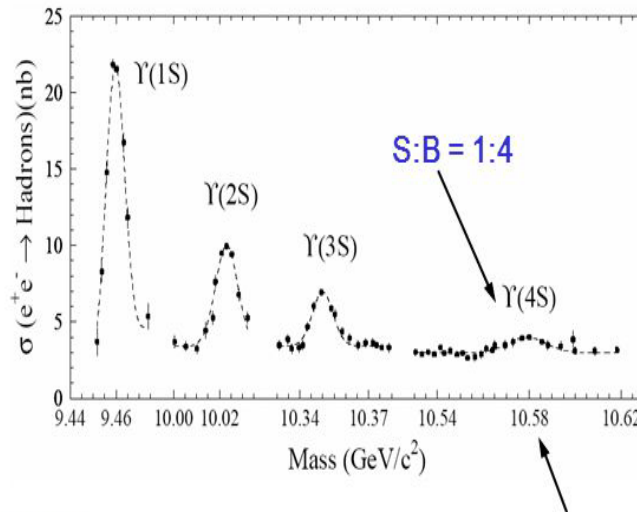
Many other physics
(Rare B decays,
Charm/tau physics
New resonances !)

Now Upgraded to
SuperKEKB/Belle II
x40 Peak Luminosity
x50 accumulated data



Belle II and B physics

Asymmetric B Factories



Y(4S) meson: $b\bar{b}$ bound state with mass $10.58 \text{ GeV}/c^2$

Just above $2 \times$ mass of B meson \rightarrow decays exclusively to $B^0 \bar{B}^0$ (50%) and $B^+ B^-$ (50%)

B factory: intense e^+ and e^- colliding beams with E_{CM} tuned to the Y(4S) mass

Use e beams with **asymmetric energy** \rightarrow **time dilation** due to relativistic speeds keeps B's alive long enough to measure them (decay length $\sim 0.25 \text{ mm}$)

Search for CPV at Belle II

CP Violation

CPV: difference in behavior of **particle** and **anti-particle**

1964: discovered in K^0 decay (J.Cronin, V.Fitch et. al.)
[PRL 13, 138]

Observation of $K_L \rightarrow \pi^+ \pi^- \rightarrow$ CP Violation

$$[K^0-\bar{K}^0 \text{ mixing}] \quad \begin{cases} |K_1\rangle = |K^0\rangle + |\bar{K}^0\rangle & [CP=+1] \\ |K_2\rangle = |K^0\rangle - |\bar{K}^0\rangle & [CP=-1] \end{cases}$$

If CP conserves $K_1=K_S, K_2=K_L$

$K_S \rightarrow \pi^+ \pi^-$ (CP=+1), $K_L \rightarrow \pi^+ \pi^- \pi^0$ (CP=-1)

Branching fraction = 2.3×10^{-3}

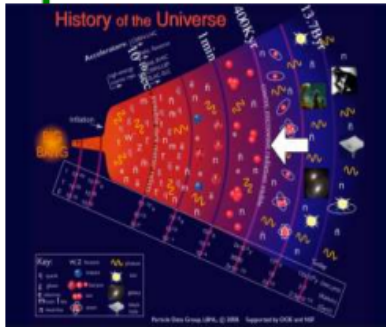


Why CPV is important ?

Difference between particle & anti-particle
(matter & anti-matter)

Universe: almost “matter” only (no anti-matter)

Big-Bang \rightarrow $N(\text{particles}) = N(\text{anti-particles})$



Sakhalov's 3 conditions (1967):

1. baryon number violation
2. **CP violation**
3. existence of non-equilibrium

CPV is a key for Existence of Universe & us !

Andrei Sakharov (1921-1989)



Search for CPV at Belle II

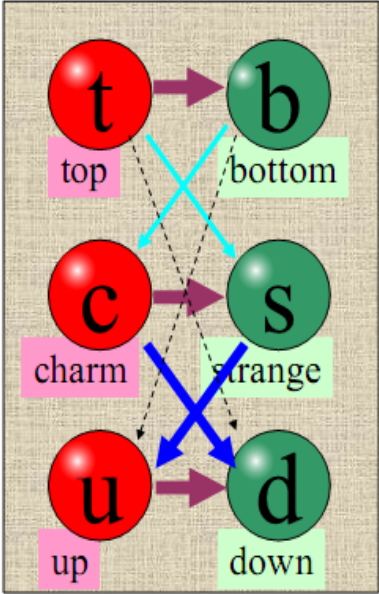
CPV: Why B ?

Size of CPV in K: $O(10^{-3}) \sim$ small
not enough information to confirm
Kobayashi-Maskawa scheme (1973)

Specialty of B
long lifetime (~ 1.5 ps)
Large B^0 - B^0 mixing
(ARGUS, 1987)
Various decay modes

Surprise
Lucky !

Sanda-Bigi-Carter (1980)
Large CPV
in B-system



Search for CPV at Belle II

- $B^0 - \bar{B}^0$ mixing

$B^0 - \bar{B}^0$ Mixing (1)

Most important role in CPV in B decays (mixing: also in K decays)

$$|B^0\rangle \equiv (\bar{b}d), \quad |\bar{B}^0\rangle \equiv (b\bar{d}), \quad \text{where } |\bar{B}^0\rangle \equiv CP|B^0\rangle$$

Flavor eigenstates and Mass eigenstates are different

Mass eigenstate

$$|B_L\rangle = p|B^0\rangle + q|\bar{B}^0\rangle \quad (\text{L : Light})$$

$$|B_H\rangle = p|B^0\rangle - q|\bar{B}^0\rangle \quad (\text{H : Heavy})$$

Basic Quantum mechanics

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - (i/2)\Gamma_{12}^*}{M_{12} - (i/2)\Gamma_{12}}}$$

Schrodinger Eq. $i\hbar \frac{d}{dt}\Psi(t) = H \Psi(t) \quad \Psi(t) = \begin{bmatrix} |B^0(t)\rangle \\ |\bar{B}^0(t)\rangle \end{bmatrix}$

$$H = M - i\Gamma/2 = \begin{bmatrix} M_{11} - i\Gamma_{11}/2 & M_{12} - i\Gamma_{12}/2 \\ M_{12}^* - i\Gamma_{12}^*/2 & M_{22} - i\Gamma_{22}/2 \end{bmatrix} \quad \text{Mass matrix Hamiltonian}$$



$$p = \frac{1}{\sqrt{2}} \frac{1 + \epsilon_B}{\sqrt{1 + |\epsilon_B|^2}}, \quad q = \frac{1}{\sqrt{2}} \frac{1 - \epsilon_B}{\sqrt{1 + |\epsilon_B|^2}}$$

CP is violated if $\epsilon_B \neq 0 \Leftrightarrow |q/p| \neq 1$

Search for CPV at Belle II

- $B^0 - \bar{B}^0$ mixing

$B^0 - \bar{B}^0$ Mixing (2)

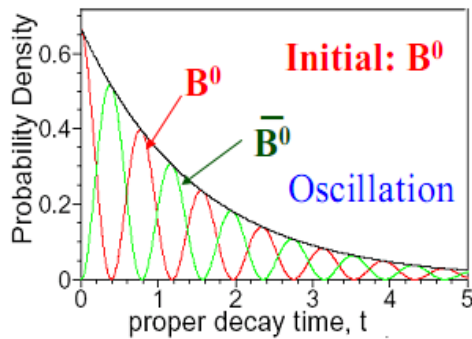
Time Evolution : $|B_{L/H}(t)\rangle \sim \exp(-im_{L/H}t - t/2\tau_{L/H}) |B_{L/H}\rangle$

Produced as pure \bar{B}^0 and B^0 $|\bar{B}^0\rangle = (|B_L\rangle + |B_H\rangle)/2p(q)$

$$|B_d^0(0)\rangle \Rightarrow |B_d^0(t)\rangle = e^{-im_s t} e^{-t/2\tau_s} \left\{ \cos(\Delta m_B t / 2) |B_d^0\rangle + \frac{iq}{p} \sin(\Delta m_B t / 2) |\bar{B}_d^0\rangle \right\}$$

$$|\bar{B}_d^0(0)\rangle \Rightarrow |\bar{B}_d^0(t)\rangle = e^{-im_s t} e^{-t/2\tau_s} \left\{ \cos(\Delta m_B t / 2) |\bar{B}_d^0\rangle + \frac{ip}{q} \sin(\Delta m_B t / 2) |B_d^0\rangle \right\}$$

$\tau_B \approx 1.5 \text{ ps } (10^{-12} \text{ Sec}) \quad \Delta m_B \equiv m_{B_H} - m_{B_L} \approx 3.1 \times 10^{-4} \text{ eV} = 0.49 \text{ ps}^{-1}$
 ($\tau_B = \tau_H = \tau_L$ assumed, very small theoretical expectation)



cf) K^0 : $\tau_{KL}(52 \text{ ns}) \gg \tau_{KS}(89 \text{ ps})$
 $\Delta m_K = 3.5 \times 10^{-6} \text{ eV} = 0.0053 \text{ ps}^{-1}$

Oscillate between B^0 and \bar{B}^0 (|q/p|=1)

Unmixed: $|\langle B^0 | B^0 \rangle|^2 = e^{-t/\tau} [1 + \cos(\Delta m t)] / 2$
 Mixed: $|\langle B^0 | \bar{B}^0 \rangle|^2 = e^{-t/\tau} [1 - \cos(\Delta m t)] / 2$

Search for CPV at Belle II

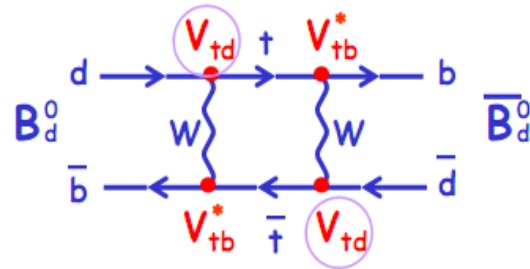
- $B^0 - \bar{B}^0$ mixing

$B^0 - \bar{B}^0$ Mixing (3)

Mechanism in SM

Box diagram

$$\frac{q}{p} \simeq \frac{V_{tb}^* V_{td}}{V_{td} V_{tb}^*}$$



$$\Delta m = \frac{G_F^2}{6\pi^4} m_B m_t^2 F(m_t^2/m_W^2) \eta_{\text{QCD}} B_{\text{Bd}} f_{\text{Bd}}^2 |V_{tb}^* V_{td}|^2$$

G_F = Fermi const., η_{QCD} = QCD correction factor,

$F(m_t^2/m_W^2)$ = Inami-Lim function

B_{Bd} = bag parameter (hadronic corr. For vac. insertion)

f_{Bd} = Bd decay constant

↗ Lattice QCD

$\Delta m \rightarrow V_{td}$: accuracy limited by uncertainty of $B_{\text{Bd}} f_{\text{Bd}}$



Statistics and the treatment of experimental data

0.1 Characteristics of probability distribution

Random processes are described by a probability density function (PDF).

PDF gives expected frequency of occurrence.

Random variable x can be continuous or discrete.

0.1.1 Cumulative distributions

Probability of finding x with $x_1 \leq x \leq x_2$ for x to be continuous

$$P(x) = \int_{x_1}^{x_2} P(x)dx \quad (1)$$

And for x to be discrete

$$P(x) = \sum_{i=1}^2 P(x_i) \quad (2)$$

The renormaliation condition for x to be continuous

$$\int P(x)dx = 1 \quad (3)$$

And for x to be discrete

$$\sum_i P(x_i) = 1 \quad (4)$$

0.1.2 Expectation values

For x to be continuous

$$E[x] = \int xP(x)dx \quad (5)$$

And for x to be discrete

$$E[x] = \sum_i x_i P(x_i) \quad (6)$$

If $f(x)$ is a continuous function of x then the expectation value of $f(x)$ is

$$E[f(x)] = \int f(x)P(x)dx \quad (7)$$

0.1.3 Distribution moments. Mean and variance

The n^{th} moment of x about some point x_0 is defined as the expectation value of $(x - x_0)^n$.

Only two first moments are important.

First moment about zero is called **mean** or **average** of x

$$\mu = E[x] = \int xP(x)dx \quad (8)$$

The second central moment is call **variance**

$$\sigma^2 = E[(x - \mu)^2] = \int (x - \mu)^2 P(x) dx \quad (9)$$

The square root of variance is call **standard deviation**

0.1.4 The covariance

The covariance measures linear correlation between two variables

$$\text{cov}(x, y) = E[(x - \mu_x)(y - \mu_y)] \quad (10)$$

- μ_x : mean of x .
- μ_y : mean of y .

Correlation coefficient

$$\rho = \frac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y} \quad (11)$$

- $-1 \leq \rho \leq 1$.
- $|\rho| = 1$: perfectly correlated linear.
- $\rho = 0$: x and y are linear independent.

0.2 Some common probability distributions

0.2.1 The binomial distribution

The binomial distribution involves repeated, independent trials, which outcome of a single trial is dichotomous.

The probability of n dichotomous trials, for example *success* and *failure* is given by

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \quad (12)$$

Where x is number of *successes* (or *failures*), p is probability of success in a single trial.

- x events occur with probability p each.
- $n - x$ events occur with probability $1 - p$ each.
- Note that $P(x)$ is the x^{th} term of binomial expansion

$$(a + b)^n = \sum_{k=0}^n C_k^n a^{n-k} b^k \quad (13)$$

Mean of binomial distribution is defined as the following

$$\mu = \sum_{x=0}^n xP(x) = \sum_{x=1}^n \frac{n!}{(n-x)!(x-1)!} p^x (1-p)^{n-x}$$

Let $N = n - 1$ and $y = x - 1$, we can rewrite the above equation as

$$\begin{aligned} \mu &= \sum_{x=1}^n \frac{n!}{(n-x)!(x-1)!} p^x (1-p)^{n-x} = \sum_{y=0}^N \frac{(N+1)!}{(N-y)!(y)!} p^{y+1} (1-p)^{N-y} \\ &= (N+1)p \sum_{y=0}^N \frac{(N)!}{(N-y)!(y)!} p^y (1-p)^{N-y} = (N+1)p [p + (1-p)]^N \\ &= (N+1)p \end{aligned}$$

Or

$$\mu = np \quad (14)$$

Variance of binomial distribution is defined as the following

#1 : By definition

$$\begin{aligned} \sigma^2 &= \sum_{x=0}^n (x - \mu)^2 P(x) = \sum_{x=0}^n (x - \mu)^2 \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^n (x - np)^2 \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \\ &= A + B + C \end{aligned}$$

Where

$$\begin{aligned} A &= \sum_{x=0}^n x^2 \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = \sum_{x=1}^n \frac{xn!}{(n-x)!(x-1)!} p^x (1-p)^{n-x} \\ &= \sum_{y=0}^N \frac{(y+1)(N+1)!}{(N-y)!y!} p^{y+1} (1-p)^{N-y} = np + n(n-1)p^2 \end{aligned}$$

$$B = -2np \sum_{x=0}^n x \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = -2(np)^2$$

And

$$C = \sum_{x=0}^n (np)^2 \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = (np)^2$$

Hence

$$\sigma^2 = \sum_x (x - \mu)^2 P(x) = np(1-p) \quad (15)$$

#2 : By using $\sigma^2 = \mu(x^2) - [\mu(x)]^2$

Where $\mu(x^2) = A = np + n(n-1)p^2$ and $[\mu(x)]^2 = (np)^2$. We get the similar result as (15)

$$\sigma^2 = \mu(x^2) - [\mu(x)]^2 = np(1-p)$$

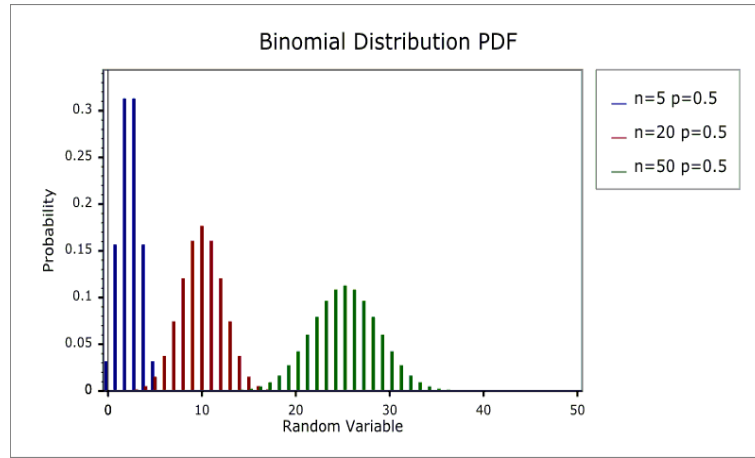


Figure 1: Binomial distribution for several values of n and p

0.2.2 The Poisson distribution

Poisson distribution is the case where we take the limits $p \rightarrow 0$ and $n \rightarrow \infty$ from binomial distribution such that $np = \mu = \text{const}$.

Poisson distribution is an appropriate model if the following assumptions are hold:

- x can take values: $0, 1, 2, \dots$
- Events occur independently.
- The rate at which events occur is constant.
- Two events cannot occur at exactly the same instant.
- Probability of an event in a small sub-interval is proportional to the length of the sub-interval.

Let $p = \frac{\mu}{n}$. When $n \rightarrow \infty$, we have following approximations

$$\frac{n!}{(n-x)!} = n(n-1)(n-2)\dots(n-x-2)(n-x-1) \approx n^x$$

And

$$(1-p)^{n-x} = \left(1 - \frac{\mu}{n}\right)^{-x} \left(1 - \frac{\mu}{n}\right)^n \approx e^{-\mu}$$

Note that: $\lim_{n \rightarrow \infty} \left(1 - \frac{\mu}{n}\right)^n \approx e^{-\mu}$

Then the Poisson distribution is defined as

$$P(x) = \lim_{n \rightarrow \infty} \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = \frac{\mu^x}{x!} e^{-\mu} \quad (16)$$

Mean of Poisson distribution

$$\mu_P = \sum_{x=0}^{\infty} x \frac{\mu^x e^{-\mu}}{x!} = \sum_{x=1}^{\infty} \frac{\mu^x e^{-\mu}}{(x-1)!} = \mu e^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!} = \mu \quad (17)$$

Variance of Poisson distribution

$$\begin{aligned} \sigma_P^2 &= \mu(x^2) - [\mu(x)]^2 = \sum_{x=0}^{\infty} x^2 \frac{\mu^x e^{-\mu}}{x!} - \mu^2 \\ &= \sum_{x=1}^{\infty} x \frac{\mu^x e^{-\mu}}{(x-1)!} - \mu^2 = \sum_{x=0}^{\infty} \frac{(x+1)\mu^{x+1}}{x!} e^{-\mu} - \mu^2 = \mu^2 + \mu - \mu^2 \\ &= \mu \end{aligned} \quad (18)$$

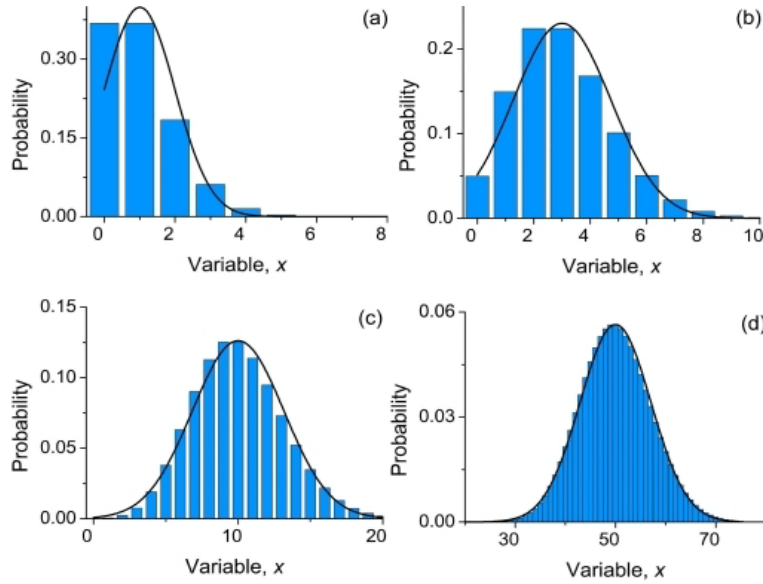


Figure 2: Poisson distribution for several values of μ

0.2.3 The Gaussian distribution

Gaussian is a continuous, symmetric distribution whose density is given by

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad (19)$$

Where μ is *expectation value* and σ^2 is *variance*.

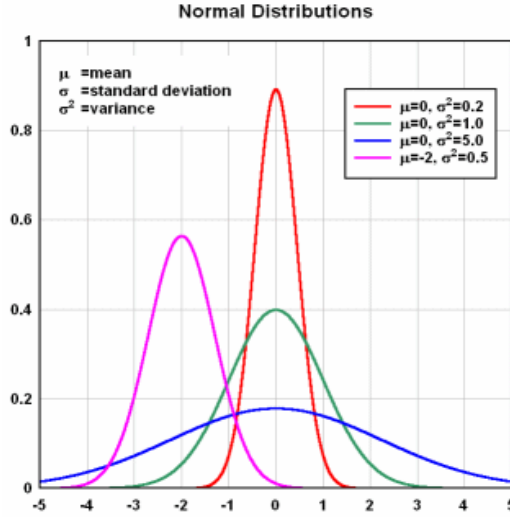


Figure 3: Gaussian distribution for several values of μ and σ^2

- **Derive Gaussian distribution from binomial distribution**

From the binomial distribution

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

By using Stirling's formula $n! = n^n e^{-n} \sqrt{2\pi n}$, we can rewrite the above equation as

$$\begin{aligned} P(x) &= \frac{n^n e^{-n} \sqrt{2\pi n}}{x^x e^{-x} \sqrt{2\pi x} (n-x)^{n-x} e^{-(n-x)} \sqrt{2\pi(n-x)}} p^x (1-p)^{n-x} \\ &= \left(\frac{np}{x}\right)^x \left(\frac{n(1-p)}{n-x}\right)^{n-x} \sqrt{\frac{n}{2\pi x(n-x)}} \end{aligned}$$

We have

$$\begin{aligned} L &= \ln \left[\left(\frac{np}{x}\right)^x \left(\frac{n(1-p)}{n-x}\right)^{n-x} \right] = -x \ln \left(\frac{x}{np}\right) - (n-x) \ln \left(\frac{n-x}{n(1-p)}\right) \\ &= -x \ln \left(\frac{x}{np}\right) - (n-x) \ln \left(1 + \frac{np-x}{n(1-p)}\right) \end{aligned}$$

$$\text{Let } \lambda = -(np-x) \quad \Rightarrow \quad x = \lambda + np.$$

By using $\ln(1+x) \approx x - \frac{1}{2}x^2 + \dots$, we get

$$\begin{aligned}
L &= -(\lambda + np) \ln \left(1 + \frac{\lambda}{np} \right) - (n(1-p) - \lambda) \ln \left(1 - \frac{\lambda}{n(1-p)} \right) \\
&= -(\lambda + np) \left(\frac{\lambda}{np} - \frac{1}{2} \frac{\lambda^2}{(np)^2} + \dots \right) - (n(1-p) - \lambda) \left(-\frac{\lambda}{n(1-p)} - \frac{1}{2} \frac{\lambda^2}{n^2(1-p)^2} + \dots \right) \\
&= -\left(\frac{\lambda^2}{np} + \lambda - \frac{1}{2} \frac{\lambda^2}{np} + \dots \right) - \left(-\lambda + \frac{\lambda^2}{n(1-p)} - \frac{1}{2} \frac{\lambda^2}{n(1-p)} + \dots \right) \\
&\approx -\lambda - \frac{\lambda^2}{2np} + \lambda - \frac{\lambda^2}{2n(1-p)} \\
&= -\frac{\lambda}{2np(1-p)}
\end{aligned}$$

Then

$$\left(\frac{np}{x} \right)^x \left(\frac{n(1-p)}{n-x} \right)^{n-x} = e^{-\frac{\lambda^2}{2np(1-p)}}$$

And for $n \rightarrow \infty$

$$\begin{aligned}
\sqrt{\frac{n}{2\pi x(n-x)}} &= \sqrt{\frac{n}{2\pi(\lambda+np)(n(1-p)-\lambda)}} \\
&\approx \frac{1}{\sqrt{2\pi np(1-p)}}
\end{aligned}$$

Therefore we finally get

$$P(x) = \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{\lambda^2}{2np(1-p)}}$$

Note that $\mu = np$ and $\sigma^2 = np(1-p)$. The Gaussian distribution is then defined as

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (20)$$

• **Derive Gaussian distribution from Poisson distribution**

The Poisson distribution is of the form

$$P(x) = \frac{\mu^x e^{-\mu}}{x!}$$

Use the Stirling's formula $x! = x^x e^{-x} \sqrt{2\pi x}$ and let $x = \mu(1+\lambda)$, where $\mu \gg 1$ and $\lambda \ll 1$

$$\begin{aligned}
P(x) &= \frac{\mu^x e^{-\mu}}{x^x e^{-x} \sqrt{2\pi x}} = \left(\frac{\mu}{x} \right)^x \frac{e^{-(\mu-x)}}{\sqrt{2\pi x}} \\
&= \left(\frac{\mu}{\mu(1+\lambda)} \right)^{\mu(1+\lambda)} \frac{e^{-(\mu-\mu(1+\lambda))}}{\sqrt{2\pi\mu(1+\lambda)}} = \frac{1}{\sqrt{2\pi\mu}} \frac{e^{-\mu\lambda}}{(1+\lambda)^{\mu(1+\lambda)} + 1/2}
\end{aligned}$$

We see that for $\mu \gg 1$ and $\lambda \ll 1$

$$\begin{aligned} \ln[(1 + \lambda)^{\mu(1+\lambda)} + 1/2] &= (\mu + \mu\lambda + 1/2) \ln(1 + \lambda) = (\mu + \mu\lambda + 1/2)(\lambda - \lambda^2/2 + \dots) \\ &\approx \mu\lambda + \frac{\mu\lambda^2}{2} \\ \Rightarrow (1 + \lambda)^{\mu(1+\lambda)} + 1/2 &\approx e^{\mu\lambda + \frac{\mu\lambda^2}{2}} \end{aligned}$$

It's then followed that

$$P(x) = \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{\mu\lambda^2}{2}} = \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{(x-\mu)^2}{2\mu}}$$

Note that, for Poisson distribution $\sigma^2 = \mu$. By substituting this into the above equation, we get exactly the same formula as (20).

• ***Derive Gaussian distribution from another way***

Consider we are aiming at the origin of a xy -plane with darts. Assume that:

- +1 : Deviation of darts not depend on the origin.
- +2 : Deviation in orthogonal directions are independent.
- +3 : Large deviation is less likely than small deviation.

Probability that the darts falls in interval $[x, x + \Delta x]$ is

$$P(x)dx$$

Similarly for interval $[y, y + \Delta y]$

Probability of falling in area dA is

$$P(x)P(y)\Delta x\Delta y$$

If we are aiming offset of the origin by a constant μ , then

$$P(x, y) = P(x + \mu)P(y + \mu)\Delta x\Delta y = g(x, y)\Delta x\Delta y$$

Where $g(x, y) = P(x + \mu)P(y + \mu)$.

In term of polar coordiantes where $x = r \cos \theta$, $y = r \sin \theta$, $g(r, \theta)$ is dependent on r , but not dependent on θ . Then

$$\begin{aligned} \frac{dg}{d\theta} = 0 &\Rightarrow \\ P(x + \mu)P'(y + \mu)\frac{dy}{d\theta} + P'(x + \mu)P(y + \mu)\frac{dx}{d\theta} &= 0 \\ \Leftrightarrow P(x + \mu)P'(y + \mu)r \cos \theta - P'(x + \mu)P(y + \mu)r \sin \theta &= 0 \\ \Leftrightarrow P(x + \mu)P'(y + \mu)x - P'(x + \mu)P(y + \mu)y &= 0 \\ \Rightarrow \frac{P'(x + \mu)}{xP(x + \mu)} = \frac{P'(y + \mu)}{yP(y + \mu)} = C; \quad \forall x, y \in R \end{aligned}$$

We solve for $P(x + \mu)$

$$\begin{aligned}\frac{P'(x + \mu)}{xP(x + \mu)} = C &\Rightarrow P(x + \mu) = Ae^{\frac{Cx^2}{2}} \\ \Rightarrow P(x) = P((x - \mu) + \mu) &= Ae^{\frac{C(x-\mu)^2}{2}}\end{aligned}$$

From assumption +3, we see that C must be negative then

$$P(x) = Ae^{\frac{-C(x-\mu)^2}{2}}; \quad C > 0$$

Normalized condition

$$\int_{-\infty}^{+\infty} Ae^{\frac{-C(x-\mu)^2}{2}} dx = 1 \quad \Rightarrow \int_{-\infty}^{+\infty} e^{\frac{-C(x-\mu)^2}{2}} dx = \frac{1}{A}$$

Use the Gaussian integral $\int_{-\infty}^{+\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \Rightarrow$

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{\frac{-C(x-\mu)^2}{2}} dx &= \frac{2\pi}{C} \frac{1}{A} \\ \Rightarrow A &= \sqrt{\frac{C}{2\pi}}\end{aligned}$$

Then

$$P(x) = \sqrt{\frac{C}{2\pi}} e^{\frac{-C}{2}(x-\mu)^2}$$

Let μ and σ^2 are the mean and the variance of the distribution, respectively. The variance is

$$\sigma^2 = \mu(x^2) - [\mu(x)]^2$$

Where

$$\begin{aligned}\mu(x^2) &= \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{\frac{-C}{2}(x-\mu)^2} dx \\ &= \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{+\infty} (x + \mu)^2 e^{\frac{-C}{2}x^2} dx \\ &= \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{\frac{-C}{2}x^2} dx + \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{+\infty} 2\mu x e^{\frac{-C}{2}x^2} dx + \sqrt{\frac{C}{2\pi}} \int_{-\infty}^{+\infty} \mu^2 e^{\frac{-C}{2}x^2} dx \\ &= \sqrt{\frac{C}{2\pi}} \sqrt{\frac{2\pi}{C}} \frac{1}{C} + \sqrt{\frac{C}{2\pi}} \cdot 0 + \sqrt{\frac{C}{2\pi}} \sqrt{\frac{2\pi}{C}} \mu^2 \\ &= \frac{1}{C} + \mu^2\end{aligned}$$

Then $\sigma^2 = \frac{1}{C}$, and therefore we find exactly the same form as (20)

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$